Moduli **2**, e9 (2025) 1–31
doi:10.1112/mod.2025.1

Asymptotic Chow stability of symmetric reflexive toric varieties

King Leung Lee

Abstract

In this note, we study the asymptotic Chow stability of symmetric reflexive toric varieties. We provide examples of symmetric reflexive toric varieties that are not asymptotically Chow semistable. On the other hand, we also show that any weakly symmetric reflexive toric varieties which have a regular triangulation (so are special) are asymptotically Chow polystable. Furthermore, we give sufficient criteria to determine when a toric variety is asymptotically Chow polystable. In particular, two examples of toric varieties are given that are asymptotically Chow polystable, but not special. We also provide some examples of special polytopes, mainly in two or three dimensions, and some in higher dimensions.

Contents

Received 10 May 2023, accepted 2 January 2025.
2020 Mathematics Subject Classification: 14J15, 14M25, 52B99.
Keywords: toric varieties, Chow stability.

1. Introduction

In Geometric invariant theory (GIT), when constructing moduli spaces one typically focuses on those varieties that are asymptotically Chow semistable [MFK94, GKZ94]. Chow stability has many relations with other stabilities in Kähler geometry (see [RT07, Yot17] for example), so it is important to study the Chow stability of singular varieties. However, unlike the smooth case, which is related to the constant scalar curvature (cscK) manifolds ([Don01, Mab04, Mab06, RT07]; see also the survey paper [PS10] for example), in general, K stability or the existence of cscK cannot imply Chow stability. Moreover, there are few examples of Chow polystable singular varieties. In general, it is very difficult to show that a variety is asymptotically Chow semistable. However, by the work of Futaki [Fut04] and Ono [Ono13], we can determine the asymptotic Chow polystability of toric varieties. We first recall the main theorem we used in [Ono13] (see also [LLSW19]).

Theorem 1.1 ([Ono13]). *Let P be an integral convex polytope of an n-dimensional toric variety X_P, and let $G < SL(n,\mathbb{Z})$ be the biggest finite group acting on P by multiplication. Then X_P is asymptotically Chow semistable if and only if for any $k \in \mathbb{N}$, and for any convex G invariant function f on kP, we have*

$$\frac{1}{\mathrm{Vol}(kP)}\int_{kP} f\,dV \leqslant \frac{1}{\chi(kP\cap\mathbb{Z}^n)}\sum_{v\in kP\cap\mathbb{Z}^n} f(v).$$

As a remark, in the original literature, Ono used concave functions instead of convex functions, so the direction of inequality in this note is different.

In this note, we mainly focus on symmetric reflexive toric varieties. One of the reasons is inspired by [BS99], which shows that if a polytope is symmetric and reflexive, then it admits a Kähler–Einstein metric. With the result of [Don02], we can see that symmetric and reflexive implies K stability. So it is natural to ask if it is true for Chow stability. The second reason is that, in this note, we define an invariant called Futaki–Ono invariant, which is

$$FO_P(a,k) := \frac{1}{\chi(kP\cap\mathbb{Z}^n)}\sum_{kp\in kP\cap\mathbb{Z}^n} a(p) - \frac{1}{\mathrm{Vol}(P)}\int_P a(x)\,dV.$$

As a rephrasing of Corollary 4.7 in [Ono13], we can see that if P is asymptotically Chow semistable, then this invariant will vanish for all $k >> k_0$ and for all affine functions a. We can see that symmetric polytopes satisfy this criterion, so it is natural to study symmetric polytopes. Also, by Claim 4.3 in [LLSW19], there is an example in which a symmetric non-reflexive polytope is not asymptotic Chow semistable. On the other hand, by the results in [LLSW19], with the fact that $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ are asymptotic Chow polystable, we can see that all two-dimensional

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

symmetric reflexive toric varieties are asymptotically Chow polystable. So it is natural to study symmetric reflexive polytopes.

However, a toric variety being symmetric and reflexive is generally not enough to guarantee that it is asymptotically Chow semistable, as illustrated by Example 3.9. Notice that this is not an isolated example. Indeed we can construct many examples using Proposition 3.7.

Therefore, to ensure asymptotic Chow polystability holds, one need more conditions on symmetric reflexive polytopes. One of the sufficient conditions is given by the following.

Definition 1.2 (See Definition 4.3 for the details). Let P be an n-dimensional integral convex polytope on $\mathbb{R}^n$. We say P has regular boundary if for any $k \in \mathbb{N}$, there exists a triangulation of ∂kP which every 'triangle' is integrally isomorphic to

$$T_{n-1} := \operatorname{conv}\{(0, \dots, 0), e_1, \dots, e_{n-1}\},$$

the standard $(n-1)$-dimensional simplex (i.e. the intersection between different T_{n-1}^i are at the boundary) such that:

(i) for any point $p \in \partial kP$, the number of simplices intersects with p, denoted as $m_k(p)$, is bounded by $n!$ (i.e. $m_k(p) \leqslant n!$); and

(ii) this is the sub-triangulation of each face.

Here, integrally isomorphic means one of the objects is obtained from another object by an integral rigid motion, i.e. the multiplication of a matrix $A \in SL(n, \mathbb{Z})$ and translation of $v \in \mathbb{Z}^n$.

We also make the following definition.

Definition 1.3 (See Definition 4.5 for the details). An integral convex polytope on $\mathbb{R}^n$ is called special if it is reflexive, weakly symmetric, and has a regular boundary.

One of our main theorems is given as follows.

Theorem 1.4 (See Theorem 6.1 for the details). *Let P be a special polytope. Then P is asymptotically Chow polystable.*

Notice that this condition is not necessary, as Theorem 8.1 gives another sufficient criterion to show when a toric variety P is asymptotically Chow polystable. The statement of the theorem is the following.

Theorem 1.5 (See Theorem 8.1 for the details). *Let P be an integral polytope with $0 \in P^0$ such that all the Futaki–Ono invariants vanish, and one has a triangulation on kP by n simplices, and a triangulation on ∂kP by $(n-1)$ simplices. We let $n(p;k)$ be the number of n simplices attached to $p \in kP$ in the first triangulation, and $m(p;k)$ be the number of $(n-1)$ simplices attached to $p \in \partial kP$ in the second. Suppose $n(p;k) \leqslant (n+1)!$ for all $p \neq 0$, and*

$$\binom{n}{2} m(p;k) < ((n+1)! - n(p;k)),$$

for all $k >> 0$ and for all $p \in \partial kP$. Then P is asymptotically Chow polystable.

As a concrete example, we have the following corollary.

Corollary 1.6 (See Corollary 8.4 for the details). *$D(X_8)$ and $D(X_9)$ are asymptotically Chow polystable, where $D(X_8)$ and $D(X_9)$ are defined in Example 4.8.*

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

This example shows that there are non-special symmetric reflexive polytopes that are asymptotically Chow polystable.

In the last section, we provide examples which are asymptotically Chow polystable, mainly of dimension 3, and have two classes of examples for higher dimensions. Notice that besides $D(X_8)$ and $D(X_9)$, the remaining examples are special. Also, the corresponding varieties of the examples are given in the Appendix.

2. Chow stability of toric varieties and criteria

2.1 GIT and Chow stabilities

In this section, we briefly recall some basic knowledge of GIT. For details, please read [GKZ94].

Definition 2.1. Let G be a reductive algebraic group and V be a finite-dimensional complex vector space such that G acts linearly on V. Let $v \neq 0 \in V$, and let $\mathcal{O}_G(v)$ be the G-orbit in V. Then:

- (i) v is called G-semistable if the Zariski closure of $\mathcal{O}_G(v)$ does not contain the origin, i.e. $0 \notin \overline{\mathcal{O}_G(v)}$;
- (ii) v is called G-polystable if $0 \notin \mathcal{O}_G(v)$ is a closed orbit;
- (iii) v is called G-stable if v is G-polystable and Stabilizer of v, G_v, is a finite group.

It is said that $[v] \in \mathbb{P}(V)$is G-polystable (respectively, semistable) if $v \in V$ is G-polystable (respectively, semistable).

Let (X, L) be an n-dimensional polarized variety of degree d, i.e. X is a complex irreducible variety with degree d, and L is an ample line bundle. Then there exists $k_0 \in \mathbb{N}$ such that, for every integer $k \geqslant k_0$, there exists an embedding map $\iota_k \colon X \to \mathbb{P}(H^0(X, kL)) \cong \mathbb{P}^{N_k}$ defined by

$$\iota_k(x) := [s_0(x), \ldots, s_{n_K(x)}],$$

where $\{s_0, \cdots, s_{N_k}\}$ is a basis of $H^0(X, kL)$. Consider $\underline{\iota_k(X)}$ be the corresponding image on $H^0(X, kL)$, i.e. $p \in \underline{\iota_k(X)}$ if and only if $[p] \in \iota_k(X)$. Then $\underline{\iota_k(X)}$ is an $(n+1)$-dimensional cone in $H^0(X, kL) \cong \mathbb{C}^{N_k+1}$. Then, for a generic linear $N_k - n + 1$ subspace $W \subset H^0(X, kL)$, the set of intersections between W and $\underline{\iota_k(X)}$ is in dimension 1, but for a generic linear $N_k - n$ subspace $L \subset H^0(X, kL)$, the set of intersections between L and $\underline{\iota_k(X)}$ is $\{0\}$.

Therefore, we can define a degree d divisor Z_X on the Grassmannian $Gr(N_k - n, N_k + 1)$ by

$$Z_k := \{L^{N_k - n} \subset H^0(X, kL) \mid L^{N_k - n} \cap (\underline{\iota_k(X)} - \{0\}) \neq \phi\}.$$

This induces a point $R_X \in \mathbb{P}(H^0(Gr(N_k - n, N_k + 1)), \mathcal{O}(d))$, which is called the Chow point.

Also, under the embedding $Gr(N_k - n, N_k + 1) \to \wedge^{n+1}(H^0(X, kL))$, where $V = H^0(X, kL)$, we have

$$R_k \in \mathrm{Sym}^d(\wedge^{n+1} H^0(X, kL)) := V_k,$$

and then the $SL(N_k + 1, \mathbb{C})$ action on $H^0(X, kL)$ induces an action on V_k.

Definition 2.2. We say that:

- (i) X is k Chow stable (respectively, polystable, semistable) if $R_X \in V_k$ is $SL(N_k + 1, \mathbb{C})$-stable (respectively, $SL(N_k + 1, \mathbb{C})$-polystable, $SL(N_k + 1, \mathbb{C})$- semistable); and

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

(ii) X is asymptotically Chow stable (respectively, polystable, semistable) if there exists k_0 such that X is k Chow stable (respectively, polystable, semistable) for all $k \geqslant k_0$.

2.2 Toric varieties

We now recall some background knowledge of toric varieties.

Definition 2.3. Let X be an n-dimensional algebraic variety. Then X is a toric variety if:

(i) X is a Zaraki closure of $(\mathbb{C}^*)^n$;

(ii) the left multiplication of $(\mathbb{C}^*)^n$ on $(\mathbb{C}^*)^n$ can extend to an action on X.

We will focus on polarized toric varieties (X, L). There exists k_0 such that for all $k \geqslant k_0$, the map $\iota_k \colon X \to \mathbb{P}(H^0(X, kL))$ we defined above is embedding. Moreover, we can choose the basis $\{s_0, \ldots, s_{N_k}\}$ such that for any $\lambda := (e^{\lambda_1}, \ldots, e^{\lambda_n}) \in (\mathbb{C}^*)^n$,

$$\lambda \cdot s_i = e^{\alpha_i^1 \lambda_1 + \cdots + \alpha_i^n \lambda_n} s_i.$$

Then for $\lambda \in (\mathbb{C}^*)^n$,

$$\lambda \cdot \iota_k(x) = \iota_k(\lambda \cdot x),$$

and, in particular, the $(S^1)^n < (\mathbb{C}^*)^n$ is a subgroup of the Hamiltonian group of $\iota_k^* \omega_{FS}$. Thus we have the moment polytopes $\mu_k \colon X \to \mathrm{Lie}((S^1)^n) \cong \mathbb{R}^n$, and the images are defined as the moment map polytope P_k. Notice that we have

$$P_{kl} = l P_k,$$

if ι_k defines an embedding. Moreover,

$$P_k = \mathrm{conv}\{(\alpha_i^1, \ldots, \alpha_i^n) \in \mathbb{R}^n \mid i = 0, \ldots, N_k\}.$$

The reason is we have a moment map $\mu_{\mathbb{P}^{N_k}} \colon \mathbb{P}^{N_k} \to \mathrm{Lie}((S^1)^{N_k})^*$, and the image is the standard simplex. Then the moment map

$$\mu = \iota_k^* \circ \mu_{\mathbb{P}^{N_k}} |_{\iota_k(X)},$$

where

$$\iota_k^* \colon \mathrm{Lie}((S^1)^{N_k})^* \to \mathrm{Lie}((S^1)^n)^*,$$

is the induced homomorphism from ι_k,

$$(\iota_k^* \eta^*)(\xi) = \eta^*((\iota_k)_*(\xi)).$$

In terms of a matrix,

$$(\iota_k)_* = \begin{pmatrix} \alpha_0^1 & \cdots & \alpha_0^n \\ \vdots & \ddots & \vdots \\ \alpha_{N_k}^1 & \cdots & \alpha_{N_k}^n \end{pmatrix},$$

and as a result, $\iota_k^* = ((\iota_k)_*)^T$ is the transpose and therefore

$$Im(\mu) = \mathrm{conv}\{(\alpha_i^1, \ldots, \alpha_i^n) \in \mathbb{R}^n \mid i = 0, \ldots, N_k\}.$$

Definition 2.4. Let (X, L) be a polarized toric variety such that $\iota \colon X \to \mathbb{P}(H^0(X, L))$ is the toric equivariant Kodaira embedding map, and let P be the corresponding polytope. Then

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

P is said to be asymptotically Chow stable (respectively, polystable, semistable) if (X, L) is asymptotically Chow stable (respectively, polystable, semistable).

2.3 Chow stability of toric variety and criteria

Recall that by Fataki and Ono ([Fut04, Ono13, OSY12], and also see [LLSW19]), a toric variety X_P is asymptotically Chow semistable if there exists C such that for any $k \geqslant C$, and for any convex G invariant function $f: kP \to \mathbb{R}$, we have

$$\frac{1}{\mathrm{Vol}(kP)} \int_{kP} f \, dV \leqslant \frac{1}{\chi(kP \cap \mathbb{Z}^n)} \sum_{kp \in kP \cap \mathbb{Z}^n} f(p), \tag{1}$$

and X_P is polystable if the equality holds only when v is affine. (In [Ono13] and [LLSW19], the inequality is on the opposite side as the inputs are concave functions.) Here $G < SL(n, \mathbb{Z})$ is the biggest group fixing P, which is a discrete group.

Notice that if there exists a toric equivariant $\mathbb{C}^*$ action on X_P, then it corresponds to an affine function on P (see [Don02]). So we can write the following definition.

Definition 2.5. Let P be an integral convex polytope. The Futaki–Ono invariant of an affine function $v(x) = a_1 x_1 + \cdots + a_n x_n + a_0$ is given by

$$FO_P(a, k) := \frac{1}{\chi(kP \cap \mathbb{Z}^n)} \sum_{kp \in kP \cap \mathbb{Z}^n} a(p) - \frac{1}{\mathrm{Vol}(P)} \int_P a(x) \, dV.$$

We can rephrase Corollary 4.7 in [Ono13] as the following lemma.

Lemma 2.6 (Corollary 4.7 in [Ono13]; also see [Fut04]). *Suppose P is asymptotically Chow semistable. Then there exists C such that for any $k \geqslant C$, and for any affine function a on kP, we have*

$$FO_P(a, k) = 0.$$

Recall the following definition.

Definition 2.7. An integral convex polytope P is symmetric if there is exactly one fixed point (which must be 0 for reflexive polytopes) of the symmetric group $G < SL(n, \mathbb{Z})$ acting on P.

In particular, any G invariant affine function on symmetric polytopes must be constant; hence it must vanish. We also define the following.

Definition 2.8. A polytope P is weakly symmetric if for any k, and for any affine function a on kP,

$$FO_P(a, k) = 0.$$

Remark 2.9. Notice that this condition is stronger than assuming $FO_P(a, k) = 0$ for all $k >> 0$. There are two questions that arise.

(i) It is easy to see that P is symmetric implies P is weakly symmetric. But is the opposite true?

(ii) If P is not weakly symmetric, does this imply P is not asymptotically Chow semistable?

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Notice that the K stability version is not true, as there are non-symmetric K stable toric varieties, for example, the toric Del Pezzo surface of degree 1. However, it is not weakly symmetric and not asymptotically Chow semistable (see [LLSW19], Section 5).

Lemma 2.10. *A weakly symmetric integral polytopes P is (asymptotically) Chow semistable if for any $k \in \mathbb{N}$ ($k \geqslant C$ for some fix C), and for any convex function $f\colon kP \to \mathbb{R}$ which* $\min_{x\in kP} f(x) = f(0) = 0$, *we have*

$$\frac{1}{\mathrm{Vol}(kP)} \int_{kP} f\, dV \leqslant \frac{1}{\chi(kP \cap \mathbb{Z}^n)} \sum_{kv \in kP\cap \mathbb{Z}^n} f(v).$$

Proof. For any convex function $f\colon kP \to \mathbb{R}$, there exists an affine function a_k such that

$$\min_{x\in kP} (f(x) - a(x)) = f(0) = 0.$$

Therefore, we have

$$\frac{\sum_{kp\in kP\cap\mathbb{Z}^n} f(p)}{\chi(kP\cap\mathbb{Z}^n)} - \frac{\int_P f(x)\, dV}{\mathrm{Vol}(P)} = \frac{\sum_{kp\in kP\cap\mathbb{Z}^n} (f-a)(p)}{\chi(kP\cap\mathbb{Z}^n)} - \frac{\int_P (f-a)(x)\, dV}{\mathrm{Vol}(P)}.$$

The result follows. □

3. Some special classes of toric varieties

3.1 Product class

The first class of polytopes is in the form $P_1 \times \cdots \times P_r$, where $P_1, \ldots, P_r$, and is Chow stable.

Lemma 3.1. *Let P_1 and P_2 be bounded convex sets. Then for any f which is a convex function on $P_1 \times P_2$, $f_{P_2}(x) := \int_{P_2} f(x,y) dV_y$ is a convex function on P_1.*

Proof. Consider $f_{P_2}(tx_1 + (1-t)x_2)$, where $0 \leqslant t \leqslant 1$. We have

$$\begin{aligned} f_{P_2}(tx_1 + (1-t)x_2) =& \int_{P_2} f(tx_1 + (1-t)x_2, y) dV_y \\ \leqslant& \int_{P_2} tf(tx_1, y) dV_y + \int_{P_2} f((1-t)x_2, y) dV_y \\ =& t f_{P_2}(x_1) + (1-t) f_{P_2}(x_2). \end{aligned}$$

□

Proposition 3.2. *Let P_1 and P_2 be integral convex polytopes.* Then *$P_1 \times P_2$ is (asymptotic) Chow polystable (semistable) if and only if P_1 and P_2 are (asymptotic) Chow polystable (semistable).*

Proof. Suppose for any $k \geqslant C_1$ and $k \geqslant C_2$ and for any convex function f_1, f_2 on P_1 and P_2, we have

$$\frac{1}{\mathrm{Vol}(kP_1)} \int_{kP_1} f_1(x) dV \leqslant \frac{1}{\chi(kP_1)} \sum_{p\in P_1} f_2(p);$$

$$\frac{1}{\mathrm{Vol}(kP_2)} \int_{kP_2} f_2(x) dV \leqslant \frac{1}{\chi(kP_2)} \sum_{p\in P_2} f_2(p).$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Then for any $k \geqslant \max\{C_1, C_2\}$, and for any convex function f, we have

$$\begin{aligned}\frac{1}{\mathrm{Vol}(kP_1 \times kP_2)} \int_{kP_1 \times kP_2} f(x,y) dV_x dV_y &= \frac{1}{\mathrm{Vol}(kP_1)} \int_{kP_1} \frac{1}{\mathrm{Vol}(kV_2)} \int_{kP_2} f(x,y) dV_y dV_x \\ &= \frac{1}{\mathrm{Vol}(kP_1)} \int_{kP_1} \frac{1}{\mathrm{Vol}(kV_2)} f_{kP_2}(x) dV_x \qquad \text{(Lemma 3.1)} \\ &\leqslant \frac{1}{\mathrm{Vol}(kP_2)} \frac{1}{\chi(kP_1)} \sum_{p_1 \in kP_1 \cap \mathbb{Z}^{n_1}} f_{P_2}(p_1) \\ &= \frac{1}{\chi(kP_1)} \sum_{p_1 \in kP_1 \cap \mathbb{Z}^{n_1}} \left(\frac{1}{\mathrm{Vol}(kP_2)} \int_{kP_2} f(p_1, y) dV_y \right) \\ &\leqslant \frac{1}{\chi(kP_1)} \sum_{p_1 \in kP_1 \cap \mathbb{Z}^{n_1}} \frac{1}{\chi(kP_2)} \sum_{p_2 \in kP_2 \cap \mathbb{Z}^{n_2}} f(p_1, p_2) \\ &= \frac{1}{\chi(k(P_1 \times P_2))} \sum_{p \in k(P_1 \times P_2) \cap \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}} f(p).\end{aligned}$$

In particular, if $C_1 = C_2 = 1$, then this inequality holds for any convex function and any k.

For the opposite, without loss of generality, assume P_1 is unstable. Then there exists a sequence of convex functions f_k on kP_1 such that for any $k >> 0$,

$$\frac{1}{\mathrm{Vol}(kP_1)} \int_{kP_1} f_k(x) dV \geqslant \frac{1}{\chi(kP_1)} \sum_{p \in kP_1 \cap \mathbb{Z}^{n_1}} f(p).$$

Define $f_k \colon kP_1 \times kP_2 \to \mathbb{R}$ such that

$$f_k(x,y) = f_k(x).$$

Then

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(kP_1 \times kP_2)} \int_{kP_1 \times kP_2} f_k(x,y) dV = \frac{1}{\mathrm{Vol}(kP_1)} \int_{kP_1} f_k(x) dv \geqslant \frac{1}{\chi(kP_1)} \sum_{p \in kP_1 \cap \mathbb{Z}^{n_1}} f_k(p) \\ =& \frac{1}{\chi(kP_1 \times kP_2)} \sum_{p \in kP_1 \cap \mathbb{Z}^{n_1}} \chi(kP_2) f_k(p) = \frac{1}{\chi(k(P_1 \times P_2))} \sum_{p \in k(P_1 \times P_2) \cap \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}} f_k(p).\end{aligned}$$

□

As a quick check, we have a computational proof of the following well-known fact.

Corollary 3.3. *The variety* $((\mathbb{P}^1)^n, -K_{(\mathbb{P}^1)^n})$ *is asymptotically Chow polystable.*

Proof. The polytope $[-1, 1]$ is asymptotically Chow polystable. A direct consequence of Proposition 3.2 implies that $[-1, 1]^n$ is asymptotically Chow polystable. □

3.2 Symmetric double cone type

We now consider a class of examples where the members are reflexive and symmetric, but not asymptotically Chow semistable. Also, it is arguably one of the simplest and yet non-trivial classes to study.

Definition 3.4. Let P be an n-dimensional integral polytope. Then we define the double cone

$$D(P) := \mathrm{conv}\{0, \ldots, 0, 1), (0, \ldots, 0, -1), (p, 0) \mid p \in P\}.$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Notice that

$$kD(P) = \{(p,q) \in \mathbb{R}^n \times \mathbb{R} \mid p \in (k-q)P, -k \leqslant q \leqslant k\}.$$

Lemma 3.5. *Suppose P is symmetric. Then $D(P)$ is symmetric.*

Proof. If G acts on P, then $G \times \mathbb{Z}/2\mathbb{Z}$ acts on $D(P)$ by

$$(g, \pm 1) \cdot (p,q) = (g \cdot p, \pm q).$$

Hence if P is symmetric, then $D(P)$ is symmetric. □

To give a counterexample, first we have the following well-known fact.

Lemma 3.6 (See [Ehr77] or [BDLD+05]). *Let P be a convex integral polytope with dim $\geqslant 2$. Then the number of points*

$$\chi(kP) := \mid kP \cap \mathbb{Z}^n \mid = \mathrm{Vol}(P)k^n + \frac{1}{2}\mathrm{Vol}(\partial P)k^{n-1} + p(k),$$

where $p(k)$ is a polynomial in k of degree $n-2$ which depends on P only. For $n=1$,

$$\chi(kP) = \mathrm{Vol}(P)k + 1;$$

and for $n=2$, we have the Pick theorem (see [Pic99]),

$$\chi(kP) := \mid kP \cap \mathbb{Z}^n \mid = \mathrm{Vol}(P)k^2 + \frac{1}{2}\mathrm{Vol}(\partial P)k + 1.$$

In particular, for $k >> 0$,

$$\chi(kP) - \mathrm{Vol}(kP) = \frac{\mathrm{Vol}(\partial P)}{2}k^{n-1} + p(k) > 0.$$

Proposition 3.7. *Let P be an n-dimensional integral polytope. Suppose $\mathrm{Vol}(P) \geqslant (n+2)(n+1)$, so then $D(P)$ is not asymptotically Chow semistable.*

Proof. For $kD(P)$, denote the point in $kD(P)$ to be (p,q), where $p \in \mathbb{R}^n$, $q \in \mathbb{R}$. Consider the function

$$f(p,q) = \begin{cases} 0 & \text{if} \quad \mid q \mid \leqslant k-1, \\ t & \text{if} \quad \mid q \mid = (1-t)(k-1) + tk = k-1+t, 0 \leqslant t \leqslant 1. \end{cases}$$

Then

$$\sum_{p \in kD(P)} f(p) = 2.$$

Let $\mathrm{Vol}(P) = (n+2)(n+1)(1+\delta)$ for some $\delta \geqslant 0$. Then

$$\int_{kD(P)} f(x)\, dV = 2\int_0^1 t(1-t)^n \mathrm{Vol}(P) dt = 2\mathrm{Vol}(P)\int_0^1 t^n(1-t)dt$$
$$= 2\mathrm{Vol}(P)\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 2\frac{\mathrm{Vol}(P)}{(n+1)(n+2)} = 2 + 2\delta,$$

for some fix $\delta > 0$. Therefore,

$$\frac{1}{\mathrm{Vol}(kD(P))}\int_{kD(P)} f(x)\, dV = \frac{2+2\delta}{\mathrm{Vol}(D(P))k^{n+1}},$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

and

$$\frac{1}{\chi(kD(P))}\sum_{p\in kD(P)} f(p) = \frac{2}{\chi(kD(P))}.$$

As a result,

$$\begin{aligned}\frac{1}{\chi(kD(P))}\sum_{p\in kD(P)} f(p) - \frac{1}{\mathrm{Vol}(kD(P))}\int_{kD(P)} f(x)\,dV &= \frac{2}{\chi(kD(P))} - \frac{2+2\delta}{\mathrm{Vol}(kD(P))}\\ &< \frac{2}{\mathrm{Vol}(kD(P))} - \frac{2+2\delta}{\mathrm{Vol}(kD(P))}\\ &= \frac{-2\delta}{\mathrm{Vol}(kD(P))}\\ &\leqslant 0.\end{aligned}$$

□

Example 3.8 (Claim 4.3 in [LLSW19]). *Let $P = [-a, a]$ for $a > 3$. Then $D(P)$ is not asymptotically Chow semistable by the Proposition 3.7.*

In the following example, we construct a toric variety which is defined by a reflexive and symmetric polytope, but it is not asymptotically Chow semistable.

Example 3.9. *Consider $P = [-1, 1]^6 = ((\mathbb{P}^1)^6, O(2, 2, 2, 2, 2, 2))$, so then*

$$\mathrm{Vol}(P) = 2^6 = 64 > 56 = 8 \times 7 = (6 + 2)(6 + 1).$$

Indeed, as $2^x - (x + 2)(x + 1)$ is increasing when $x \geqslant 6$, so for all $n \geqslant 6$,

$$2^n - (n - 2)(n - 1) \geqslant 64 - 56 = 8 > 0,$$

which implies that $D([-1, 1]^n)$ are not asymptotically Chow semistable for all $n \geqslant 6$.

Remark 3.10. In the previous example, we provided a sequence of functions $f_k\colon kD([-1, 1]^n) \to \mathbb{R}$ such that the inequality (1) does not hold for all $k >> 0$, and hence we show that $D([-1, 1]^n)$ are not asymptotically Chow semistable for all $n \geqslant 6$. We can generalize this construction to any d-dimensional toric variety with the polytope Δ. To be precise, for any $p \in \Delta$, we define a sequence of piecewise linear functions $f_{p,k}\colon k\Delta \to \mathbb{R}$ such that:

(i) $f_k(kp) = 1$ and $f_k(q) = 0$ for any $q \in k\Delta \cap \mathbb{Z}^n$; and
(ii) for any piecewise linear function g_k satisfies the condition (i),

$$\int_{k\Delta} f_{p,k}(x)dV \geqslant \int_{k\Delta} g_k(x)dV.$$

This is equivalent to defining hyperplanes $H_k := \left\{L\left(\frac{x}{k}\right) = 0\right\}$ with $L(p) > 0$ such that:

(i) $R^o_{p,k} := \{L\left(\frac{x}{k}\right) > 0\} \cap k\Delta = \{kp\}$; and
(ii) for any ${R'_{p,k}}^o := \{\hat{L}\left(\frac{x}{k}\right) > 0\} \cap k\Delta = \{kp\}$,

$$\mathrm{Vol}(R^o_k) \geqslant \mathrm{Vol}({R'_k}^o).$$

We denote

$$Q_p := \{L\,(x) = 0\} \cap \Delta,$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

which is the base of the cone $\overline{R^o_{p,1}}$. By the same argument as in the proof of Proposition 3.7, if there exists p such that $\mathrm{Vol}(Q_p) \geqslant d(d+1)$, then f_k do not satisfy the inequality (1) for all $k >> 0$. The whole construction is called the cut a vertex technique as $k\Delta = R^o_{p,k} \cap \{f_k = 0\}$, where we separate $k\Delta$ into a cone $\overline{R_{p,k}}$ near the vertex kp and the remaining, and we can show that Δ is not asymptotically Chow semistable by studying the properties of $\overline{R_{p,k}}$ or even only $\overline{R_{p,1}}$.

Additionally, for $(\mathbb{P}^n, O(n+1))$ and $((\mathbb{P}^1)^n, O(2,\ldots,2))$, under the above construction, $\overline{R^o_k}$ must be an n-dimensional simplex. Hence Q_p are $(n-1)$-dimensional simplices for all p, and the volume of Q_p is

$$\mathrm{Vol}(Q_p)\frac{1}{(n)!} < (n+2)(n+1),$$

which is expected as we know that they are asymptotically Chow polystable.

In § 4, we will define a more restrictive type of polytopes, which are asymptotically Chow polystable.

4. Special polytopes

We first recall some definitions from toric geometry.

Definition 4.1. An integral polytope P is reflexive if the boundary is given by the equations

$$\sum_{i=1}^{n} a_i x_i = \pm 1,$$

where $a_i \in \mathbb{Z}$. Or equivalently, there exists exactly one interior point $(0,\ldots,0)$.

Definition 4.2. An integral polytope P is symmetric if there is exactly one fixed point of the symmetric group G acting on P.

Notice that if P is reflexive, then the fixed point is 0, and the $G < SL(n,\mathbb{Z})$ action is given by the matrix multiplication. We now add one extra restriction on the symmetric reflexive polytopes.

Definition 4.3. Let P be an n-dimensional integral convex polytope on $\mathbb{R}^n$. We say P has regular boundary if for any $k \in \mathbb{N}$, there exists a triangulation of ∂kP which every 'triangle' is integrally isomorphic to

$$T_{n-1} := \mathrm{conv}\{(0,\ldots,0), e_1,\ldots,e_{n-1}\},$$

the standard $(n-1)$-dimensional simplex, (i.e. the intersection between different T^i_{n-1} are at the boundary) such that:

(i) for any point $p \in \partial kP$, the number of simplices intersecting with p, denoted as $m_k(p)$, is bounded by $n!$ (i.e. $m_k(p) \leqslant n!$); and

(ii) this is the sub-triangulation of each face.

Here, integrally isomorphic means one of the objects is obtained from another object by an integral rigid motion, i.e. the multiplication of a matrix $A \in SL(n,\mathbb{Z})$ and translation of $v \in \mathbb{Z}^n$.

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Remark 4.4. If two objects P_1, P_2 are integrally isomorphic, then for all k, kP_1 has the same number of integral points as kP_2. Indeed, integral isomorphism is obtained by a bijection map $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^n$. So for each compact object $U \subset \mathbb{R}^n$, the map $\varphi\colon U \cap \mathbb{Z}^n \to \varphi(U \cap \mathbb{Z}^n)$ is a bijection.

Definition 4.5. An integral convex polytope on $\mathbb{R}^n$ is called special if it is reflexive, weakly symmetric, and has a regular boundary. A Fano toric variety $(X, -K_X)$ is called special if the corresponding polytope is special.

Example 4.6. *Suppose P is a two-dimensional symmetric reflexive polytope, so then it is special. This is because the boundary of P is a loop, so every point must connect with two segments hence the boundary has a regular triangulation.*

Remark 4.7. The two-dimensional symmetric reflexive polytopes are

$$X_3 := \text{conv}\{(-1,-1),(1,0),(0,1)\}, X_4 := \text{conv}\{(\pm 1,0),(0,\pm 1)\},$$
$$X_6 := \text{conv}\{(0,\pm 1),(\pm 1,0),(1,-1),(-1,1)\}, X_8 := \text{conv}\{(\pm 1,\pm 1)\},$$
$$X_9 := \text{conv}\{(-1,-1),(2,-1),(-1,2)\}$$

Example 4.8. *The polytopes $D(X_3)$, $D(X_4)$, $D(X_6)$, $D(X_8)$ and $D(X_9)$ are symmetric and reflexive. However, among these five polytopes, only $D(X_3)$, $D(X_4)$ and $D(X_6)$ are special. For instance, the faces of $D(X_8)$ are given by the triangles integrally isomorphic to* conv$\{(-1,0),(1,0),(0,1)\}$. *As a result, for any k, and for the point $(0,0,\pm k)$, there must be* 2 *simplices attaching the vertex for each face. Therefore,*

$$n(0,0,\pm k) = 2 \cdot 4 = 8.$$

Similarly, we can see that for any triangulation for $D(X_9)$,

$$n(0,0,\pm k) = 3 \cdot 3 = 9.$$

There are pictures indicating how to triangulate a face of $D(X_8)$ and a face of $D(X_9)$ in § 10.

5. Properties of special polytopes

In this section, we study what results each assumption in the definition of the special toric varieties can provide, starting with reflexivity.

Lemma 5.1. *Let P* be *a reflexive polytopes. Then, for all $k \in \mathbb{N}$,*

$$kP \cap \mathbb{Z}^n = \bigcup_{i=0}^{k} (\partial iP \cap \mathbb{Z}^n).$$

Proof. Let P be reflexive. Then $(0,\dots,0) \in \partial(0P)$ by definition. Notice that, for any $p = (p_1,\dots,p_n) \neq 0 \in kP$, there exists α and $0 < c_\alpha < k$ such that

$$a_{1,\alpha}p_1 + \cdots + a_{n,\alpha}p_n = c_\alpha.$$

But $p \in \mathbb{Z}^n$ implies $c_{\alpha \in \mathbb{Z}}$, and hence $p \in \partial c_\alpha P \cap \mathbb{Z}^n$. □

Also, we have the following.

Lemma 5.2. *Let P be a reflexive n-dimensional polytope. Then*

$$\frac{\text{Vol}(\partial P)}{n} = \text{Vol}(P).$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Proof. Let $\bigcup_{i=1}^{r} Q_i = \partial P$, where the Q_i are faces of P. Then define

$$C(Q_i) := \operatorname{conv}\{(0,\ldots,0), Q_i\} = \{tx \in P \mid x \in Q_i, 0 \leqslant t \leqslant 1\}.$$

Then

$$P = \bigcup_{i=1}^{r} C(Q_i),$$

and

$$\operatorname{Vol}(P) = \sum_{i=1}^{r} \operatorname{Vol}(C(Q_i)).$$

The assumption that P is reflexive implies the height is 1 for any $C(Q_i)$, so

$$\operatorname{Vol}(P) = \sum_{i=1}^{r} \operatorname{Vol}(C(Q_i)) = \sum_{i=1}^{r} \frac{\operatorname{Vol}(Q_i)}{n} = \frac{\operatorname{Vol}(\partial P)}{n}.$$

□

Lemma 5.3. *Suppose $f\colon P \to \mathbb{R}$ is a G-invariant convex function such that*

$$\min_{p \in P} f(x) = f(0) \geqslant 0.$$

Then

$$F_f(t) := \int_{t\partial P} f(tx) d\sigma_P$$

is convex, where $\sigma_{\partial P}|_x = d(l_{Q_i})|_x$ for $x \in Q_i$, the defining boundary function of the face $Q_i \subset \partial P$.

Proof. First, we have a map $\varphi\colon \partial P \times [0,1] \to P$ defined by

$$\varphi(x,t) = tx.$$

Notice that this map is surjective, that $\varphi(x,0) = 0$ and that $\varphi|_{\partial P \times (0,1]}$ is bijective. Hence any function f on P can be represented by the function

$$g(x,t) := f \circ \varphi(tx).$$

Notice that $f(0)$ is the minimum, so $f(x) \geqslant 0$. We find that a (decreasing) sequence of smooth G-invariant convex functions f_i, with $f_i(0) \geqslant 0$, converges to f. Denote $Q = \partial P$. We define $g_i\colon Q \times [0,1] \to \mathbb{R}$ by

$$g_i(x,t) := f_i \circ \varphi(x,t).$$

Now, by convexity, and since $f(0)$ is minimum, f is increasing along the segment
$\{(tx,t) \mid 0 \leqslant t \leqslant 1\}$, so it implies

$$\frac{dg_i}{dt}(x,t) \geqslant 0.$$

Also, convexity of f_i implies

$$\frac{d^2 g_i}{dt^2}(x,t) \geqslant 0.$$

As

$$\int_{tQ} f_i(tx) d\sigma_Q = t^{n-1} \int_Q g_i\,(x,t)\, d\sigma_Q,$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

we now compute the second derivative of F_i. For $n \geqslant 3$, the second derivative of F_i is given by

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{tQ} f_i(x,t)d\sigma_Q &= \frac{d^2}{dt^2}t^{n-1}\int_Q g_i(x,t)d\sigma_Q \\
&= \frac{d}{dt}\left((n-1)t^{n-2}\int_Q g_i\,(x,t)\,d\sigma_Q + t^{n-1}\int_Q \frac{dg_i}{dt}\,(x,t)\,d\sigma_Q\right) \\
&= (n-1)(n-2)t^{n-3}\int_Q g_i\,(x,t)\,d\sigma_Q \\
&\quad +2(n-1)t^{n-2}\int_Q \frac{dg_i}{dt}\,(x,t)\,d\sigma_Q + t^{n-1}\int_Q \frac{d^2g_i}{dt^2}d\sigma_Q \\
&\geqslant 0,
\end{aligned}$$

so all F_i are convex. Thus F is convex.

Also, for $n=2$,

$$F_i''(t) = 2(n-1)\int_Q \frac{dg_i}{dt}\,(x,t)\,d\sigma_Q + t\int_Q \frac{d^2g_i}{dt^2}d\sigma_Q.$$

Finally, for $n=1$, $F(t)=f(-ta)+f(tb)$ for $P=[-a,b]$, so

$$F_i''(t) = a^2 f''(-ta) + b^2 f''(tb) \geqslant 0.$$

So $F_i''(t)\geqslant 0$ for all i, which implies that $F(t)$ is convex. □

As a remark, when we put $f(x)=c$, then $F_c(t)=c\mathrm{Vol}(\partial P)t^{n-1}$, in which we can see that if $c<0$ and $n\geqslant 3$, F_c is not convex on $[0,1]$.

Corollary 5.4. *Suppose P is symmetric. Then for all $k\in\mathbb{R}$, for all G invariant convex functions $f\colon kP\to\mathbb{R}$ with $\min_{x\in kP} f(x)=f(0)=0$, we have*

$$\int_{kP} f(x,t)\,dV \leqslant \frac{1}{2}F(0)+F(1)+\cdots+F(k-1)+\frac{1}{2}F(k),$$

where

$$F(t) := \int_{t\partial P} f(x,t)d\sigma_{\partial P}.$$

Also, equality holds if and only if $f=0$.

Proof. Now

$$\int_{kP} f(tx)dV = \int_0^1\int_{t\partial kP} f(tx)d\sigma dt = \int_0^1 F_{f,kP}(t)dt.$$

By Lemma 5.3, $F(t)$ is convex, and hence by the trapezoid rule, we have

$$\int_{kP} f(x,t) \leqslant \frac{1}{2}F(0)+F(1)+\cdots+F(k-1)+\frac{1}{2}F(k).$$

□

The final lemma is a property of a regular boundary.

Lemma 5.5. *Let P have a regular boundary. Then for any k, and for any convex function f, we have*

$$\int_{\partial kP} f(x)d\sigma \leqslant \sum_{v\in\partial kP} f(v).$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Proof. Let n be the dimension of P. Then its boundary can be triangulated by the $(n-1)$ simplex T_{n-1}. Let the vertex point of T^{α}_{n-1} be $p^{\alpha}_0, \dots, p^{\alpha}_{n-1}$, so then convexity implies all simplex T_{n-1} have the property

$$\int_{T_{n-1}} f(x) d\sigma \leqslant \mathrm{Vol}(T_{n-1}) \sum_{i=0}^{n-1} \frac{f(p_i)}{n} = \sum_{i=0}^{n-1} \frac{f(p_i)}{n!}.$$

Therefore, if we denote $n(p)$ to be the number of simplex touching the point p, then the regular boundary assumption means $n(p) \leqslant n!$, which implies

$$\begin{aligned}\int_{\partial kP} f(x) d\sigma &= \sum_{\alpha} \int_{T^{\alpha}_{n-1}} f(x) d\sigma \leqslant \sum_{\alpha} \sum_{i=0}^{n-1} \frac{f(p^{\alpha}_i)}{n!} = \sum_{p \in \partial kP \cap \mathbb{Z}^n} \frac{n(p) f(p)}{n!} \\ &\leqslant \sum_{p \in \partial kP \cap \mathbb{Z}^n} \frac{n! f(p)}{n!} = \sum_{p \in \partial kP \cap \mathbb{Z}^n} f(p).\end{aligned}$$

□

6. Chow stabilities of special polytopes

We now show that a special polytope is asymptotically Chow polystable.

Theorem 6.1. *Let P be a special polytope. Then P is asymptotically Chow polystable.*

Proof. First, denote $\chi(kP) := \#\{kP \cap \mathbb{Z}^n\}$, so then

$$\frac{1}{\mathrm{Vol}(kP)} \int_{kP} c \, dV = \frac{1}{\chi(kP)} \sum_{p \in kP \cap \mathbb{Z}^n} c.$$

Second, P is reflexive and symmetric implies that for any G invariant convex function f,

$$\min_{x \in kP} f(x) = f(0).$$

Therefore, to show that the inequality (1) holds for any G-invariant convex function f, we only need to show that the inequality (1) holds for all G-invariant convex functions f satisfying

$$\min_{x \in kP} f(x) = f(0) \geqslant 0.$$

.

Let f be a G-invariant convex function satisfying

$$\min_{x \in kP} f(x) = f(0) \geqslant 0.$$

By Corollary 5.4,

$$\int_{kP} f(x) dV \leqslant \frac{f(0)}{2} + \sum_{r=1}^{k-1} \int_{\partial rP} f(x) d\sigma + \frac{1}{2} \int_{\partial kP} f(x) d\sigma.$$

Lemma 5.5 implies

$$\begin{aligned}\int_{kP} f(x) dV &\leqslant \frac{f(0)}{2} + \sum_{r=1}^{k-1} \sum_{\partial rP \cap \mathbb{Z}^n} f(p) + \frac{1}{2} \sum_{\partial kP \cap \mathbb{Z}^n} f(p) \\ &= \sum_{r=0}^{k} \sum_{p \in \partial rP \cap \mathbb{Z}^n} f(p) - \frac{f(0)}{2} - \frac{1}{2} \sum_{p \in \partial kP \cap \mathbb{Z}^n} f(p).\end{aligned}$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Therefore, Lemma 5.1 implies

$$\int_{kP} f(x)dV \leqslant \sum_{kP\cap\mathbb{Z}^n} f(p) - \frac{1}{2}f(0) - \frac{1}{2}\sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p).$$

Therefore,

$$\begin{aligned}
\frac{1}{\mathrm{Vol}(kP)}\int_{kP} f(x)dV =& \frac{1}{\chi(kP)}\int_{kP} f(x)dV + \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \\
&\leqslant \frac{1}{\chi(kP)}\left(\sum_{r=0}^{k}\int_{\partial rP} f(x)d\sigma - \frac{1}{2}f(0) - \frac{1}{2}\sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p)\right) \\
&+ \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \\
&\leqslant \frac{1}{\chi(kP)}\sum_{r=0}^{k}\sum_{p\in\partial rP} f(p) - \frac{1}{2\chi(kP)}\left(f(0) + \sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p)\right) \\
&+ \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \qquad \text{(Lemma 5.5)} \\
&= \frac{1}{\chi(kP)}\sum_{p\in kP} f(p) - \frac{1}{2\chi(kP)}\left(f(0) + \sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p)\right) \\
&+ \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \qquad \text{(Lemma 5.1).}
\end{aligned}$$

So we only need to show

$$-\frac{1}{2\chi(kP)}\left(f(0) + \sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p)\right) + \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \leqslant 0.$$

That is,

$$\left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right)\int_{kP} f(x)dV \leqslant \frac{1}{2\chi(kP)}\left(f(0) + \sum_{p\in\partial kP\cap\mathbb{Z}^n} f(p)\right). \tag{2}$$

Now, we can triangulate kP by $C_\alpha := \mathrm{conv}\{(0,\dots,0), T^\alpha_{n-1}\}$, where $\bigcup_\alpha T^\alpha_{n-1}$ is the regular triangulation on ∂kP, $\mathrm{Vol}(C_\alpha) = \frac{k}{n(n-1)!} = \frac{k}{n!}$, and by convexity,

$$\begin{aligned}
\int_{kP} f(x)dV &\leqslant \sum_\alpha \mathrm{Vol}(C_\alpha)\sum_i \frac{f(0) + f(p^\alpha_0) + \cdots + f(p^\alpha_{n-1})}{n+1} \\
&= \sum_{p\in\partial kP}\frac{kn(p)f(p)}{(n!)(n+1)} + \frac{\mathrm{Vol}(\partial kP)}{\mathrm{Vol}(C_{n-1})}\frac{k}{n!(n+1)}f(0) \\
&\leqslant \sum_{p\in\partial kP}\frac{kf(p)}{n+1} + \mathrm{Vol}(\partial kP)\frac{k}{n(n+1)}f(0).
\end{aligned}$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Therefore, in order to show equation (2), it suffices to show that we have

$$\left(\frac{1}{\mathrm{Vol}(kP)}-\frac{1}{\chi(kP)}\right)\left(\sum_{p\in\partial kP}\frac{kf(p)}{n+1}+\mathrm{Vol}(\partial kP)\frac{k}{(n+1)n}f(0)\right)\leqslant\frac{1}{2\chi(kP)}\left(f(0)+\sum_{p\in\partial kP\cap\mathbb{Z}^n}f(p)\right),$$

or

$$\left[\left(\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}\right)\left(\frac{k}{n(n+1)}\mathrm{Vol}(\partial kP)\right)-\left(\frac{1}{2}\right)\right]f(0)$$
$$\leqslant\left(\frac{1}{2}-\left(\frac{k}{n+1}\right)\left(\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}\right)\right)\sum_{p\in\partial kP}f(p).$$

By assumption, $f(0)=\min_{p\in kP}f(p)=0$, so we only need to show

$$0\leqslant\left(\frac{1}{2}-\left(\frac{k}{n+1}\right)\left(\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}\right)\right).$$

By Lemma 3.6, $\chi(kP)=\mathrm{Vol}(P)k^n+\frac{1}{2}\mathrm{Vol}(\partial P)k^{n-1}+r(k)$, where $r(k)=a_{n-2}k^{n-2}+\cdots+a_1k+1$ is a polynomial, and a_i depends on P only, so

$$\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}=\frac{\mathrm{Vol}(\partial P)}{2k\mathrm{Vol}(P)}+r(k)k^{-n}.$$

Using Lemma 5.2,

$$\begin{aligned}\left(\frac{k}{n+1}\right)\left(\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}\right)&=\frac{k}{n+1}\left(\frac{\mathrm{Vol}(\partial P)}{k\mathrm{Vol}(P)}+r(k)\frac{k^{-n}}{\mathrm{Vol}(P)}\right)\\&=\frac{k}{n+1}\left(\frac{n}{k}+r(k)\frac{k^{1-n}}{\mathrm{Vol}(P)}\right)\\&=\frac{n}{2(n+1)}+r(k)\frac{k^{1-n}}{\mathrm{Vol}(P)}.\end{aligned}$$

Therefore, there exists C such that

$$\frac{|\,r(k)k^{1-n}\,|}{\mathrm{Vol}(P)}=\frac{1}{\mathrm{Vol}(P)}\,|\,a_{n-2}k^{-1}+\cdots+a_1k^{2-n}+k^{1-n}\,|<\frac{1}{2(n+1)}$$

for all $k\geqslant C$, and hence

$$\left(\frac{k}{n+1}\right)\left(\frac{\chi(kP)-\mathrm{Vol}(kP)}{\mathrm{Vol}(kP)}\right)\leqslant\frac{n}{2(n+1)}+\frac{|\,r(k)k^{1-n}\,|}{\mathrm{Vol}(P)}<\frac{1}{2},$$

which shows our theorem. □

Example 6.2 (See also [LLSW19]). *By Example 4.6 and Remark 4.7, all two-dimensional symmetric reflexive polytopes are special, which are X_i for $i=3,4,6,8,9$, and hence the above varieties are asymptotically Chow polystable.*

7. Regular triangulation of an n simplex

To find higher-dimensional examples, we first need to know how to triangulate a polytope in higher dimensions. In general, it may be very difficult, but at least we can triangulate a polytope kP by the following.

(i) Triangulate P into simplices.

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

(ii) Triangulate kP, by first enlarging the triangulation on P, then triangulating kP by enlarged simplices kT_n. After that, further triangulate every enlarged n simplex kT_n into simplices.

So we need to know how to triangulate a simplex $kT_n := \mathrm{conv}\{(0,\dots,0), ke_i \mid i = 1,\dots,n\}$, where $ke_1 = (k,0,\dots,0),\dots,ke_n = (0,\dots,0,k)$. As a remark, the Lemma 7.1 is also proven in [LY24], but for completeness, we will provide the proof here as well.

Lemma 7.1 (Lemma A1 in [LY24]). *For any $p \in \mathbb{Z}^n$, there exists a simplex triangulation T of $\mathbb{R}^n$ such that $n(p) = (n+1)!$. Moreover, this triangulation T can triangulate the simplex*

$$k\Delta_n = \mathrm{conv}\{(0,\dots,0),(k,0,\dots,0),\dots,(0,\dots,0,k)\},$$

so that

$$n(p) = \frac{(n+1)!}{(k+1)!},$$

for all $p \in ((n-k)$-faces of $k\Delta_n)^o \cap \mathbb{Z}^n$.

Proof. We modify an idea from [Hat02, p. 112] for the construction. Let $I = [0,1]$ be the unit interval in $\mathbb{R}$. After taking an appropriate parallel transformation, we pick up one vertex p from $\mathcal{V}(I^n)$.

First, we triangulate the n-dimensional cube I^n into exactly $n!$ copies of an n-simplex Δ_n. For the vertex $p \in \mathcal{V}(I^n)$, we construct such a triangulation by induction on n. Since the vertex p has n hyperfaces $F_1,\dots,F_n \subset \Delta_n$ opposite it, we regard each F_i as an $(n-1)$-cube. By the assumption of inductive argument on n, each F_i can be triangulated into $(n-1)!$ copies of an $(n-1)$-simplex such as

$$F_i = \bigcup_{j=1}^{(n-1)!} \Delta_{n-1}^{(j)}.$$

Let $\mathcal{V}(\Delta_{n-1}^{(j)}) = \{q_1^{(j)},\dots,q_n^{(j)}\}$. Then, $\mathrm{conv}\{p, q_1^{(j)},\dots,q_n^{(j)}\}$ gives an n-simplex for each j, and hence we have $n \times (n-1)! = n!$ copies of an n-simplex by considering all n hyperfaces $F_1,\dots,F_n$.

Second, we denote by $T(I^n)$ this triangulation of I^n into exactly $n!$ simplices. Then we use parallel transformations of $T(I^n)$ for obtaining a triangulation of $\mathbb{R}^n$ such that $n(p) = (n+1)!$.

For $n = 1$, this is obvious. For $n = 2$, $T(I^2)$ consists of two triangles (see Figure 1). Keeping this and taking parallel transformations of $T(I^2)$ around the vertex p, we obtain the triangulation of $\mathbb{R}^2$ with $n(p) = 3!$. See Figure 2, also the Figure 3 for the parallel transformation in finite steps.

For arbitrary $n \in \mathbb{N}$, let us denote the set of 2^n vertices of the n-cube by

$$\mathcal{V}(I_n) = \{p_1, p_2, \dots, p_{2^n}\}.$$

Then we see that $n(p)$ is given by the number of all simplices in $T(I^n)$ whose vertices lie in $\mathcal{V}(I^n)$. If we denote by $T(\mathbb{R}^n)$ the triangulation of $\mathbb{R}^n$ induced by $T(I^n)$, we see that $n(p)$ coincides with the value of the *characteristic function* $\varphi_{T(I^n)} : \mathbb{Z}^n \to \mathbb{R}$ defined by

$$\varphi_{T(I^n)}(p) = \sum_{S : p \in \mathcal{V}(S)} n!\mathrm{vol}(S),$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

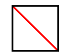


Figure 1. Triangulation of I^2.

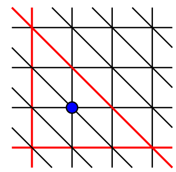


Figure 2. Triangulation of $\mathbb{R}^2$ induced from I^2.

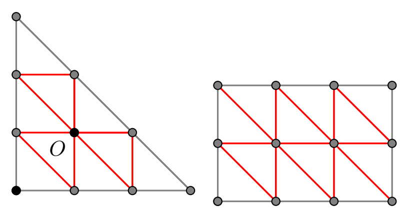


Figure 3. Triangulation of a 2-simplex and rectangle.

where the summation is over all n-simplices of $T(\mathbb{R}^n)$ for which p is a vertex (see, [GKZ94, p. 220]). Consequently, we have $n(p) = (n+1)!$, which proves the first part.

For the second part, by taking a suitable $k\Delta_n \subset \mathbb{R}^n$, the triangulation of $\mathbb{R}^n$ induced a triangulation of $k\Delta_n$ (see the red triangle in Figure 2). Moreover, an action of the permutation group $\mathcal{S}_{n+1}$ on $k\Delta_n$ (which permute the vertices) induces the action on the triangulation T in $k\Delta_n$. Consequently, the stabilizer group of p in $\mathcal{S}_{n+1}$ is $\mathcal{S}_{k+1}$, which implies that

$$n(p) = \frac{(n+1)!}{(k+1)!},$$

for $p \in ((n-k)\text{-faces of } k\Delta_n)^o \cap \mathbb{Z}^n$. □

8. Another sufficient condition of Chow stabilities on reflexive toric varieties

Theorem 8.1. *Let P be a reflexive polytope such that all the Futaki–Ono invariants vanish, and one has a triangulation on kP by n simplices, and a triangulation on ∂kP by $(n-1)$simplices, We let $n(p;k)$ be the number of n simplices attached to $p \in kP$ in the first triangulation, and $m(p;k)$ be the number of $(n-1)$ simplices attached to $p \in \partial kP$ in the second. Suppose $n(p;k) \leqslant (n+1)!$ for all $p \neq 0$ and*

$$\binom{n}{2} m(p;k) < ((n+1)! - n(p;k)),$$

for all $k >> 0$ and for all $p \in \partial kP$. Then P is asymptotically Chow polystable.

Proof. First, P is weakly symmetric implies that we can assume $f(0) = \min_{x\in P} f(x) \geqslant 0$. Now,

$$\frac{1}{\mathrm{Vol}(kP)} \int_{kP} f(x)\, dV = \frac{1}{\chi(kP)} \int_{kP} f(x) dV + \left(\frac{1}{\mathrm{Vol}(kP)} - \frac{1}{\chi(kP)}\right) \int_{kP} f(x) dV.$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Notice that

$$\int_{kP} f(x)dV \leqslant \sum_{p\in kP} \frac{n(p;k)f(p)}{(n+1)!} \leqslant \sum_{p\in kP} f(p) - \sum_{p\in \partial kP} \frac{(n+1)!-n(p;k)}{(n+1)!} f(p).$$

Also, as in the proof of theorem 6.1, using the triangulation of ∂kP, we triangulate kP such that each component is the convex hull of the origin and the simplex on $\partial(kP)$, so we have

$$\int_{kP} f(p)\, dV \leqslant \sum_{p\in\partial kP} \frac{m(p;k)kf(p)}{(n)!(n+1)} + \mathrm{Vol}(kP)\frac{f(0)}{(n+1)}.$$

Also,

$$\chi(kP) - \mathrm{Vol}(kP) = \frac{\mathrm{Vol}(\partial P)k^{n-1}}{2} + O(k^{n-2}) = \frac{n\mathrm{Vol}(P)k^{n-1}}{2} + O(k^{n-2}).$$

By assumption, we may assume $f(0)=0$ is the minimum, and therefore

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}(kP)}\int_{kP} f(x)\, dV \\
&\leqslant \frac{1}{\chi(kP)}\left(\sum_{p\in kP} f(p) - \sum_{p\in\partial kP}\frac{(n+1)!-n(p;k)}{(n+1)!}f(p)\right) \\
&\quad + \left(\frac{nk^{-1}}{2\chi(kP)} + \frac{O(k^{-2})}{\chi(kP)}\right)\left(\sum_{p\in\partial kP}\frac{m(p;k)kf(p)}{(n)!(n+1)} + \mathrm{Vol}(kP)\frac{f(0)}{(n+1)}\right) \\
&= \frac{1}{\chi(kP)}\sum_{p\in kP} f(p) + \frac{1}{(n+1)!\chi(kP)}\left(\left(\frac{n}{2}+O(k^{-1})\right)m(p;k) - ((n+1)!-n(p;k))\right)\sum_{p\in\partial kP} f(p).
\end{aligned}$$

Therefore, if

$$\left(\left(\frac{n}{2}+O(k^{-1})\right)m(p;k) - ((n+1)!-n(p;k))\right) \leqslant 0,$$

then the inequality holds. Therefore, if for all k, for all $p\in\partial kP$,

$$\frac{n}{2}m(p;k) < ((n+1)!-n(p;k)),$$

then P is asymptotically Chow polystable. □

Remark 8.2. We cannot generalize this statement from a reflexive polytope to an integral polytope as the inequalities

$$\int_{kP} f(p)\, dV \leqslant \sum_{p\in\partial kP} \frac{m(p;k)kf(p)}{(n)!(n+1)} + \mathrm{Vol}(kP)\frac{f(0)}{(n+1)}$$

and

$$\chi(kP) - \mathrm{Vol}(kP) = \frac{\mathrm{Vol}(\partial P)k^{n-1}}{2} + O(k^{n-2}) = \frac{n\mathrm{Vol}(P)k^{n-1}}{2} + O(k^{n-2})$$

do not hold in general. However, suppose there exists a positive integer c such that

$$P = \bigcap_{i=1}^{M}\{l_i(x)\leqslant c\},$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

where $l_i(x) = a_i^1 x_1 + \cdots + a_i^n x_n$ are linear functions with integral coefficients such that $(a_i^1, \ldots, a_i^n, c)$ are coprime. Then the inequalities become

$$\int_{kP} f(p)\,dV \leqslant \sum_{p\in\partial kP} \frac{cm(p;k)kf(p)}{(n)!(n+1)} + \mathrm{Vol}(kP)\frac{f(0)}{(n+1)}$$

and

$$\chi(kP) - \mathrm{Vol}(kP) = \frac{\mathrm{Vol}(\partial P)k^{n-1}}{2} + O(k^{n-2}) = \frac{n\mathrm{Vol}(P)k^{n-1}}{2c} + O(k^{n-2}).$$

Then, by the same calculation as in the proof of Theorem 8.1, if we have other assumptions in Theorem 8.1, then the result still holds.

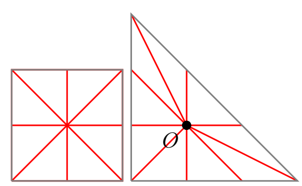
8.1 $D(X_8)$ and $D(X_9)$

Recall that $D(X_8)$ and $D(X_9)$ are given by

$$D(X_8) := \mathrm{conv}\{(1,1,0), (-1,1,0), (1,-1,0), (-1,-1,0), (0,0,1), (0,0,-1)\},$$
$$D(X_9) := \mathrm{conv}\{(-1,-1,0), (2,-1,0), (-1,2,0), (0,0,1), (0,0,-1)\}.$$

In Example 4.8, we showed that $D(X_8)$ and $D(X_9)$ are not special. Therefore, to show that $D(X_8)$ and $D(X_9)$ are asymptotically Chow polystable, we have to triangulate the whole polytopes and compute the inequality directly.

Notice that the only way to triangulate $D(X_8)$ and $D(X_9)$ into simplices is the following. We triangulate X_8 and X_9 as follows.



Then we connect any small triangle to $(0,0,1)$ and $(0,0,-1)$ to get 3-simplex. Therefore, we can triangulate $D(X_8)$ into 16 simplices and $D(X_9)$ into 18 simplices. Then, by triangulation of each simplex, we have a triangulation of $kD(X_8)$ and $kD(X_9)$.

As a consequence of Lemma 7.1, we have the following lemma.

Lemma 8.3. *For $kD(X_i)$, under the above triangulation,*

$$n(p) \begin{cases} = i & \text{if } p = (0,0,\pm k), \\ \leqslant 24 & \text{if } p \in kD(X_i)^o, \\ \leqslant 12 & \text{if } p \in \partial kD(X_i). \end{cases}$$

Moreover, the triangulation on kT_2 combined with the induced triangulation on $D(X_i)$ onto $\partial kD(X_i)$ gives

$$m(p) \begin{cases} = i & \text{if } p = (0,0,\pm k), \\ \leqslant 6 & \text{otherwise.} \end{cases}$$

As a remark, for each $D(X_i)$, $n_{kP}(0,\ldots,0) = 2i$, also for the triangulation of ∂kD_i, $n(p) = 4$ for $p \in \partial kP$ intersect with the red line.

Corollary 8.4. *For $i = 3,4,6,8,9$, $D(X_i)$ are asymptotically Chow polystable.*

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

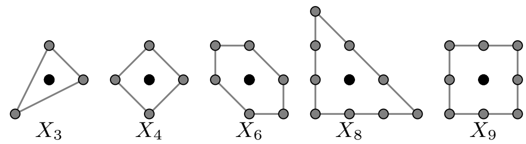


Figure 4. X_i for $i=3,4,6,8,9$.

Proof. For $p\in\partial kP$ such that $p\neq(0,\dots,\pm1)$, $n(p;k)\leqslant\frac{(n+1)!}{2}$ and $m(p;k)\leqslant n!$, the inequality becomes

$$n(n!)\leqslant(n+1)!,$$

which is true. Also, at $p=(0,\dots,0,\pm1)$, we have

$$n(p;k)=m(p;k)=i,$$

and then we need

$$(n+1)!>\frac{(n+2)}{2}i.$$

That is,

$$1>\frac{(n+2)i}{2(n+1)!}.$$

If $n=3$, then it becomes

$$1>\frac{5i}{48},$$

and hence this inequality holds for $i\leqslant 9$. Therefore, by Lemma 8.3 and Theorem 8.1, for $i=3,4,6,8,9$, $D(X_i)$ are asymptotically Chow polystable. □

9. Examples of the stability of symmetric reflexive polytopes

9.1 One- and two-dimensional symmetric reflexive polytopes

Example 9.1. *The only one-dimensional symmetric reflexive polytope is* $[-1,1]$, *which is Chow stable (see* [LLSW19]*).*

Example 9.2. *Suppose P is a two-dimensional symmetric reflexive polytope, so then it is special, and hence it is asymptotic Chow stable. Indeed, by Theorem 1.2 and Corollary 3.3 in* [LLSW19], *combined with the fact that $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ (by Proposition 3.2) are Chow stable, so indeed, all two-dimensional special polytopes are Chow stable.*

As a remark, they are given by

$$X_3:=\mathbb{P}^2/(\mathbb{Z}/3\mathbb{Z}),\,X_4:=\mathbb{P}^1\times\mathbb{P}^1/(\mathbb{Z}/2\mathbb{Z}),\,X_6:=\mathbb{P}^2\text{ blow up 3 points},\,X_8:=\mathbb{P}^1\times\mathbb{P}^1,\,X_9:=\mathbb{P}^2,$$

and all the line bundles to define the polytopes are $-K_{X_i}$. The polytopes are given in Figure 4.

Notice that in [LLSW19] there are some examples of the non-reflexive polytopes which we will not discuss in detail in this note.

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

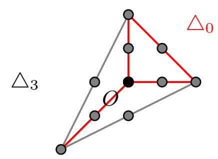


Figure 5. $\triangle_0 \subset X_3$, $\triangle_0 \subset X_4$ and $\triangle_0 \subset X_6$.

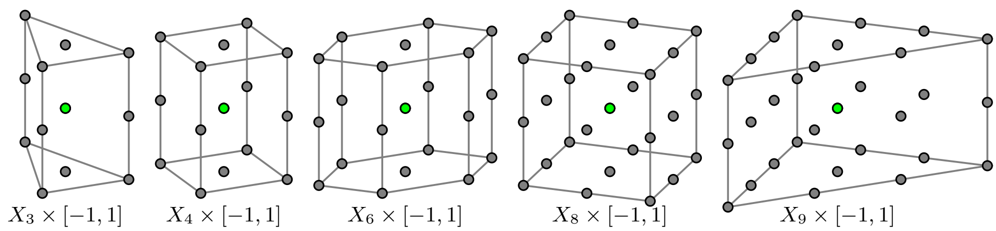


Figure 6. $X_i \times [-1,1]$.

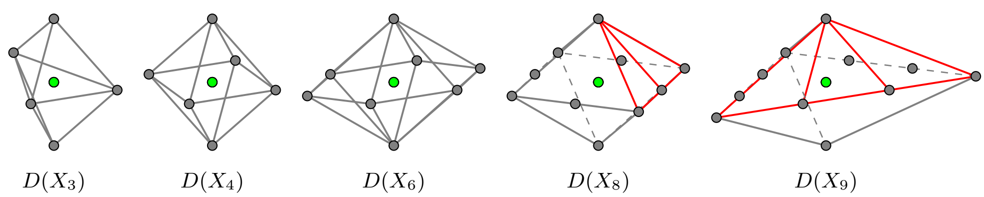


Figure 7. $D(X_i)$.

9.2 Three-dimensional polytopes

To study the higher-dimensional polytopes, we first recall that, given a reflexive polytope P, we can define the *dual* polytope $\hat{P}$ as follows. Let

$$\hat{P} := \{y \in \mathbb{R}^n \ \langle x, y\rangle \geqslant -1, \ \text{for all } x \in P\}.$$

If P is symmetric and reflexive, so is $\hat{P}$. For example, $D(P) = \widehat{\hat{P} \times \mathbb{P}^1}$. However, the duality may not share the stability.

By Lemma 7.1, for faces that are 2-simplices or rectangles, then $m(p) = 6$ for $p \in P^o$, $m(p) = 3$ if $p \in (\partial P)^o$, and $m(p) = 1$ if p is the vertex. For any $p \in (\partial P)^o$, there are at most two faces connected to p, so in order to check if a polytope is special, we only need that there are $i \leqslant 6$ simplices connecting each vertex under the triangulation on the boundary.

Denote Δ_0 as the triangulation of a 2-simplex. As in Figure 5, if the faces are given by X_i, for $i = 3, 4, 6$, then $n(0) = i$, $n(p) = 6$ for $p \in P^o - \{0\}$, $n(p) = 3$ if $p \in (\partial P)^o$, and $n(p) = 2$ if p is the vertex using the rotation of Δ_0 as the triangulation.

Therefore, if the polytopes whose faces are a combination of the above, then the only possible problem is the vertex, and we can study those polytopes case by case.

Proposition 9.3. *The following symmetric reflexive three-dimensional polytopes are asymptotically chow polystable.*

(i) $X_i \times [-1,1]$ *for* $i = 3, 4, 6, 8, 9$ *(Figure 6).*

(ii) $D(X_i)$, *where* $i = 3, 4, 6, 8, 9$ *(Figure 7).*

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

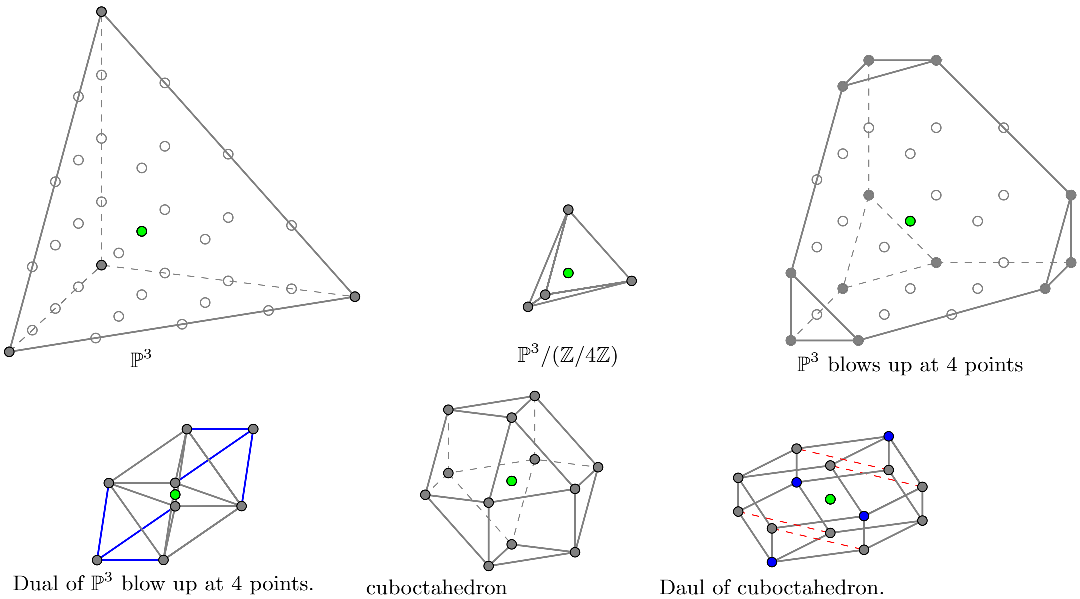


Figure 8. Other special polytopes, where the red line indicates part of the triangulation.

(iii) *Other polytopes (Figure 8):*

(a) *the polytope of* $(\mathbb{P}^3, O(4)) := \operatorname{conv}\{(-1,-1,-1),(3,-1,-1),(-1,3,-1),(-1,-1,3)\}$ *(tetrahedron) and its dual,* $A_3 = \operatorname{conv}\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\}$ *(tetrahedron;*

(b) *the polytope of* $\mathbb{P}^3$ *blow up 4 points, which is a convex set of the points* $(0,-1,-1)$, $(-1,0,-1)$, $(-1,-1,0)$, $(2,-1,-1)$, $(2,-1,0)$, $(2,0,-1)$, $(-1,2,-1)$, $(-1,2,0)$, $(0,2,-1)$, $(-1,-1,2)$, $(-1,0,2)$, $(0,-1,2)$. *Hence this polytope is a truncated tetrahedron, in which the boundaries contain 4* X_6 *and 4-simplices. Each vertex is connected with one 2-simplex and two* X_6*; and its dual:*

(c) *The dual polytope of (B),* $\operatorname{conv}\{(\pm 1,0,0)(0,\pm 1,0),(0,0,\pm 1),(-1,-1,-1),(1,1,1)\}$, *which is* $D(X_4)$ *glue with two standard 3-simplices, and the faces are all standard 2-simplices.*

(d) *The polytope* $\operatorname{conv}\{(\pm 1,0,0),(0,\pm 1,0),(\pm 1,\mp 1,0),(0,0,\pm 1),(\pm 1,0,\mp 1),(0,\pm 1,\mp 1)\}$, *which is a cuboctahedron, with eight triangular faces and six square faces, and each vertex is connected to two 2-simplex and 2-square, and its dual, which is given by:*

(e) *the convex hull of the points:*

$$(1,0,0),(1,1,0),(0,1,0),(-1,0,0),(-1,-1,0),(0,-1,0),$$
$$(0,0,1),(1,0,1),(1,1,1),(0,1,1),(0,0,-1),(-1,0,-1),(-1,-1,-1),(0,-1,-1),$$

which is a rhombic dodecahedron.

Proof.

(i) $X_i \times [-1,1]$ for $i = 3,4,6,8,9$: All two-dimensional reflexive symmetric polytopes are special, and hence X_i are asymptotically chow polystable. Also, as stated in Example 9.1, $[-1,1]$ is asymptotically chow polystable. By Proposition 3.2, they are asymptotically chow polystable.

(ii) $D(X_i)$, where $i=3,4,6,8,9$: This is a direct consequence of Corollary 8.4.

(iii) To show the remaining polytopes are asymptotically Chow polystable, we only need to show that the remaining polytopes are special. As we explained right before this statement, if the faces of the boundary consist of X_i for $i=3,4,6$, 2-simplices and rectangles, then $m_k(p)\leqslant 6$ for all non-vertexes p. Also, for any vertex p, denote $N(S)$ to be the number of S connected to p, so then

$$m_k(p)=\sum_{i=3,4,6} 2N(X_i)+N(2\text{ simplices})+aN(\text{rectangles}),$$

where a can be 1 or 2 depending on the triangulation. In particular, if all the faces of the boundary are given by X_3, X_4, X_6, simplices or rectangles, and the number of faces connected to a vertex p is smaller than 3, then $m_k(p)\leqslant 6$. Also, see Figures 8 in § 10, which provide the detail of the boundary of the polytopes.

(a) The polytope of $(\mathbb{P}^3, O(4)):=\mathrm{conv}\{(-1,-1,-1),(3,-1,-1),(-1,3,-1),(-1,-1,3)\}$. The boundary consists of four enlarged two-dimensional simplices, hence $m_k(p)\leqslant 6$, and for all the vertexes p, $m_k(p)=3$. For $A_3=\mathrm{conv}\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\}$, the boundary consists of 4 X_3. Also, each vertex p is connected to 3 X_3, and hence

$$m_k(p)=6.$$

Therefore, A_3 is special.

(b) The convex set of the points $(0,-1,-1),(-1,0,-1),(-1,-1,0).(2,-1,-1),(2,-1,0),(2,0,-1),(-1,2,-1),(-1,2,0),(0,2,-1),(-1,-1,2),(-1,0,2),(0,-1,2)$ is a truncated tetrahedron, in which the boundary contains four X_6 and four 2-simplices. Each vertex is connected with 2 simplices and 2 X_6, so for any vertex p,

$$m_k(p)=2+2\times 2=6.$$

The symmetric group acting on it is the permutation group S_4, which is the induced action from the permutation of the vertexes of the enlarged simplex, and hence it is symmetric. Also, by Figures 8 in § 10, we can see that it is reflexive. So, it is special.

(c)

$$\mathrm{conv}\{(\pm 1,0,0)(0,\pm 1,0),(0,0,\pm 1),(-1,-1,-1),(1,1,1)\},$$

which is $D(X_4)$ glue with two standard 3-simplices, for which the faces are all standard 2-simplices. By Figure 8 in § 10, we can see that the number of faces connected to each vertex is less than six, the symmetric group acting on it is the permutation group S_4, and it is reflexive. So it is special.

(d) $\mathrm{conv}\{(\pm 1,0,0),(0,\pm 1,0),(\pm 1,\mp 1,0),(0,0,\pm 1),(\pm 1,0,\mp 1),(0,\pm 1,\mp 1)\}$, which is a cuboctahedron, with eight triangular faces and six square faces, and each vertex is connected to one 2-simplices and two squares. Hence, for the surface, and for each vertex p,

$$m_k(p)\leqslant 2+2\cdot 2=6.$$

Also, the group acting on the polytope contains $\mathbb{Z}_3\times\mathbb{Z}_2$, the rotation group rotates along the z-axis, times the reflection group $\{\mathrm{id},\sigma\}$, which $\sigma(p)=-p$. Therefore this polytope is symmetric. Also, from Figure 8 in § 10, we can see it is reflexive. Therefore, it is special, and its dual, which is given by

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

(e) the convex hull of the points,

$$(1,0,0),(1,1,0),(0,1,0),(-1,0,0),(-1,-1,0),(0,-1,0),$$
$$(0,0,1),(1,0,1),(1,1,1),(0,1,1),(0,0,-1),(-1,0,-1),(-1,-1,-1),(0,-1,-1),$$

is a rhombic dodecahedron. This is symmetric as the symmetric group is the same as its dual. By Figures 8 in § 10, we can see that it is reflexive. Also, the vertices $(0,1,1),(1,0,1),(0,-1,-1),(-1,0,-1)$ have four squares touching the points, and the others have only three. So when we triangulate the surface, if we choose the triangulation such that two of the squares do not bisect along those points, then for any point p in it,

$$m_k(p) \leqslant 6,$$

and hence this polytope is special. (See § 10 for the above triangulation.) □

Remark 9.4. Indeed, except for $D(X_8)$ and $D(X_9)$, all the polytopes given in Proposition 9.3 are special. Also, we conjecture that all three-dimensional symmetric reflexive polytopes are asymptotically Chow polystable, and they are special except $D(X_8)$ and $D(X_9)$. The difficulty in showing this conjecture is to find all three-dimensional symmetric reflexive polytopes up to the integral isomorphism.

9.3 Higher-dimensional polytopes

One knows that in high dimensions that not every symmetric reflexive polytope is asymptotic Chow stable, for example $D([-1,1]^n)$ for $n \geqslant 6$. On the other hand, we can provide two classes of polytopes that are special.

Example 9.5. *Consider $A_n := \{[z_0, \dots, z_{n+1}] \in \mathbb{P}^{n+1} \mid z_1 \dots z_{n+1} = z_0^{n+1}\}$. The corresponding polytopes, also denoted as A_n, are given by*

$$A_n = \operatorname{conv}\{(1,\dots,0),(0,1,\dots,0),\dots,(0,\dots,0,1),(-1,-1,\dots,-1)\}.$$

As all the faces of A_n are simplices, with all the codimension 2 or above boundary intersecting with less than n, it is asymptotic Chow polystable.

Notice that $A_2 = X_3$ in our notation on symmetric reflexive polygons. Also, as a polytope, each A_n is the dual polytope of the polytope corresponding to $(\mathbb{P}^n, O(n+1))$.

To show A_n are special for all $n \geqslant 3$, notice that the boundry of A_n is given by n piece of $(n-1)$-simplices. So we only need to know how many simplicies will be attached to a point in the codimension k skeleton.

Let $a_i = e_i$ and $a_{n-1} = (-1,\dots,-1)$, so then we can represent any codimensional k piece by the following: a_1 represents the point e_1, $\{a_1, a_2\}$ represents the segment containing a_1, a_2, etc. Also, as the symmetric group S_{n+1} acts on A_n, we only need to consider how many faces contain the $n-r$ skeleton $\{a_1 \dots a_{n-r}\}$. But the faces containing $a_1 \dots a_{n-r}$ are represented by the set *$\{a_{n-r+1} \dots a_{n+1}\}$ removing one element. Therefore, there are $r+1$ faces connecting the skeleton containing $a_1 \dots a_{n-r}$. Therefore, Lemma 7.1 implies that, for any point $p \in \partial kP$ which is in the interior of the $n-r$ skeleton,*

$$n(p) = \frac{(n)!}{(r)!}(r) = \frac{n!}{(r-1)!} \leqslant n!.$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Therefore, A_n has a regular triangulation. Also, all A_n are symmetric and reflexive, and hence they are special, which implies all A_n for $n \geqslant 2$ are asymptotically Chow polystable.

Example 9.6. *Consider $(\mathbb{P}^n, O(n+1))$. The boundary of $k(\mathbb{P}^n, O(n+1))$ is defined by*

$$\bigcap_{i=1}^{n}\{x_i = -k\} \cap \{x_1 + \cdots + x_n = k\}.$$

Up to an S_{n+1} action, a point p is in the interior of a codimensional r skeleton if

$$p = (-k, \ldots, -k, v),$$

where $v \in \mathbb{R}^{n-r}$ such that

$$-r + v_1 + \cdots + v_{n-r} < k,$$
$$v_i > -k,$$

for all $i = 1, \ldots, n-k$. Hence $n(p) = r$. So we have the same calculation of A_n, which implies it is special, and therefore it is asymptotically Chow polystable.

Example 9.7. *Define $D_n := \mathrm{conv}\{(\pm 1, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}$ so D_n is special for all n. Notice that $D_2 = X_4$ and $D_3 = D(X_4)$.*

Proof. Notice that $kD_n = \mathbb{Z}_2^n \cdot kT_n$, where $\mathbb{Z}_2^n = \{1, -1\}^n$ with the group action to be multiplication, and the action is multiplication to the corresponding coordinate. D_n is symmetric and reflexive. To show that D_n has a regular boundary, p is in the interior of the codimension $r+1$ skeleton if $p = (x_1, \ldots, x_n) \in \partial k D_n$ with

$$x_{i_1} = \cdots = x_{i_r} = 0,$$

for $r = 0, \ldots, n-1$. We denote these points as p_r. Hence, similar to Example 9.6, as a consequence of Lemma 7.1, we have

$$n(p_r) = \frac{(n)!}{(r+1)!}(2^r) = \left(\frac{2}{r+1}\right) \cdots \left(\frac{2}{2}\right) n! \leqslant n!,$$

and hence D_n is special. □

Example 9.8. *Notice that D_6 is the dual polytope of $[-1,1]^6$, and thus $D_6 \times [-1,1]$ (i.e. $D_6 \times \mathbb{P}^1$ as the corresponding variety) is asymptotically Chow polystable. However, its dual is $D([-1,1]^6)$; therefore, a dual of an asymptotically Chow polystable polytope need not be asymptotically Chow polystable (or even semistable).*

Appendix A. From integral polytypes to varieties

In this appendix, we will briefly explain how we obtain a toric variety from an integral polytope. Then we will write down the corresponding varieties of the toric varieties occurring in this note.

A.1 General procedure

Let P be a integral polytope containing $(0, \ldots, 0)$. Let $\{p_0 = (0, \ldots, 0), p_1, \ldots, p_N\}$ be all the integral points in P. Then we can define a toric subvariety in $\mathbb{P}^N$ by the following equations.

Suppose we have

$$c_1 p_{i_1} + \cdots + c_r p_{i_r} = b_1 p_{j_1} + \cdots + b_s p_{j_s},$$

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

and without loss of generality, we may assume

$$c_1 + \cdots + c_r = b_1 + \cdots + b_s + a,$$

for some $a \geqslant 0$. Then we have a homogeneous polynomial defined by

$$z_{i_1}^{c_1} \cdots z_{i_r}^{c_r} - z_{j_1}^{b_1} \cdots z_{j_s}^{b_s} z_0^a.$$

Then the zero set $\{[z_0, \dots z_N] \in \mathbb{P}^N \mid z_{i_1}^{c_1} \cdots z_{i_r}^{c_r} - z_{j_1}^{b_1} \cdots z_{j_s}^{b_s} z_0^a = 0\}$ is a divisor in $\mathbb{P}^N$, and it is a toric subvariety.The toric action is given by

$$(\mathbb{C}^*)^{N-1} \cong \{(\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid \lambda_{i_1}^{c_1} \cdots \lambda_{i_r}^{c_r} = \lambda_{j_1}^{b_1} \cdots \lambda_{j_s}^{b_s}\}.$$

By intersecting all these divisors, we can obtain a toric subvariety.

Notice that some of the equations are repeated in $(\mathbb{C}^*)^N$, so in the following we will define the variety only by those which are different equations in $(\mathbb{C}^*)^N$, and the variety is the closure of this.

A.2 Examples

Example A.1 (A_n). *Denote $p_0 = (0, \dots, 0)$, $p_i = e_i$ for $i = 1, \dots, n$ and $p_{n+1} = (-1, \dots, -1)$. Then we have*

$$p_1 + \cdots + p_n = (0, \dots, 0) = p_0,$$

so the corresponding varieties, also denoted as A_n, are given by

$$A_n = \{[z_0, \dots, z_{n+1}] \in \mathbb{P}^{n+1} \mid z_1 \cdots z_{n+1} = z_0^n\}.$$

Example A.2 (D_n). *Recall that $D_n := \operatorname{conv}\{\pm e_i\}$. Denote $p_0 = (0, \dots, 0)$, $p_{2i-1} = e_i$, $p_{2i} = -e_i$ for $i = 1, \dots, n$. Then $p_{2i-1} + p_{2i} = 0$ for $i = 1, \dots, n$ gives n equations $z_{2i-1} z_{2i} = z_0^2$, and it gives a n codimension subvariety of $\mathbb{P}^{2n}$; hence these equations define D_n,*

$$D_n = \{[z_0, \dots, z_{2n}] \in \mathbb{P}^{2n} \mid z_{2i-1} z_{2i} = z_0^2\}.$$

Given

$$z_{i_1}^{c_1} \cdots z_{i_r}^{c_r} = z_{j_1}^{b_1} \cdots z_{j_s}^{b_s} z_0^a,$$

we denote

$$f(z_0, \dots, z_N) = z_{i_1}^{c_1} \cdots z_{i_r}^{c_r} - z_{j_1}^{b_1} \cdots z_{j_s}^{b_s} z_0^a.$$

Example A.3 ($D(P)$). *Let P be defined by*

$$\{[z_0, \dots, z_N] \in \mathbb{P}^N \mid f_1 = \cdots = f_r = 0\}.$$

Then we define $\hat{f}_i(z_0, \dots, z_{N+2}) = f_i(z_0, \dots, z_N)$. Then by denoting $p_{N+1} = (0, \dots, 0, 1)$, $p_{N+2} = (0, \dots, 0, -1)$, we have a new equation,

$$z_{N+1} z_{N+2} = z_0^2.$$

Then the variety of $D(P)$ is given by

$$D(P) = \{[z_0, \dots, z_{N+1} z_{N+2}] \in \mathbb{P}^N \mid f_1 = \cdots = f_r = z_{N+1} z_{N+2} - z_0^2 = 0\}.$$

In order to define $D(P)$, we need to know P as a subvariety of $\mathbb{P}^N$. Therefore, in order to compute all the examples, we need to write down what X_i is as a subvariety.

Example A.4. *As a subvariety of $\mathbb{P}^i$, restricted in $(\mathbb{C}^*)^i \subset \mathbb{P}^i$, X_i are given by:*

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

(i) $X_3 = A_2 = \{[z_0, z_1, z_2, z_3] \in \mathbb{P}^3 \mid z_1z_2z_3 = z_0^3\}$;

(ii) $X_4 = D_2 = \{[z_0, z_1, z_2, z_3, z_4] \in \mathbb{P}^4 \mid z_1z_2 = z_0^2, z_3z_4 = z_0^2\}$;

(iii) $X_6 = \{[z_0, z_1, \dots, z_6] \in \mathbb{P}^6 \mid z_1z_4 = z_2z_5 = z_3z_6 = z_0^2, z_2z_4 = z_3z_0\}$, *where this last equation comes from* $(0,1)+(-1,0)=(-1,1)$ *and* $z_2z_4 = z_3z_0$ *can be replaced by* $z_1z_3z_5 = z_0^3$ *with other equations to get the same variety;*

(iv) $X_8 = \{[z_0, z_1, \dots, z_8] \in \mathbb{P}^8 \mid z_rz_{r+4} = z_0^2, \text{ where } r = 1,2,3,4; z_1z_3 = z_2z_0, z_3z_5 = z_4z_0\}$; *and*

(v) $X_9 = \{[z_0, z_1, \dots, z_9] \in \mathbb{P}^9 \mid z_rz_{r+3}z_{r+6} = z_0^3, \text{ where } r = 1,2,3; z_1z_3 = z_2^2, z_2z_4 = z_3^2, z_2z_6 = z_0^2, z_3z_7 = z_0^2\}$. *Another way to write this uses* X_6 *plus 3 points, hence we need two more relations, namely*

$$\{[z_0, z_1, \dots, z_9] \in \mathbb{P}^9 \mid z_1z_4 = z_2z_5 = z_3z_6 = z_0^2, z_2z_4 = z_3z_0, z_7z_8z_9 = z_0^3, z_7z_8 = z_1z_2\}.$$

With this, we can write $D(X_i)$ *as subvarieties of* $\mathbb{P}^{i+2}$*. For example,*

(i) $D(X_3) = \{[z_0, z_1, z_2, z_3, z_4, z_5] \in \mathbb{P}^5 \mid z_1z_2z_3 = z_0^3, z_4z_5 = z_0^2\}$;

(ii) $D(X_4) = \{[z_0, z_1, \dots, z_6] \in \mathbb{P}^6 \mid z_1z_2 = z_0^2, z_3z_4 = z_0^2, z_5z_6 = z_0^2\}$;

(iii) $D(X_6) = \{[z_0, z_1, \dots, z_8] \in \mathbb{P}^8 \mid z_1z_4 = z_2z_5 = z_3z_6 = z_0^2 z_2z_4 = z_3z_0, z_7z_8 = z_0^2\}$;

(iv) $D(X_8) = \{[z_0, z_1, \dots, z_{10}] \in \mathbb{P}^{10} \mid z_rz_{r+4} = z_0^2, \text{ where } r = 1,2,3,4; z_1z_3 = z_2z_0, z_3z_5 = z_4z_0, z_9z_{10} = z_0^2\}$; *and*

(v) $D(X_9) = \{[z_0, z_1, \dots, z_{11}] \in \mathbb{P}^{11} \mid z_rz_{r+3}z_{r+6} = z_0^3, \text{ where } r = 1,2,3; z_1z_3 = z_2^2, z_2z_4 = z_3^2, z_2z_6 = z_0^2, z_3z_7 = z_0^2, z_{10}z_{11} = z_0^2\}$.

Example A.5. *Let* P_{X_i} *be the polytope of* X_i*. For each* i*, the toric variety corresponding to* $P_{X_i} \times [-1,1]$ *is* $X_i \times \mathbb{P}^1$*.*

Example A.6.

(i) $P_1 = \operatorname{conv}\{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1),(-1,-1,-1),(1,1,1)\}$, *which is* $D(X_4)$ *glue with two standard 3-simplices, which the faces are all standard 2-simplices;*

(ii) $P_2 = \operatorname{conv}\{(\pm1,0,0),(0,\pm1,0),(\pm1,\mp1,0),(0,0,\pm1),(\pm1,0,\mp1),(0,\pm1,\mp1)\}$;

(iii) P_3*, which is given by the convex hull of the points*

$$(1,0,0),(1,1,0),(0,1,0),(-1,0,0),(-1,-1,0),(0,-1,0),$$
$$(0,0,1),(1,0,1),(1,1,1),(0,1,1),(0,0,-1),(-1,0,-1),(-1,-1,-1),(0,-1,-1),$$

which is a rhombic dodecahedron.

The subvarieties are given by the following:

(i) $P_1 = \overline{\{[z_0, z_1, \dots, z_8] \in (\mathbb{C}^*)^8 \subset \mathbb{P}^8 \mid z_1z_2 = z_0^2, z_3z_4 = z_0^2, z_5z_6 = z_0^2, z_7z_8 = z_0^2, z_1z_3z_5 = z_7z_0^2\}}$, *where the last equation is deduced from* $e_1+e_2+e_3 = (1,1,1)$*;*

(ii) P_2 *is a subvariety of* $\mathbb{P}^{12}$*, so we need nine equations:* $z_{2r-1}z_{2r} = z_0^2$ *for* $r = 1,\dots,6$, $z_1z_4 = z_5z_0$, $z_7z_9 = z_1z_0$ *and* $z_7z_{11} = z_3z_0$*;*

(iii) P_3 *is a subvariety of* $\mathbb{P}^{14}$*. Following the order above, we have* $z_1z_4 = z_2z_5 = z_3 = z_6 = z_0^2$, $z_1z_3 = z_2z_0$, $z_7z_9 = z_8z_{10}$, $z_{11}z_{13} = z_{12}z_{14}$, $z_7z_{11} = z_8z_{12} = z_9z_{13} = z_{10}z_{14} = z_0^2$ *and* $z_8z_{11} = z_0z_1$.

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

Acknowledgements

The author would like to thank Universidad autónoma de Madrid (UAM), Instituto de Ciencias Matemáticas (ICMAT) and L'Institut Montpelliérain Alexander Grothendieck (IMAG) for providing an excellent environment for the author to focus on research. The author is also very grateful to his post-doc advisors Thibaut Delcroix and Mario Garcia-Fernandez, and their post-doc and students for many fruitful discussions and advice. Moreover, the author would like to express his deepest gratitude to Jacob Sturm and Xiaowei Wang for introducing the author to the topic of Chow stabilities on toric surfaces, as well as for their very helpful suggestions, valuable guidance and encouragement. Furthermore, the author is deeply grateful to the anonymous referees for their constructive advice. Last but not least, the author is very thankful to his classmates from the Chinese University of Hong Kong for their enthusiastic discussions and advice.

Financial Support

The author is supported by grant no. EUR2020-112265 funded by MCIN/AEI/10.13039/501100011033, by the European Union NextGenerationEU/PRTR project, grant no. CEX2019-000904-S funded by MCIN/AEI/10.13039/501100011033, and by the ANR-21-CE40-0011 JCJC project MARGE.

Conflicts of Interest

None

Journal Information

Moduli is published as a joint venture of the Foundation Compositio Mathematica and the London Mathematical Society. As not-for-profit organisations, the Foundation and Society reinvest 100% of any surplus generated from their publications back into mathematics through their charitable activities.

References

[BDLD+05] M. Beck, J.ús A. De Loera, M. Develin, J. Pfeifle and R. P. Stanley, *Integer points in polyhedra—Geometry, number theory, algebra, optimization*, Contemp. Math. **374** (2005), 15–36, MR2134759.

[BS99] V. V. Batyrev and E. N. Selivanova, *Einstein-Kähler metrics on symmetric toric Fano manifolds*, J. Reine Angew. Math. **512** (1999), 225–236.

[Don01] S. K. Donaldson, *Scalar curvature and projective embeddings, I*, J. Differential Geom. **59** (2001), 479–522.

[Don02] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349, MR1988506.

[Ehr77] Ehrhart, *Polynomes arithmetiques et methode de polyedres en combinatoire*, *International Series of Numerical Mathematics* (Birkhäuser, Basel, 1977).

[Fut04] A. Futaki, *Asymptotic Chow semi-stability and integral invariants*, *Int. J. Math.* **15** (2004), 967–979.

[GKZ94] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants* (Birkhäuser Boston, Inc., Boston, MA, 1994).

[Hat02] A. Hatcher, Algebraic topology (Cambridge University Press, Cambridge, 2002).

https://doi.org/10.1112/mod.2025.1 Published online by Cambridge University Press

[LLSW19] K. L. Lee, Z. Li, J. Sturm and X. Wang, *Asymptotic Chow stability of toric Del Pezzo surfaces*, Math. Res. Lett. **26** (2019), 1759–1787.

[LY24] K. L. Lee and N. Yotsutani, *Chow stability of λ-stable toric varieties* (2024), preprint: arXiv: 2405.06883.

[Mab04] T. Mabuchi, *An obstruction to asymptotic semistability and approximate critical metrics*, Osaka J. Math. **41** (2004), 463–472, MR2069096.

[Mab06] T. Mabuchi, *Chow-stability and Hilbert-stability in Mumford's geometric invariant theory*, Osaka J. Math. **45** (2006), 833–846.

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, in Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in mathematics and related Areas 2)], vol. **34**, 3rd ed. (Springer-Verlag, Berlin, 1994), MR1304906.

[Ono13] H. Ono, *Algebro-geometric semistability of polarized toric manifolds*, Asian J. Math. **17** (2013), 609–616.

[OSY12] H. Ono, Y. Sano and N. Yotsutani, *An example of an asymptotically Chow unstable manifold with constant scalar curvature*, Ann. Inst. Fourier (Grenoble) **62** (2012), 1265–1287, (English, with English and French summaries), MR3025743.

[Pic99] G. A. Pick, *Geometrisches zur Zahlentheorie*, Sitzungber Lotos (Prague) **19**, (1899), 311–319.

[PS10] D. H. Phong and J. Sturm, *Lectures on stability and constant scalar curvature*, in Handbook of geometric analysis, vol. **14**, Adv. Lect. Math. (ALM) (Int. Press, Somerville, MA, 2010), 357–436, MR2743451.

[RT07] J. Ross and R. Thomas, *A study of the Hilbert–Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. **16** (2007), 201–255.

[Yot17] N. Yotsutani, The delta invariant and the various GIT-stability notions of toric Fano varieties, *Special Issue: New Trends in Algebraic Geometry and Its Applications*, 2nd Edn., vol. **11**, no 19, (2023), p. 4114.

King Leung Lee king-leung.lee@umontpellier.fr
L'Institut Montpelliérain Alexander Grothendieck, IMAG, IMAG – UMR 5149, Université de Montpellier, Montpellier, 34090, France