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PROJECTIVE ORTHOMODULAR LATTICES

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ABSTRACT. We introduce sectional projectivity, which appears to be the correct notion of projectivity when working with orthomodular lattices. We prove some positive results for varieties of OMLs satisfying various finiteness conditions, namely that every finite OML in such a variety is sectionally projective. In contrast, we prove that the eight element modular ortholattice, MO 3, is not projective in the variety of modular ortholattices.

1. **Introduction.** An algebra P is said to be *projective* in some class $\bar{k}$ to which it belongs if for every homomorphism $\alpha: P \longmapsto B \in \bar{k}$ and every onto homomorphism $\beta: C \longmapsto B$, $C \in \bar{k}$, there exists a homomorphism $\gamma: P \longmapsto C$ such that $\beta \circ \gamma = \alpha$. If $\bar{k}$ is a nondegenerate variety, more generally: if $\bar{k}$ has enough free algebras, then this is equivalent with the condition that P is a retract of every algebra of which it is a homomorphic image. Explicitly, if $A \in \bar{k}$ and $\phi: A \longmapsto P$ is an onto homomorphism then there exists a homomorphism $\psi: P \longmapsto A$ such that $\phi \circ \psi = \mathrm{id}_P$. Since we are only dealing with varieties we will take this last property as the definition of projectivity.

In this paper we start investigating projectivity in varieties of orthomodular lattices. In the first section we show that the well known result that every at most countable Boolean algebra is projective in the class of all Boolean algebras extends to the class of all orthomodular lattices. We found it convenient to weaken the concept of projective to s-projective. In Section 2 we examine the relationship between these two concepts and study their behaviour under the formation of products and factors. In Section 3 we show that every finite orthomodular lattice is s-projective in every variety satisfying certain finiteness conditions, in particular every finite orthomodular lattice is s-projective in the variety it generates. In the final section we prove that MO 3 is not s-projective in the variety of all modular ortholattices. Here we make use of an example constructed in [4], and thus indirectly of an analogous result for modular lattices [5].

Throughout the paper we abbreviate ortholattice as OL, orthomodular lattice as OML and modular ortholattice as MOL.

2. **Projective Boolean algebras.** For an element a of an OML L we define $a^0 = a$ and $a^1 = a'$.

LEMMA 2.1. *Let L be an* OML, *B an at most countable Boolean algebra and $\phi: L \longmapsto B$*

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a homomorphism onto B. Then there is a Boolean subalgebra C of L such that $\phi(C) = B$.

PROOF. Let $b_0, b_1, \ldots$ be the elements of B. For every n choose $a_n \in L$ such that $\phi(a_n) = b_n$ and define recursively elements c_n by $c_0 = a_0$ and $c_n = \bigwedge\{a_n \vee \bigvee_{i=0}^{n-1} c_i^{\alpha(i)} \mid \alpha\colon \{0, \ldots, n-1\} \longmapsto \{0,1\}\}$ for $n \geq 1$. Since for $m < n$, either $c_m \leq a_n \vee \bigvee_{i=0}^{n-1} c_i^{\alpha(i)}$ or $c'_m \leq a_n \vee \bigvee_{i=0}^{n-1} c_i^{\alpha(i)}$, any two of the c_n commute and hence the subalgebra C of L generated by the c_n is Boolean, see [8] pages 38, 39. We show by induction on n that $\phi(c_n) = b_n$ which obviously proves the claim. Clearly $\phi(c_0) = b_0$. For $n \geq 1$ by inductive hypothesis: $\phi(c_n) = \bigwedge\{b_n \vee \bigvee_{i=0}^{n-1} b_i^{\alpha(i)} \mid \alpha\colon \{0, \ldots, n-1\} \longmapsto \{0,1\}\} = b_n \vee \bigwedge\{\bigvee_{i=0}^{n-1} b_i^{\alpha(i)} \mid \alpha\colon \{0, \ldots, n-1\} \longmapsto \{0,1\}\} = b_n \vee \bigvee_{i=0}^{n-1}(b_i \wedge b'_i) = b_n$, completing the proof.

It is well known, see [6], that every at most countable Boolean algebra is projective in the variety of Boolean algebras. This with Lemma 2.1 gives,

THEOREM 2.2. *Every nondegenerate, at most countable Boolean algebra is projective in every nondegenerate variety of* OML*s*.

Lemma 2.1 and Theorem 2.2 were obtained in collaboration with R. Greechie. We are sure that these simple results were also known elsewhere.

3. **General results.** We say that an OML L is *sectionally projective*, or *s-projective*, in a class $\bar{k}$ of OMLs to which it belongs iff for every $M \in \bar{k}$ and every homomorphism $\phi\colon M \longmapsto L$ onto L, there exists $u \in M$ and a homomorphism $\psi\colon L \longmapsto [0, u]$, where $[0, u]$ is equipped with the usual orthocomplementation inherited from L, such that $\phi \circ \psi = \mathrm{id}_L$.

Clearly every OML L which is projective in some class $\bar{k}$ is s-projective in $\bar{k}$, but the converse is not true. A trivial example of this is the one-element OML. A more relevant example is the smallest non-Boolean OML, MO 2, *cf.* Figure 1, as we will show.

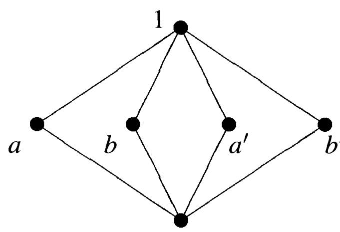

FIGURE 1: MO 2

PROPOSITION 3.1. *An* OML *L is projective in a non-degenerate variety* $\bar{k}$ *of* OML*s iff it is s-projective in* $\bar{k}$ *and contains a non-degenerate Boolean homomorphic image (or, equivalently, it contains a prime ideal).*

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PROOF. If L is projective it is clearly s-projective and there exists a homomorphism $\phi: L \longmapsto \mathbf{2} \times L$ such that $\mathrm{pr}_2 \circ \phi = \mathrm{id}_L$. Then $\mathrm{pr}_1 \circ \phi$ is a homomorphism of L onto $\mathbf{2}$. Assume conversely that L is s-projective and contains a prime ideal P. Let $\phi: M \longmapsto L$ be an onto homomorphism. By assumption, there exist $u \in M$ and a homomorphism $\psi: L \longmapsto [0, u]$ such that $\phi \circ \psi = \mathrm{id}_L$. Define a map $\alpha: L \longmapsto M$ by $\alpha(a) = \psi(a)$ if $a \in P$ and $\alpha(a) = u' \vee \psi(a)$ if $a \notin P$. It is easily checked that α is a homomorphism and that $\phi \circ \alpha = \mathrm{id}_L$. Thus L is projective in $\bar{k}$.

PROPOSITION 3.2. *If $L_1 \times L_2$ is s-projective in a variety $\bar{k}$ of* OMLs *then L_1 and L_2 are s-projective in $\bar{k}$.*

PROOF. We show that L_2 is s-projective. Let $\phi: M \longmapsto L_2$ be a homomorphism onto L_2. Define an onto homomorphism $\alpha: L_1 \times M \longmapsto L_1 \times L_2$ by $\alpha(x, y) = \big(x, \phi(y)\big)$, *i.e.* $\alpha = \mathrm{id}_{L_1} \times \phi$. Since $L_1 \times L_2$ is s-projective there exist $u \in L_1 \times M$ and a homomorphism $\beta: L_1 \times L_2 \longmapsto [0, u]$ such that $\alpha \circ \beta = \mathrm{id}_{L_1 \times L_2}$. Since $(0, 1) = \alpha\big(\beta(0, 1)\big), \beta(0, 1) = (0, v)$ for some $v \in M$. Let $\gamma = \beta|_{\{0\} \times L_2}$. Then γ is a homomorphism of $\{0\} \times L_2$ into $[(0, 0), (0, v)]$. Define $\psi: L_2 \longmapsto [0, v]$ by $\psi(y) = \mathrm{pr}_2\big(\gamma(0, y)\big)$. Then ψ is obviously a homomorphism. Since $\phi \circ \mathrm{pr}_2 = \mathrm{pr}_2 \circ \alpha$ we obtain for every $y \in L_2$, $\phi\big(\psi(y)\big) = \phi\Big(\mathrm{pr}_2\big(\beta(0, y)\big)\Big) = \mathrm{pr}_2(0, y) = y$, *i.e.* $\phi \circ \psi = \mathrm{id}_{L_2}$, proving the claim.

PROPOSITION 3.3. *If L_1 and L_2 are s-projective in a variety $\bar{k}$ of* OMLs *then $L_1 \times L_2$ is s-projective in $\bar{k}$.*

PROOF. Let $\phi: M \longmapsto L_1 \times L_2$ be an onto homomorphism. There exists an element $c \in M$ such that $\phi(c) = (1, 0)$ and hence $\phi(c') = (0, 1)$. Let ϕ_1 be the restriction of ϕ to $[0, c]$ and let ϕ_2 be the restriction of ϕ to $[0, c']$. Then $\phi_1: [0, c] \longmapsto L_1 \times \{0\}$ and $\phi_2: [0, c'] \longmapsto \{0\} \times L_2$ are onto homomorphisms. By assumption there exist elements $u \leq c$ and $v \leq c'$ and homomorphisms $\psi_1: L_1 \times \{0\} \longmapsto [0, u]$ and $\psi_2: \{0\} \times L_2 \longmapsto [0, v]$ such that $\phi_1 \circ \psi_1 = \mathrm{id}_{L_1 \times \{0\}}$ and $\phi_2 \circ \psi_2 = \mathrm{id}_{\{0\} \times L_2}$. Define $\psi: L_1 \times L_2 \longmapsto [0, u \vee v]$ by $\psi(x, y) = \psi_1(x, 0) \vee \psi_2(0, y)$. Clearly ψ preserves $\vee$. To prove that it preserves orthocomplementation note first that $\psi_1(x, 0) \leq u \leq c \leq v' \leq \psi_2(0, y)'$ and hence any two of $\psi_1(x, 0), u, v, \psi_2(0, y)$ commute. Thus, $\psi\big((x, y)'\big) = \psi(x', y') = \psi_1(x', 0) \vee \psi_2(0, y') = \big(u \wedge \psi_1(x, 0)'\big) \vee \big(v \wedge \psi_2(0, y)'\big) = (u \vee v) \wedge \big(v \vee \psi_1(x, 0)'\big) \wedge \big(u \vee \psi_2(0, y)'\big) \wedge \big(\psi_1(x, 0)' \vee \psi_2(0, y)'\big) = (u \vee v) \wedge \psi_1(x, 0)' \wedge \psi_2(0, y)' = (u \vee v) \wedge \big(\psi_1(x, 0) \vee \psi_2(0, y)\big)' = (v \vee v) \wedge \psi(x, y)'$, which is the orthocomplement of $\psi(u, v)$ in $[0.u \vee v]$. Finally, $\phi\big(\psi(x, y)\big) = \phi\big(\psi_1(x, 0) \vee \psi_2(0, y)\big) = \phi_1\big(\psi_1(x, 0)\big) \vee \phi_2\big(\psi_2(0, y)\big) = (x, 0) \vee (0, y) = (x, y)$, completing the proof.

These basic results have some interesting consequences.

COROLLARY 3.4. *An* OML *L is s-projective in a non-degenerate variety $\bar{k}$ of* OMLs *iff $\mathbf{2} \times L$ is projective in $\bar{k}$.*

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PROOF. If L is s-projective in $\bar{k}$ then $\mathbf{2} \times L$ is s-projective by Proposition 3.3 and Theorem 2.2. Since it has a non-degenerate Boolean homomorphic image it is projective by Proposition 3.1. Conversely, if $\mathbf{2} \times L$ is projective then L is s-projective by Proposition 3.2.

It is an easy exercise to show that MO 2 is s-projective in the variety of all OMLs. It is not projective by Proposition 3.1. Thus,

COROLLARY 3.5. MO 2 *is s-projective in the variety of all* OML*s but is not a projective* OML. *It even fails to be projective in the variety it generates.*

COROLLARY 3.6. *If B is a non-degenerate Boolean algebra, L an* OML *and if $B \times L$ is s-projective in a variety $\bar{k}$ then B is projective in $\bar{k}$.*

PROOF. This is a consequence of Propositions 3.1 and 3.2.

It is not true that a non-Boolean factor of a projective OML is projective, $\mathbf{2} \times$ MO 2 is a counter-example by Corollaries 3.4 and 3.5.

COROLLARY 3.7. *If L_1 is projective and L_2 is s-projective in a variety $\bar{k}$ of* OML*s then $L_1 \times L_2$ is projective in $\bar{k}$.*

PROOF. We may assume that $\bar{k}$ is non-degenerate. $L_1 \times L_2$ is s-projective by Proposition 3.3. Since L_1 has a non-degenerate Boolean homomorphic image, the same is true for $L_1 \times L_2$ and the claim follows from Proposition 3.1.

A *generalized* OML is an algebra $(L; \vee, \wedge, *, 0)$ of type $(2, 2, 2, 0)$ where, in particular, $(L; \vee, \wedge, 0)$ is a lattice with 0 and $a * b$ computes an orthocomplement of $a \wedge b$ in the interval $[0, b]$. These algebras correspond exactly to ideals in OMLs and have been studied extensively in [1], and papers cited therein. In much the same way that some people study rings without unit so that ideals become subrings, the ideals of a generalized OML are still generalized OMLs. It is easy to see that there is an isomorphism between the lattice of varieties of generalized OMLs and the lattice of varieties of OMLs. It is perhaps worth observing also that s-projectivity in OMLs corresponds exactly to projectivity in generalized OMLs.

4. **Finiteness conditions.** The following is a slight reformulation of Proposition 6 of [2].

PROPOSITION 4.1. *Let L be an irreducible* OML *having exactly n elements $a_1, \ldots, a_n$. Then there exists an* OL *polynomial q in n variables satisfying:*

1. If $b_1, \ldots, b_n$ are elements of any OML *and $b_i = b_j$ for any distinct indices i, j then $q(b_1, \ldots, b_n) = 0$.*

2. If $b_1, \ldots, b_n$ are elements of any OML *M and if $q(b_1, \ldots, b_n) = 1$ then $a_i \longmapsto b_i$ is an* OL*-embedding of L into M.*

3. If $b_1, \ldots, b_n \in L$ and if $a_i \longmapsto b_i$ is an automorphism of L then $q(b_1, \ldots, b_n) = 1$ and $q(b_1, \ldots, b_n) = 0$ otherwise.

LEMMA 4.2. *Let* $(L_k)_{k\in K}$ *be a family of* OMLs *of height at most m, let M be a subalgebra of* $\prod_{k\in K} L_k$ *and let* $\phi\colon M \longmapsto L$ *be a homomorphism onto a finite, irreducible* OML *L. Then there exists* $u \in M$ *and a homomorphism* $\psi\colon L \longmapsto [0,u]$ *such that* $\phi\circ\psi = \mathrm{id}_L$.

PROOF. Let $a_1,\ldots,a_n$ be the elements of L and q the polynomial of Lemma 4.1. Let $\alpha\colon L\longmapsto M$ be a map (not necessarily a homomorphism) such that $\phi\circ\alpha = \mathrm{id}_L$. In the following it is crucial in which algebra the polynomial q is applied. We add this in parentheses where it is not obvious. Define recursively elements $u_0, u_1, \ldots$ of M by:

$$u_0 = q\big(\alpha(a_1),\ldots,\alpha(a_n)\big) \quad (q \text{ applied in } M)$$
$$u_{i+1} = q\big(u_i\wedge\alpha(a_1),\ldots,u_i\wedge\alpha(a_n)\big) \quad (q \text{ applied in the interval } [0,u_i] \text{ of } M)$$

Clearly $u_{i+1}\le u_i$ holds for every i. We show by induction on i that $\phi(u_i)=1$ holds for every i.

$$\phi(u_0) = \phi\Big(q\big(\alpha(a_1),\ldots,\alpha(a_n)\big)\Big) = q\Big(\phi\big(\alpha(a_1)\big),\ldots,\phi\big(\alpha(a_n)\big)\Big) = q(a_1,\ldots,a_n) = 1.$$

Assume now that, for some i, $\mathrm{pr}_k(u_i) = \mathrm{pr}_k(u_{i+1})$. Then,

$$\begin{aligned}
\mathrm{pr}_k(u_{i+2}) &= \mathrm{pr}_k(q(u_{i+1}\wedge\alpha(a_1),\ldots,u_{i+1}\wedge\alpha(a_n)) \quad (q \text{ applied in } [0,u_{i+1}])\\
&= q\Big(\mathrm{pr}_k(u_{i+1})\wedge\mathrm{pr}_k\big(\alpha(a_1)\big),\ldots,\mathrm{pr}_k(u_{i+1})\wedge\mathrm{pr}_k\big(\alpha(a_n)\big)\Big) \quad (\text{in } [0,\mathrm{pr}_k(u_{i+1})])\\
&= q\Big(\mathrm{pr}_k(u_i)\wedge\mathrm{pr}_k\big(\alpha(a_1)\big),\ldots,\mathrm{pr}_k(u_i)\wedge\mathrm{pr}_k\big(\alpha(a_n)\big)\Big) \quad (\text{in } [0,\mathrm{pr}_k(u_i)])\\
&= \mathrm{pr}_k\Big(q\big(u_i\wedge\alpha(a_1),\ldots,u_i\wedge\alpha(a_n)\big)\Big) \quad (\text{in } [0,u_i])\\
&= \mathrm{pr}_k(u_{i+1}).
\end{aligned}$$

Since every chain in any L_k has at most $m+1$ elements it follows that $u_{m+1} = u_m$. Define $\psi\colon L\longmapsto[0,u_m]$ by $\psi(a_j) = u_m\wedge\alpha(a_j)$. Then, in $[0,u_m]$, $q\big(\psi(a_1),\ldots,\psi(a_n)\big) = q\big(u_m\wedge\alpha(a_1),\ldots,u_m\wedge\alpha(a_n)\big) = u_{m+1} = u_m$ (the unit of $[0,u_m]$). It follows from Proposition 4.1 that ψ is an OL embedding of L into $[0,u_m]$. For every a_j: $\phi\big(\psi(a_j)\big) = \phi\big(u_m\wedge\alpha(a_j)\big) = \phi(u_m)\wedge\phi\big(\alpha(a_j)\big) = a_j$, thus $\phi\circ\psi = \mathrm{id}_L$, proving the lemma.

THEOREM 4.3. *If a variety* $\bar{k}$ *of* OMLs *is generated by* OMLs *of height at most m or if every finitely generated member of* $\bar{k}$ *has finite height then every finite member of k is s-projective in* $\bar{k}$.

PROOF. By Proposition 3.3 it is enough to show that every finite irreducible member L of $\bar{k}$ is s-projective in $\bar{k}$. Let $\phi\colon M\longmapsto L$ be a homomorphism of $M\in\bar{k}$ onto L. By Birkhoff's subdirect representation Theorem we may assume that M is a subalgebra of a product $\prod_{k\in K} L_k$ of subdirectly irreducibles. If $\bar{k}$ is generated by OMLs of height less than or equal to m then, by Jónsson, every L_k has height less than or equal to m and the claim follows directly from our Lemma 4.2. If every finitely generated member of

$\bar{k}$ has finite height and if the $\bar{k}$-free algebra generated by an n-element set has height m then obviously every n-generated member of $\bar{k}$ has height at most m. Assume now that L has n elements. Choose a subset S of M the elements of which are, via ϕ, in a one-one correspondence with the elements of L. Let N be the subalgebra of M generated by S and let L'_k be the subalgebra of L_k generated by $\mathrm{pr}_k(S)$. Then every L'_k has height at most m, N is a subalgebra of $\prod_{k\in K} L'_k$ and the restriction of ϕ to N maps N onto L. We thus again are in a position where our Lemma 4.2 applies, completing the proof of the theorem.

As an obvious consequence of this we obtain

COROLLARY 4.4. *Every finite* OML *is s-projective in the variety it generates.*

5. MO 3 is not s-projective.

LEMMA 5.1. *Assume that x, y, z are elements of an* MOL, *L which generate* MO 3 *in the interval* $[0, u]$, *that $u < v$ and that $v \wedge u' = p \vee q = q \vee r = q \vee r$, where p, q, r are atoms, no two of which commute. Define $a = x \vee p$, $b = y \vee q$ and $c = z \vee r$. Then a, b, c generate* MO 3 *in* $[0, v]$.

PROOF. Since $x, y, z \leq u \leq p', q', r'$ each of x, y, z commutes with each of p, q, r. Let $^\#$ be the orthocomplementation in $[0, v]$. Then $a\wedge b = (x\vee p)\wedge(y\vee q) \leq (x\vee p)\wedge(y\vee p\vee q) = p\vee\big(x\wedge(y\vee p\vee q)\big) = p\vee(x\wedge y)\vee\big(x\wedge(p\vee q)\big) \leq p\vee\big(u\wedge(p\vee q)\big) = p$. By symmetry, $a\wedge b \leq q$ and thus $a\wedge b = 0$. Furthermore, $a\wedge b^\# = (x\vee p)\wedge y'\wedge q' = \big((x\wedge y')\vee(p\wedge y')\big)\wedge q' = p\wedge q' = 0$. and $a^\# \wedge b^\# = x' \wedge p' \wedge y' \wedge q' \wedge v = x' \wedge y' \wedge v \wedge (v' \vee u) = x' \wedge y' \wedge u = 0$. The rest follows by symmetry.

For elements a, b of a modular lattice we define $a <_n b$ to mean that $a \leq b$ and every maximal chain in the interval $[a, b]$ has exactly $n+1$ elements. We define $a \leq_n b$ to mean that $a \leq b$ and that every chain in $[a, b]$ has at most $n + 1$ elements. For intervals $[a, b]$, $[c, d]$ in a lattice we define, as usual, $[a, b] \nearrow [c, d]$, or equivalently, $[c, d] \searrow [a, b]$ to mean that $b \vee c = d$ and $b \wedge c = a$.

LEMMA 5.2. *If, in a modular lattice, $x <_k y$, $u <_l v$ and $y \wedge v = 0$ then $x \vee u <_{k+l} y \vee v$.*

PROOF. It is easily checked that $[x, y] \nearrow [x \vee u, y \vee u]$ and $[u, v] \nearrow [y \vee u, y \vee v]$, which gives the claim.

In a modular lattice L we define a congruence relation $\equiv$ by $a \equiv b$ iff the interval $[a \wedge b, a \vee b]$ has finite height. If L is complemented then this congruence relation corresponds to the p-ideal of all elements of finite height. For elements a, b, c, d, T of a lattice L define $\Lambda(a, b, c, d, T)$ to mean that $a \vee b = a \vee c = a \vee d = b \vee d = c \vee d = T$, $a\wedge b = a\wedge c = a\wedge d = b\wedge c = b\wedge d = c\wedge d = 0$, and $b\vee c <_1 T$. We define $M(a, b, c, d, T)$ to mean the same except we replace the last condition with $b \vee c = T$.

LEMMA 5.3. *In a bounded modular lattice, $\Lambda(a, b, c, d, T)$, $M(\alpha, \beta, \gamma, \delta, \Gamma)$, $T \leq_1 1$, $\Gamma \leq_1 1$, $\beta \leq b$, $a \equiv \alpha$, $b \equiv \beta$, $c \equiv \gamma$, $d \equiv \delta$ is impossible.*

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PROOF. Define k, l, m, n, p, q by: $a \wedge \alpha <_k a$, $a \wedge \alpha <_l \alpha$, $c \wedge \gamma <_m c$, $c \wedge \gamma <_n \gamma$, $d \wedge \delta <_p d$, $d \wedge \delta <_q \delta$. The following transpositions are easily checked.

(1) $[a\wedge\alpha, a] \nearrow [b\vee(a\wedge\alpha), T]; [b\vee(a\wedge\alpha), b\vee\alpha] \searrow [(b\wedge\alpha)\vee(a\wedge\alpha), \alpha] \subseteq [a\wedge\alpha, \alpha]$,

(2) $[c\wedge\gamma, c] \nearrow [b\vee(c\wedge\gamma), b\vee c]; [b\vee(c\wedge\gamma), b\vee\gamma] \searrow [(b\wedge\gamma)\vee(c\wedge\gamma), \gamma] \subseteq [c\wedge\gamma, \gamma]$,

(3) $[d\wedge\delta, d] \nearrow [b\vee(d\wedge\delta), T]; [b\vee(d\wedge\delta), b\vee\delta] \searrow [(b\wedge\delta)\vee(d\wedge\delta), \delta] \subseteq [d\wedge\delta, \delta]$.

Since, $a \wedge d = \alpha \wedge \delta = c \wedge d = \gamma \wedge \delta = 0$ we obtain from Lemma 5.2:

(4) $(a \wedge \alpha) \vee (d \wedge \delta) <_{k+p} a \vee d$; $(a \wedge \alpha) \vee (d \wedge \delta) <_{l+q} \alpha \vee \delta$,

(5) $(c \wedge \gamma) \vee (d \wedge \delta) <_{m+p} c \vee d$; $(c \wedge \gamma) \vee (d \wedge \delta) <_{n+q} \gamma \vee \delta$.

Assume now first that $b \not\leq \Gamma$. Then $\Gamma <_1 1$, $b \vee \gamma \geq \beta \vee \gamma = \Gamma$, $b \vee \delta \geq \beta \vee \delta = \Gamma$ and hence $b \vee \gamma = b \vee \delta = 1$. From (2) and (3) we obtain $m < n$ and $p \leq q$, and $m+p < n+q$. From (5) we get $m + p = n + q$, again a contradiction.

This with (5) gives $m + p \geq n + q > m + p$, a contradiction.

We next deal with the case $b \leq \Gamma$ and $T <_1 1$. Then $b \vee \gamma = b \vee \delta = \Gamma$ and from (2) and (3) we obtain $m < n$ and $p \leq q$. This with (5) gives $n + q \leq m + p + 1 < n + q + 1$, hence $n = m + 1$ and $p = q$. From (2) we now obtain that $\Gamma = b \vee \gamma <_1 1$ and with (5) we get $m + p = n + q = m + p + 1$, again a contradiction.

The third case we deal with is $\Gamma = T = 1$. Here (2), (3) and (5) yield $m < n$, $p \leq q$ and $n + q = m + p$, which is also impossible. The only remaining case is $b \leq \Gamma < 1 = T$. In this case (2), (3) and (5) give $m \leq n$, $p \leq q + 1$ and $n + q = m + p - 1 \leq n + q$. Thus, $p = q + 1$ and $m = n$. With (3) this gives, $b \wedge \delta \leq d \wedge \delta$ and hence $b \wedge \delta = 0$. We thus have $\beta \leq b$, $b \wedge \delta = \beta \wedge \delta = 0$ and $b \vee \delta = \beta \vee \delta = \Gamma$ and hence $b = \beta$. We apply (1) to obtain $l = k - 1$. This with (4) gives $l + q = k + p - 1 = l + 1 + q + 1 - 1$, again a contradiction, proving the lemma.

For elements x, y of a modular lattice we define $x \sqsubseteq y$ to mean $x \equiv y$, $x \wedge y <_m x$, $x \wedge y <_n y$ and $m \leq n$.

LEMMA 5.4. *In a modular* OL *assume* $\Lambda(a, b, c, d, 1)$, $M(\alpha, \beta, \gamma, \delta, \Gamma)$, $\Gamma \leq_1 1$, $b = a'$, $\beta = \alpha' \wedge \Gamma$, $a \equiv \alpha$, $b \equiv \beta$, $c \equiv \gamma$, $d \equiv \delta$ *and* $\beta \not\sqsubseteq b$. *Define,* $a_1 = b$, $b_1 = a \wedge (b \vee c)$, $c_1 = d \wedge (b \vee c)$, $d_1 = c$, $T = b \vee c$. *Then* $\Lambda(a_1, b_1, c_1, d_1, T)$, $T <_1 1$, $M(\beta, \alpha, \delta, \gamma, \Gamma)$, $a_1 \equiv \beta$, $b_1 \equiv \alpha$, $c_1 \equiv \delta$, $d_1 \equiv \gamma$ *and* $\alpha \sqsubseteq b_1$.

PROOF. It is easily checked that $a_1 \vee b_1 = a_1 \vee c_1 = a_1 \vee d_1 = b_1 \vee d_1 = c_1 \vee d_1 = T$ and $a_1 \wedge b_1 = a_1 \wedge c_1 = a_1 \wedge d_1 = b_1 \wedge c_1 = b_1 \wedge d_1 = c_1 \wedge d_1 = 0$. The transpositions $[b\vee c, 1] \searrow [d\wedge(b\vee c), d] \nearrow \left[a\vee\left(d\wedge(b\vee c)\right), 1\right] \searrow [b_1\vee c_1, T]$ show that $b_1 \vee c_1 <_1 T$, proving $\Lambda(a_1, b_1, c_1, d_1, T)$. $T = b \vee c <_1 1$ is part of the definition of $\Lambda(a, b, c, d, 1)$ and $M(\beta, \alpha, \delta, \gamma, \Gamma)$ is also obvious as are $a_1 \equiv \beta$, $b_1 \equiv \alpha$, $c_1 \equiv \delta$, $d_1 \equiv \gamma$. It remains to show that $\alpha \sqsubseteq b_1$.

If $\beta \wedge b <_n b$ and $\beta \wedge b <_k \beta$ then, by the assumption $\beta \not\sqsubseteq b$, $n < k$. It follows that $a = b' <_n \beta' \vee a$, $\beta' <_k \beta' \vee a$, $\beta' \wedge a <_n \beta'$ and $\beta' \wedge a <_k a$. Let $\beta' \wedge \Gamma \wedge a <_p \beta' \wedge a$. Assume now first that $\alpha \wedge a \leq b \vee c$. Then $\alpha \wedge b_1 = \alpha \wedge a \wedge (b \vee c) = \alpha \wedge a = \beta' \wedge \Gamma \wedge a <_p \beta' \wedge a <_n \beta' \geq_1 \beta' \wedge \Gamma = \alpha$ and $\alpha \wedge b_1 <_p \beta' \wedge a <_k a >_1 a \wedge (b \vee c) = b_1$. Thus $\alpha \wedge b_1 \leq_{p+n} \alpha$ and $\alpha \wedge b_1 <_{p+k-1} b_1$. Since $n \leq k - 1$, this gives $\alpha \sqsubseteq b_1$.

Assume next that $\alpha \wedge a \not\leq b \vee c$. Then $\alpha \wedge b_1 = \alpha \wedge a \wedge (b \vee c) <_1 \alpha \wedge a = \beta' \wedge \Gamma \wedge a <_p \beta' \wedge a <_n \beta' \geq_1 \beta' \wedge \Gamma = \alpha$ and $\alpha \wedge b_1 <_{p+1} \beta' \wedge a <_k a >_1 a \wedge (b \vee c) = b_1$. Thus $\alpha \wedge b_1 \leq_{1+p+n} \alpha$ and $\alpha \wedge b_1 <_{p+k} b_1$ and again $\alpha \sqsubseteq b_1$, proving the lemma.

We are now in a position to prove

THEOREM 5.5. MO 3 *is not s-projective in the variety of all* MOL*s*.

PROOF. In [4] we found closed subspaces a, c, d of a separable real Hilbert space H with the following properties:

> If x' is the orthogonal complement of x then $a \vee c = a \vee c' = a \vee d = a \vee d' = a' \vee c' = a' \vee d = a' \vee d' = c \vee d = c \vee d' = c' \vee d = c' \vee d' = H$ and $a' \vee c <_1 H$. If F is the p-ideal of all finite-dimensional subspaces of H and L the set of all closed subspaces of H which are congruent modulo F with one of $0, a, a', c, c', d, d', H$ then L is a subortholattice of the orthomodular lattice $\mathcal{C}(H)$ of all closed subspaces of H. For all $x, y \in L$, $x \vee y$ (taken in $\mathcal{C}(H)$) is just the algebraic (or vector space) sum of x and y, in particular L is modular.

It is clear from these properties that the quotient L/F is MO3. If MO 3 was s-projective there would exist $\alpha, \gamma, \delta, \Gamma \in L$ such that $\alpha \equiv a, \gamma \equiv c, \delta \equiv d$, and $\Gamma \equiv 1$ (= H, the unit of $\mathcal{C}(H)$) and such that α, γ, δ generate MO3 in the interval $[0, \Gamma]$. Using Lemma 5.1, perhaps repeatedly, we may assume that $\Gamma \leq_1 1$. With $b = a'$ and $\beta = \alpha' \wedge \Gamma$ the assumptions of Lemma 5.4 are satisfied with the possible exception of $\beta \not\sqsubseteq b$. If $\beta \not\sqsubseteq b$ we apply Lemma 5.4.

Renaming the elements, if necessary, we obtain elements $a \equiv \alpha$, $b \equiv \beta$, $c \equiv \gamma$, $d \equiv \delta$, $\Gamma \leq_1 1$, $T \leq_1 1$ such that $\Lambda(a, b, c, d, T)$, $M(\alpha, \beta, \gamma, \delta, \Gamma)$ and $\beta \sqsubseteq b$. Assume $\beta \wedge b <_m \beta <_n \beta \vee b$, $\beta \wedge b <_k b <_l \beta \vee b$. Then $m \leq k$ and $m + n = k + l$. Let $\{e_1, \ldots, e_m\}$ be a basis of $\beta \wedge (\beta \wedge b)'$. Extend this to a basis $\{e_1, \ldots, e_m, e_{m+1}, \ldots e_{m+n}\}$ of $(\beta \vee b) \wedge (\beta \wedge b)'$. Let $\{f_1, \ldots, f_k\}$ be a basis of $b \wedge (\beta \wedge b)'$. Extend this to a basis $\{f_1, \ldots, f_{k+l}\}$ of $(\beta \vee b) \wedge (\beta \wedge b)'$. Clearly there is an automorphism ϕ of the vector space H (not necessarily preserving orthogonality) which maps $\beta \wedge b$ and $(\beta \vee b)'$ identically and satisfies $\phi(e_i) = f_i$. In the lattice of (not necessarily closed) subspaces of H we then have $\phi(\alpha) \equiv \alpha$, $\phi(\beta) \equiv \beta$, $\phi(\gamma) \equiv \gamma$, $\phi(\delta) \equiv \delta$, and $\phi(\beta) \leq b$. Thus, if we replace $\alpha, \beta, \gamma, \delta, \Gamma$ by their images under ϕ, the assumptions of Lemma 5.3 are fulfilled, which, by Lemma 5.3, is impossible. This proves the theorem.

6. Concluding remarks.

We are painfully aware that our investigations provide little information regarding the basic question: 'Which OMLs are s-projective in the variety of all OMLs?' We have seen that MO 2 is s-projective in the variety of all OMLs and that MO 3 is not even s-projective in the variety of all MOLs. We suspect that MO 2 and **2** are the only finite, irreducible, s-projective OMLs but we see no way of proving it. We have in fact a stronger conjecture. It is well known, see [3], that the free OML generated by an n-element set is of the form $B \times N$ where B is the free Boolean algebra on n elements and N is an OML without a non-trivial Boolean image. It follows from Proposition 3.1 that N is s-projective. Our conjecture is: The N are the only finitely generated, irreducible, non-Boolean s-projective OMLs.

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