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# NONTRIVIAL RATIONAL POLYNOMIALS IN TWO VARIABLES HAVE REDUCIBLE FIBRES

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We show that every $f : \mathbb{C}^2 \to \mathbb{C}$ which is a rational polynomial map with irreducible fibres is a coordinate.

We shall call a polynomial map $f: \mathbb{C}^2 \to \mathbb{C}$ a "coordinate" if there is a $g$ such that $(f,g): \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial automorphism. Equivalently, by Abhyankar–Moh and Suzuki [**1, 12**], $f$ has one, and therefore all, fibres isomorphic to $\mathbb{C}$. Following [**7**] we call a polynomial $f: \mathbb{C}^2 \to \mathbb{C}$ "rational" if the general fibres of $f$ (and hence all fibres of $f$) are rational curves. The following theorem, which says that a rational polynomial map with irreducible fibres cannot be part of a counterexample to the 2-dimensional Jacobian Conjecture, has appeared in the literature several times. It appears with an algebraic proof in Razar [**10**]. It appears in [**4**, Theorem 2.5] (as corrected in the Corrigendum), and Lê and Weber, who give a geometric proof in [**6**], also cite the reference Friedland [**3**], which we have not seen.

**THEOREM 1.** *If $f: \mathbb{C}^2 \to \mathbb{C}$ is a rational polynomial map with irreducible fibres and is not a coordinate then $f$ has no jacobian partner (that is, there is no polynomial $g$ such that the jacobian of $(f,g)$ is a nonzero constant).*

In this note we prove the above theorem is empty:

**THEOREM 2.** *There is no $f$ satisfying the assumptions of the above theorem. That is, a rational $f$ with irreducible fibres is a coordinate.*

PROOF: This theorem is implicit in [**7**]. Suppose $f$ is rational. As in [**7, 6, 9**], et cetera, we consider a nonsingular compactification $Y = \mathbb{C}^2 \cup E$ of $\mathbb{C}^2$ such that $f$ extends to a holomorphic map $\overline{f}: Y \to \mathbb{P}^1$. Then $E$ is a union of smooth rational curves $E_1, \dots, E_n$ with normal crossings. An $E_i$ is called *horizontal* if $\overline{f} \mid E_i$ is nonconstant. Let $\delta$ be the number of horizontal curves. Then we have

$$\delta - 1 = \sum_{a \in \mathbb{C}} (r_a - 1),$$

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where $r_a$ is the number of irreducible components of $f^{-1}(a)$. This is Miyanishi and Sugie [**7**, Lemma 1.6] who attribute it to Saito [**11**], and Lê and Weber [**6**, Lemma 4] who attribute it to Kaliman [**5**, Corollary 2]. The proof is simple arithmetic from the topological observation that on the one hand the Euler characteristic of $Y$ is $n+2$ and on the other hand it is $4 + \sum_{a \in \mathbb{P}^1} (\overline{r}_a - 1)$, where $\overline{r}_a$ is the number of components of $\overline{f}^{-1}(a)$, $a \in \mathbb{P}^1$.

By this formula, if $f$ has irreducible fibres then there is just one horizontal curve. [**7**, Lemma 1.7] now says that $f$ is a coordinate. This also follows from the following proposition, which implies that the generic fibres of $f$ have just one point at infinity and are thus isomorphic to $\mathbb{C}$. ∎

**Proposition 3.** *Let $f: \mathbb{C}^2 \to \mathbb{C}$ be any polynomial map and $\overline{f}: Y \to \mathbb{P}^1$ an extension as above. Denote by $d$ the greatest common divisor of the degrees of $\overline{f}$ on the horizontal curves of $Y$ and $D$ the sum of these degrees. Then the general fibre of $f$ has $d$ components (so $f = h \circ f_1$ for some polynomials $f_1: \mathbb{C}^2 \to \mathbb{C}$ and $h: \mathbb{C} \to \mathbb{C}$ with* degree $(h) = d$*), each of which is a compact curve with $D/d$ punctures.*

Proof: Let $E_1, \ldots, E_\delta$ be the horizontal curves and $d_1, \ldots, d_\delta$ be the degrees of $\overline{f}$ on these. Note that the points at infinity of a general fibre $f^{-1}(a)$ are the points where $\overline{f}^{-1}(a)$ meet the horizontal curves $E_i$, so there are $d_i$ such points on $E_i$ for $i = 1, \ldots, \delta$. The relationship between plumbing diagram and splice diagram (see [**9**, **2**]) says that the splice diagram $\Gamma$ for a regular link at infinity for $f$ (see [**8**]) has $\delta$ nodes with arrows at them, and the number of arrows at these nodes are $d_1, \ldots, d_\delta$ respectively. Let $\Gamma_0$ be the same splice diagram but with $d_1/d, \ldots, d_\delta/d$ arrows at these nodes. Then a minimal Seifert surface $S$ for the link represented by $\Gamma$ will consist of $d$ parallel copies of a minimal Seifert surface for the link represented by $\Gamma_0$, so this $S$ has $d$ components. But the general fibre of $f$ is such a minimal Seifert surface [**8**, Theorem 1], completing the proof. (It also follows that $\Gamma_0$ is the regular splice diagram for the polynomial $f_1$ of the proposition.) ∎

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