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GENERATING SYSTEMS OF SUBGROUPS IN SU(2, 1)

XI FU✉ and LIULAN LI

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Abstract

Let $G \subset \mathbf{SU}(2, 1)$ be nonelementary and S be its minimal generating system. In this paper, we show that if S satisfies some conditions, then S can be replaced by a minimal generating system S_1 consisting only of loxodromic elements.

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1. Introduction

Let G be a nonelementary Möbius subgroup and S be its generating system. Whether S can be replaced by a generating system S_1 consisting only of loxodromic elements is an interesting problem which has been studied extensively. For instance, Doyle and James proved in [3] that every nonelementary subgroup G of $\mathrm{SL}(2, \mathbb{R})$ has a generating system consisting only of hyperbolic elements. In [9], Rosenberger proved further that such a system of generators can be chosen to be minimal. Isachenko [6] and Rosenberger [10] extended these results to the case of $\mathrm{PSL}(2, \mathbb{C})$ and obtained the following theorem.

Theorem 1.1. *Let G be a nonelementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$. *Then there exists a minimal system of generators of G consisting only of loxodromic elements.*

In 2002, Wang and Yang [11] generalised Theorem 1.1 to the setting of $\mathrm{PSL}(2, \Gamma_n)$ and proved the following theorem.

Theorem 1.2. *Let G be a nonelementary subgroup of* $\mathrm{PSL}(2, \Gamma_n)$. *If G contains no elliptic element which is not strict, then there is a minimal generating system of G consisting only of loxodromic elements.*

In this note, we study the corresponding problem in the setting of $\mathbf{SU}(2, 1)$.

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2. Complex hyperbolic geometry

2.1. Complex hyperbolic space. Let $\mathbb{C}^{2,1}$ be the complex vector space of dimension three equipped with a nondegenerate, indefinite Hermitian form $\langle .,. \rangle$ of signature (2, 1) defined to be

$$\langle z, w \rangle = w^* J z = z_1 \overline{w}_3 + z_2 \overline{w}_2 + z_3 \overline{w}_1$$

with matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We consider the subspaces

$$V_- = \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$
$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}$$

and the canonical projection

$$\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{C}P^2$$

onto the complex projective space. The complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}(V_-)$ and its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}(V_0)$. That is,

$$\mathbf{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\mathrm{Re}(z_1) + |z_2|^2 < 0\}$$

and

$$\partial \mathbf{H}^2_{\mathbb{C}} - \{\infty\} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\mathrm{Re}(z_1) + |z_2|^2 = 0\}.$$

Given a point $z \in \mathbb{C}^2 \subset \mathbb{C}P^2$, we can lift $z = (z_1, z_2)$ to a point $\mathbf{z}$ in $\mathbb{C}^{2,1}$, called the standard lift of z, where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}.$$

There are two distinguished points in V_0 which are denoted by $\mathbf{0}$ and ∞, respectively. They are

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \infty = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2.2. Isometries. Denote by $\mathbf{U}(2,1)$ the group of unitary matrices for the Hermitian product $\langle .,. \rangle$. Each such matrix A satisfies the relation $A^{-1} = JA^*J$, where A^* is the Hermitian transpose of A. The full group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is the projective unitary group $\mathbf{PU}(2,1) = \mathbf{U}(2,1)/\mathbf{U}(1)$, where $\mathbf{U}(1) = \{e^{i\theta} I : \theta \in [0, 2\pi)\}$ and I is the 3×3 identity matrix. In this paper, we shall consider the group $\mathbf{SU}(2,1)$ of matrices which are unitary with respect to $\langle .,. \rangle$ and have determinant 1. Following [5], holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ are classified as follows.

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(1) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.
(2) An isometry is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$.
(3) An isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$.

LEMMA 2.1. *Let*

$$f = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \in \mathbf{SU}(2,1).$$

Then:

(1) *f is loxodromic iff f is conjugate to*

$$\begin{pmatrix} re^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & r^{-1}e^{i\theta} \end{pmatrix},$$

where $r > 1$*;*

(2) *f is elliptic iff f is conjugate to*

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta e^{i\theta} & 0 & i\sin\theta e^{i\theta} \\ 0 & e^{i\phi} & 0 \\ i\sin\theta e^{i\theta} & 0 & \cos\theta e^{i\theta} \end{pmatrix};$$

(3) *f is parabolic iff f is conjugate to*

$$\begin{pmatrix} 1 & -\sqrt{2}\bar{\zeta} & -|\zeta|^2 + iv \\ 0 & 1 & \sqrt{2}\zeta \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{i\theta} & 0 & ie^{i\theta}t \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix},$$

where $\zeta \in \mathbb{C}$, $t, v \in \mathbb{R}$.

2.3. Totally geodesic manifolds and Fuchsian groups. Unlike real hyperbolic space, there are two kinds of totally geodesic manifolds with codimension two in $\mathbf{H}^2_{\mathbb{C}}$. In the first place there are *complex lines* which have constant curvature -1. Every complex line L is the image of the complex line

$$L_0 = \{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : z_2 = 0\}$$

under some element of $\mathbf{SU}(2,1)$. The subgroup of $\mathbf{SU}(2,1)$ stabilising L is thus conjugate to the subgroup $\mathbf{S}(U(1) \times U(1,1)) \subset \mathbf{SU}(2,1)$. Secondly, we have totally real *Lagrangian planes* which have constant curvature $-\frac{1}{4}$. Every Lagrangian plane is the image of the standard real Lagrangian plane

$$R_{\mathbb{R}} = \{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : z_i = x_i \in \mathbb{R}, 2x_1 + x_2^2 < 0\}$$

under some element of $\mathbf{SU}(2,1)$. The group stabilising $R_{\mathbb{R}}$ is denoted by $\mathbf{SO}(2,1)$, which is the subgroup of $\mathbf{SU}(2,1)$ comprising elements with real entries. We say that a group G is *nonelementary* if there are two loxodromic elements in G with distinct fixed points. Following [2], for any nonelementary complex hyperbolic Kleinian group $G \subset \mathbf{SU}(2,1)$,

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(1) G is called $\mathbb{C}$-*Fuchsian* if it preserves a complex line;
(2) G is called $\mathbb{R}$-*Fuchsian* if it preserves a Lagrangian plane.

Otherwise, G is called *non-Fuchsian*.

We call a nonelementary Kleinian group G *Fuchsian* if G is either $\mathbb{C}$-Fuchsian or $\mathbb{R}$-Fuchsian.

See [1, 5, 8] for more details about complex hyperbolic geometry and complex hyperbolic isometric groups.

3. Generating systems

In order to prove our main result, we need the following lemmas.

Lemma 3.1. *Let $f, g \in \mathbf{SU}(2,1)$ and f be loxodromic. If g does not interchange the two fixed points of f, then there is an integer $n_0 \in \mathbb{N}$ such that $f^m g$ or $f^{-m} g$ is loxodromic for all $m \geq n_0$.*

Proof. Without loss of generality, we assume that

$$f = \begin{pmatrix} re^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & r^{-1}e^{i\theta} \end{pmatrix}, \quad g = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix},$$

where $r > 1$. Then

$$\mathrm{tr}(f^m g) = r^m e^{im\theta} a + e^{-2im\theta} e + r^{-m} e^{im\theta} j$$

and

$$\mathrm{tr}(f^{-m} g) = r^{-m} e^{-im\theta} a + e^{2im\theta} e + r^m e^{-im\theta} j.$$

Since the fixed points of f are $\mathbf{0}$ and ∞ and g does not interchange them, we know that at least one of a or j is not zero. This implies that

$$\max\{|\mathrm{tr}(f^m g)|, |\mathrm{tr}(f^{-m} g)|\} > 3,$$

when m is large enough. It follows from [2] that at least one of $f^m g$ or $f^{-m} g$ is loxodromic. □

By the same method used in the proof of Lemma 3.1, we can prove the following.

Lemma 3.2. *Let $f, g \in \mathbf{SU}(2,1)$ and f be parabolic. If g does not fix the fixed point of f, then for all m large enough, the elements $f^m g$ are loxodromic.*

Theorem 3.3. *Let G be a nonelementary subgroup of $\mathbf{SU}(2,1)$ and S be a minimal generating system of G.*

(1) *If S contains an element which is not elliptic, then S can be replaced by a minimal generating system S_1 consisting only of loxodromic elements.*
(2) *If S contains a sub-generating system S_2 which generates a $\mathbb{C}$-Fuchsian group, then S can be replaced by a minimal generating system S_1 consisting only of loxodromic elements.*

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PROOF. The proof of (1) can be divided into the following two cases.

Case I. S contains a loxodromic element f. We may assume that

$$f = \begin{pmatrix} re^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & r^{-1}e^{i\theta} \end{pmatrix},$$

where $r > 1$. If every element in $S \setminus \{f\}$ is loxodromic there is nothing to prove. Let $g \in S$ be parabolic or elliptic. If g does not interchange 0 and ∞, then by Lemma 3.1, we can find a positive integer n such that $f^n g$ (or $f^{-n}g$) is loxodromic. Replace g by $f^n g$ (or $f^{-n}g$). If g interchange 0 and ∞, then

$$g = \begin{pmatrix} 0 & 0 & u \\ 0 & v & 0 \\ s & 0 & 0 \end{pmatrix}, \quad uvs = -1.$$

Since G is nonelementary, there exists $h \in S$ such that

$$h = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

and at least three of the numbers a, c, g, j are not zero. Then

$$gh = \begin{pmatrix} ug & uh & uj \\ vd & ve & vf \\ sa & sb & sc \end{pmatrix}.$$

First, replace g by gh and then replace gh by $f^n gh$ (or $f^{-n}gh$). Repeating the above procedure on each nonloxodromic element in S, we can obtain a minimal generating system S_1 consisting only of loxodromic elements.

Case II. S contains a parabolic element f.

By Lemma 3.2 and a discussion similar to Case I, we can obtain a minimal generating system S_1 consisting only of loxodromic elements.

We now prove (2). Since S_2 generates a $\mathbb{C}$-Fuchsian group, by conjugation, we may assume that

$$S_2 \subset \mathbf{S}(U(1) \times U(1, 1)).$$

Because the group PU(1, 1) is isomorphic to PSL(2, $\mathbb{R}$), by [9], S_2 can be replaced by S_2' consisting only of loxodromic elements with card$[S_2]$ = card$[S_2']$, where for a set M, card$[M]$ denotes its cardinality. Now by arguing similarly to the proof of (1), we can prove (2). □

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4. Two criteria for Fuchsian groups

In [7], Maskit proved that a nonelementary subgroup G of $\mathbf{SL}(2, \mathbb{C})$ is Fuchsian if and only if each element in G has real trace. In [4], the authors considered the corresponding problem in the setting of $\mathbf{SU}(2, 1)$ and obtained the following theorem.

THEOREM 4.1. *Let $G \subset \mathbf{SU}(2, 1)$ be nonelementary. If each loxodromic element in G is hyperbolic, then G is Fuchsian.*

REMARK 4.2. In [4], the authors constructed an $\mathbb{R}$-Fuchsian group and a $\mathbb{C}$-Fuchsian group in which each loxodromic element is hyperbolic. Note that the converse of Theorem 4.1 is false, that is, there exists some $\mathbb{C}$-Fuchsian group in which loxodromic elements are not hyperbolic (see [4]).

In this section, we prove two 'if and only if' criteria for Fuchsian groups.

THEOREM 4.3. *Let $G \subset \mathbf{SU}(2, 1)$ be nonelementary and $f \in G$ be loxodromic. Then G is $\mathbb{R}$-Fuchsian if and only if each nonelementary subgroup $\langle f, g\rangle$ is $\mathbb{R}$-Fuchsian, where $g \in G$ is loxodromic.*

PROOF. We claim that each loxodromic element in G is hyperbolic. Let $g \in G$ be loxodromic. If $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$, then by the assumption, we know that g is hyperbolic. If $\mathrm{fix}(f) \cap \mathrm{fix}(g) \neq \emptyset$, we can find a loxodromic element h in G such that $\mathrm{fix}(f) \cap \mathrm{fix}(hgh^{-1}) = \emptyset$; then the subgroup $\langle f, hgh^{-1}\rangle$ is $\mathbb{R}$-Fuchsian. This implies that g is hyperbolic. It follows from Theorem 4.1 that G is Fuchsian. Since G contains two-generator $\mathbb{R}$-Fuchsian subgroups, it follows that G is $\mathbb{R}$-Fuchsian. □

It is known that every complex line is uniquely determined by two points in $\overline{\mathbf{H}}^2_{\mathbb{C}}$, so the following theorem is obvious.

THEOREM 4.4. *Let $G \subset \mathbf{SU}(2, 1)$ be nonelementary and $f \in G$ be loxodromic. Then G is $\mathbb{C}$-Fuchsian if and only if each nonelementary subgroup $\langle f, g\rangle$ is $\mathbb{C}$-Fuchsian, where $g \in G$ is loxodromic.*

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XI FU, Department of Mathematics, Shaoxing University,
Shaoxing, Zhejiang 312000, PR China
e-mail: fuxi1000@yahoo.com.cn

LIULAN LI, Department of Mathematics and Computational Science,
Hengyang Normal University, Hengyang, Hunan 421008, PR China
e-mail: lanlimail2008@yahoo.com.cn

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