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DIRECTION OF VORTICITY AND A NEW REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS

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Abstract

In this paper, we prove a new regularity criterion in terms of the direction of vorticity for the weak solution to 3-D incompressible Navier-Stokes equations. Under the framework of Constantin and Fefferman, a more relaxed regularity criterion in terms of the direction of vorticity is established.

1. Introduction

We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times [0, T]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\ \operatorname{div} u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = \big(u_1(x, t), u_2(x, t), u_3(x, t)\big)$ is the velocity field, $p(x, t)$ is a scalar pressure, and $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering works [4] and [3], Leray and Hopf proved the existence of weak solutions $u(x, t) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$ for any given $u_0(x) \in L^2(\mathbb{R}^3)$. But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [2], they considered the direction of vorticity

$$\xi(x, t) = \frac{\omega(x, t)}{|\omega(x, t)|} \tag{1.2}$$

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and proved that the solution of the Navier-Stokes equations (1.1) corresponding to u_0, which is divergence-free, smooth and has compact support, is strong and hence smooth (C^∞) on the time interval $[0, T)$ if the following assumption holds.

Assumption (A). There exist constants $K > 0$ and $\rho(t) > 0$ such that

$$\left|P^{\perp}_{\xi(x,t)}(\xi(x+y,t))\right| \le \frac{|y|}{\rho(t)} \quad \text{for } \rho(t)^{-12} \in L^1(0,T) \tag{1.3}$$

holds if both $|\omega(x,t)| > K$ and $|\omega(x+y,t)| > K$, and $0 \le t \le T$, where $P^{\perp}_{\xi(x,t)}(\xi(x+y,t))$ denotes the projection of $\xi(x+y,t)$ orthogonal to $\xi(x,t)$.

In this paper, we want to prove regularity under a more relaxed assumption than (1.3). Our assumption reads as follows.

(H) There exist $\beta \in [1/2, 3/2)$, a positive constant K, and $g(x,t) \in L^{\alpha,\gamma} \equiv L^\alpha(0,T; L^\gamma(\mathbb{R}^3))$ such that

$$\left|P^{\perp}_{\xi(x,t)}(\xi(x+y,t))\right| \le g(x,t)|y|^\beta \tag{1.4}$$

holds in the region both $|\omega(x,t)| > K$ and $|\omega(x+y,t)| > K$, and $0 \le t \le T$, with

$$\frac{2}{\alpha} + \frac{3}{\gamma} \le \beta - \frac{1}{2} \quad \text{for } \alpha \in \left[\frac{4}{2\beta-1}, \infty\right]. \tag{1.5}$$

The main result of this paper is given in the following theorem.

THEOREM 1.1. *Let* $u_0 \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. *Suppose* u *is a Leray-Hopf weak solution to* (1.1) *corresponding to* u_0. *If* (H) *is satisfied, then* u *is a strong solution on* $[0, T]$.

REMARK 1.1. We explain the motivation for establishing (1.4) as follows. First, from a mathematical viewpoint, $P^{\perp}_{\xi(x,t)}(\xi(x+y,t))$ is a function which depends on x, y and t, so it is reasonable to find a condition in terms of x, y and t also to control it; that is, $g(x,t)$ should depend on both x and t. In this sense, (1.4) is a more natural (and relaxed) condition than (1.3). Secondly, it is well known that if u solves the Navier-Stokes equations, then so does u_λ for all $\lambda > 0$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$. So the ratio of the dimension of space to time is 3/2 [1]. So the condition on $g(x,t)$ is $g \in L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma$. Finally, there is a balance between $g(x,t)$ and β. When β is bigger, the condition on $g(x,t)$ is more relaxed. Hence we let $\beta \in [1/2, 3/2)$.

REMARK 1.2. One can find that even for $\beta = 1$, assumption (H) is weaker than assumption (A). When $\beta = 1/2$, (1.5) implies that $g(x,t) \in L^\infty(\mathbb{R}^3 \times (0,T))$.

REMARK 1.3. For recent progress on regularity criteria in terms of velocity and pressure, see [6–8] and references therein.

Before going to the proof, let us recall the definition of Leray-Hopf weak solutions.

DEFINITION. A measurable vector u is called a Leray-Hopf weak solution to the Navier-Stokes equations (1.1), if u satisfies the following properties:

(i) u is weakly continuous from $[0, \infty)$ to $L^2(\mathbb{R}^3)$.

(ii) u verifies (1.1) in the sense of distribution, that is,

$$\int_0^\infty \int_{\mathbb{R}^3} \left(\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi \right) u \, dx\, dt + \int_{\mathbb{R}^3} u_0 \phi(x, 0)\, dx = \int_0^\infty \int_{\mathbb{R}^3} \nabla u : \nabla \phi \, dx\, dt$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0, \infty))$ with $\operatorname{div} \phi = 0$. Here $A : B = \sum_{i,j}^3 a_{ij} b_{ij}$, $A = (a_{ij})$ and $B = (b_{ij})$ are 3×3 matrices and

$$\int_0^\infty \int_{\mathbb{R}^3} u \cdot \nabla \phi \, dx\, dt = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^3 \times [0, \infty))$.

(iii) The energy inequality holds, that is,

$$\|u(\cdot, t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(\cdot, s)\|_{L^2}^2 \, ds \le \|u_0\|_{L^2}^2, \quad t \ge 0.$$

By a strong solution we mean a weak solution u such that

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2).$$

It is well known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

2. Proof of Theorem 1.1

Since the theorem is proved under the framework of [2], let us recall a few observations regarding the relationship between divergence-free velocities, the associated vorticities and strain matrices in [2].

Let ω be the vorticity, $w = \operatorname{curl} u$. The strain matrix $S(x)$ in terms of ω is given by

$$S(x) = S[\omega](x) \equiv \frac{1}{2} \left(\nabla u + (\nabla u)^T \right) = \frac{3}{4\pi} \text{P. V.} \int_{\mathbb{R}^3} M\left(\hat{y}, \omega(x + y)\right) \frac{dy}{|y|^3},$$

where $M(\hat{y}, \omega) = [\hat{y} \otimes (\hat{y} \times \omega) + (\hat{y} \times \omega) \otimes \hat{y}]/2$ and $\hat{y} = y/|y|$. Let

$$\eta(x) = S(x)\xi(x) \cdot \xi(x),$$

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where ξ is the direction of the vorticity defined by (1.2). Both η and ξ are defined in the region $\{x : |\omega(x)| > 0\}$. It was derived in [2] that

$$\eta(x) = \frac{3}{4\pi}\,\mathrm{P.V.}\int_{\mathbb{R}^3} D\big(\hat{y}, \xi(x+y), \xi(x)\big)|\omega(x+y)|\,\frac{dy}{|y|^3},$$

where D is given by $D(e_1, e_2, e_3) = (e_1 \cdot e_3)(\mathrm{Det}(e_1, e_2, e_3))$. The Det in D is the determinant of the matrix whose columns are the three unit column vectors e_1, e_2 and e_3. The geometric significance of D is that it is a multiple of the volume of the prism of edges equal to $\hat{y}$, $\xi(x+y)$, $\xi(x)$. In particular, it depends on $\xi(x+y)$ only through $P^{\perp}_{\xi(x)}\xi(x+y)$, thus

$$\big|D(\hat{y}, \xi(x+y), \xi(x))\big| \le \big|P^{\perp}_{\xi(x)}\xi(x+y)\big|. \tag{2.1}$$

For solutions of the Navier-Stokes equations the dynamical significance of the expression

$$\big(S(x,t)\omega(x,t)\big)\cdot\omega(x,t) = \eta(x,t)|\omega(x,t)|^2$$

is that it presents the stretching term in evolution of the vorticity magnitude:

$$\big(\partial_t + u\cdot\nabla - \Delta\big)|\omega|^2 + |\nabla\omega|^2 = \eta|\omega|^2. \tag{2.2}$$

Equation (2.2) allows one to understand how local alignment of the vorticity direction depletes the nonlinearity.

After this review of important formulas we turn our attention to the proof. As in the argument in [7], one must have an *a priori* estimate for the strong solution under the assumption (H). The key lemma reads as follows.

LEMMA 2.1. *Let $u_0 \in H^1(\mathbb{R}^3)$ with* $\operatorname{div} u_0 = 0$. *Suppose that u is a strong solution on* $(0, T)$. *If* (H) *is satisfied, then for all* $0 \le t \le T$,

$$\|\omega\|_{L^2}^2 + \int_0^t \|\nabla\omega\|_{L^2}^2\, d \le \|\omega_0\|_{L^2}^2 \exp\Big\{AT + BT^{3/5} + C\|g\|_{L^{\alpha,\gamma}}^{2\alpha/(\alpha+2)}\Big\}$$
$$\times\Big(1 + AT + BT^{3/5} + C\|g\|_{L^{\alpha,\gamma}}^{c2\alpha/(\alpha+2)}\Big), \tag{2.3}$$

where the constant A depends on K, B depends on K and $\|u_0\|_{L^2}$, while C depends on α, γ and $\|u_0\|_{L^2}$.

PROOF. The vorticity field satisfies

$$\begin{cases} \dfrac{\partial\omega}{\partial t} + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u + \Delta\omega, \\ \operatorname{div} u = 0, \\ \operatorname{curl} u = \omega, \\ \omega(x,0) = \omega_0(x). \end{cases} \tag{2.4}$$

Multiplying the first equation of (2.4) by ω, and integrating on $\mathbb{R}^3$, after suitable integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2+\|\nabla\omega(\cdot,t)\|_{L^2}^2=\int_{\mathbb{R}^3}\big(S(x,t)\omega(x,t)\big)\cdot\omega(x,t)\,dx\equiv I. \tag{2.5}$$

Let K be the positive constant in (H) and split $\omega(x,t)$ as

$$\begin{aligned}\omega(x,t)&=\chi\left(\frac{|\omega(x,t)|}{K}\right)\omega(x,t)+\left(1-\chi\left(\frac{|\omega(x,t)|}{K}\right)\right)\omega(x,t)\\&=w_1(x,t)+w_2(x,t),\end{aligned}$$

where the smooth bump function $\chi(\lambda)\in[0,1]$ is identically equal to one for $0\le\lambda\le 1$ and identically equal to zero for $\lambda\ge 2$ or $\lambda\le -1$.

So we can decompose I into

$$\begin{aligned}I&=\int_{\mathbb{R}^3}\big(S(x,t)\omega(x,t)\big)\cdot\omega(x,t)dx\\&=\sum_{i=1}^{2}\int_{\mathbb{R}^3}\sum_{k=1}^{2}\big(S_i(x,t)\omega_1(x,t)\big)\cdot\omega_k(x,t)\,dx\\&\quad+\sum_{i=1}^{2}\int_{\mathbb{R}^3}\big(S_i(x,t)\omega_2(x,t)\big)\cdot\omega_1(x,t)\,dx\\&\quad+\int_{\mathbb{R}^3}\big(S_1(x,t)\omega_2(x,t)\big)\cdot\omega_2(x,t)\,dx\\&\quad+\int_{\mathbb{R}^3}\big(S_2(x,t)\omega_2(x,t)\big)\cdot\omega_2(x,t)\,dx\equiv I_1+I_2+I_3+I_4,\end{aligned}$$

where $S_i(x)=S[w_i](x)$, for $i=1,2$. We will estimate the above terms one by one:

$$\begin{aligned}|I_1|&=\left|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\sum_{k=1}^{2}\big(S_i(x,t)\omega_1(x,t)\big)\cdot\omega_k(x,t)\,dx\right|\\&\le 2K\sum_{i=1}^{2}\sum_{k=1}^{2}\|S_i(x,t)\|_{L^2}\|\omega_k\|_{L^2}\le C_1\|\omega\|_{L^2}^2,\end{aligned}\tag{2.6}$$

where C_1 is a constant depending only on K, and we used Hölder's inequality $|\omega_1|\le 2K$ and the Calderón-Zygmund inequality

$$\|S_i(x,t)\|_{L^q}\le C_2\|w_i\|_{L^q} \tag{2.7}$$

with $1<q<\infty$ and $i=1,2$, with C_2 a constant depending only on q.

The term I_2 can be treated similarly as I_1, so that

$$|I_2| \leq C_1 \|\omega\|_{L^2}^2. \tag{2.8}$$

Also I_3 is not a difficult term, and it can be treated as

$$\begin{aligned}
|I_3| &= \left| \int_{\mathbb{R}^3} \big(S_1(x,t)\omega_2(x,t)\big) \cdot \omega_2(x,t)\,dx \right| \\
&\leq C_2 \|\omega_1\|_{L^4} \|\omega\|_{L^4} \|\omega\|_{L^2} \quad \text{(Hölder's inequality and (2.7))} \\
&\leq C_3 \|\omega_1\|_{L^4} \|\omega\|_{L^2}^{1/4} \|\nabla\omega\|_{L^2}^{3/4} \|\omega\|_{L^2} \quad \text{(the Gagliardo-Nirenberg inequality)} \\
&\leq \frac{1}{4} \|\nabla\omega\|_{L^2}^2 + C_4 \|\omega_1\|_{L^4}^{8/5} \|\omega\|_{L^2}^2 \quad \text{(Young's inequality)} \\
&\leq \frac{1}{4} \|\nabla\omega\|_{L^2}^2 + C_5 \|\omega\|_{L^2}^{4/5} \|\omega\|_{L^2}^2. \qquad (2.9)
\end{aligned}$$

In the last inequality, we used the L^∞-bound of $|\omega_1|$. We note that C_5 depends only on K.

Actually, as one can see, I_4 is the crucial term. First, note that

$$\xi_2(x,t) = \frac{\omega_2(x,t)}{|\omega_2(x,t)|} = \xi(x,t),$$

just as was done in [2, pp. 785]. Then

$$\begin{aligned}
&\left|\big(S_2(x,t)\omega_2(x,t)\big) \cdot \omega_2(x,t)\right| \\
&\quad = |\omega_2(x,t)|^2 |S_2(x,t)\xi_2(x,t) \cdot \xi_2(x,t)| \\
&\quad = \frac{3}{4\pi} |\omega_2(x,t)|^2 \left| \mathrm{P.V.} \int_{\mathbb{R}^3} D\big(\hat{y}, \xi_2(x+y,t), \xi_2(x,t)\big) |\omega_2(x+y,t)| \frac{dy}{|y|^3} \right| \\
&\quad \leq \frac{3}{4\pi} |\omega(x,t)|^2 |g(x,t)| f(x,t), \quad \text{(by (1.4) and (2.1))}, \qquad (2.10)
\end{aligned}$$

where $f(x,t) = \int_{\mathbb{R}^3} \big(|\omega(x+y,t)|/|y|^{3-\beta}\big)\,dy$. Therefore, due to (2.10), I_4 can be estimated as

$$\begin{aligned}
|I_4| &\leq \frac{3}{4\pi} \int_{\mathbb{R}^3} |\omega(x,t)|^2 |g(x,t)| f(x,t)\,dx \\
&\leq \frac{3}{4\pi} \|\omega\|_{L^a}^2 \|f\|_{L^b} \|g\|_{L^r} \quad \text{(Hölder's inequality)} \\
&\leq C_6 \|\omega\|_{L^a}^2 \|\omega\|_{L^p} \|g\|_{L^r} \quad \text{(the Hardy-Littlewood-Sobolev inequality)} \\
&\leq C_7 \|\omega\|_{L^2}^{2(1-\theta)} \|\nabla\omega\|_{L^2}^{2\theta} \|\omega\|_{L^2}^{1-\delta} \|\nabla\omega\|_{L^2}^{\delta} \|g\|_{L^r}
\end{aligned}$$

(the Gagliardo-Nirenberg inequality for $\|\omega\|_{L^a}$ and $\|\omega\|_{L^a}$, repectively)

$$\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C_8\|\omega\|_{L^2}^{2(3-2\theta-\delta)/(2-2\theta-\delta)}\|g\|_{L^\gamma}^{2/(2-2\theta-\delta)} \quad \text{(Young's inequality)}$$
$$= \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C_8\|\omega\|_{L^2}^{2/(2-2\theta-\delta)}\|g\|_{L^\gamma}^{2/(2-2\theta-\delta)}\|\omega\|_{L^2}^2, \tag{2.11}$$

where C_8 depends on α and γ, and in the above inequality (2.11) we have used the following identities:

$$\begin{aligned} &2/a + 1/b + 1/\gamma = 1, && 1/b = 1/p - \beta/3, \\ &1/a = (1-\theta)/2 + \theta\,(1/2 - 1/3), && 1/p = (1-\delta)/2 + \delta\,(1/2 - 1/3). \end{aligned} \tag{2.12}$$

Actually, we can solve (2.12) with

$$\begin{gathered} \frac{1}{p} = \frac{\beta}{3} + \frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{3}\right), \quad \frac{1}{a} = \frac{1}{2} - \frac{1}{2}\frac{1}{\gamma} - \frac{1}{4}\left(\frac{1}{2} - \frac{\beta}{3}\right), \\ \frac{1}{b} = \frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{3}\right), \quad \theta = \frac{3}{2}\left(\frac{1}{\gamma} + \frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{3}\right)\right), \quad \delta = \frac{3}{4} - \frac{\beta}{2}, \end{gathered} \tag{2.13}$$

where α and γ satisfy (1.5). And from (2.12), one has

$$\beta/3 < 1/b + \beta/3 = 1/p = 1/2 - \delta/3 \le 1/2,$$

which implies $\beta < 3/2$.

Substituting (2.13) into (2.11), we obtain

$$|I_4| \le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C_8\|\omega\|_{L^2}^{2/(\beta+1/2-3/\gamma)}\|g\|_{L^\gamma}^{2/(\beta+1/2-3/\gamma)}\|\omega\|_{L^2}^2, \tag{2.14}$$

where C_8 is a constant depending only on α and γ.

Putting (2.6), (2.8), (2.9) and (2.14) into (2.5), we have

$$\begin{aligned} \frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &\le 4C_1\|\omega\|_{L^2}^2 + 2C_5\|\omega\|_{L^2}^{4/5}\|\omega\|_{L^2}^2 \\ &\quad + 2C_8\|\omega\|_{L^2}^{2/(\beta+1/2-3/\gamma)}\|g\|_{L^\gamma}^{2/(\beta+1/2-3/\gamma)}\|\omega\|_{L^2}^2. \end{aligned} \tag{2.15}$$

So we can use Gronwall's inequality on $\|\omega\|_{L^2}$ and it follows from (2.15) that

$$\begin{aligned} \|\omega\|_{L^2}^2 &\le \|\omega_0\|_{L^2}^2 \exp\left\{\int_0^t 4C_1 + 2C_5\|\omega\|_{L^2}^{4/5} + 2C_8\|\omega\|_{L^2}^{2/(\beta+1/2-3/\gamma)}\|g\|_{L^\gamma}^{2/(\beta+1/2-3/\gamma)}\,ds\right\} \\ &\le \|\omega_0\|_{L^2}^2 \exp\left\{4C_1T + C_9T^{3/5} + C_{10}\|g\|_{L^{\alpha,\gamma}}^{2\alpha/(\alpha+2)}\right\}, \end{aligned} \tag{2.16}$$

where C_1 depends on K, C_9 depends on K and $\|u_0\|_{L^2}$, while C_{10} depends on α, γ and $\|u_0\|_{L^2}$. In (2.16), we have used the energy inequality for u, and Hölder's inequality with

$$\frac{1}{2}\frac{2}{\beta + 1/2 - 3/\gamma} + \frac{1}{\alpha}\frac{2}{\beta + 1/2 - 3/\gamma} \le 1,$$

for $2/\alpha + 3/\gamma \le \beta - 1/2$ and $1/2 \le \beta < 3/2$.

Finally (2.3) follows from (2.15) and (2.16). This completes the proof.

After we have an *a priori* estimate for ω, the proof of Theorem 1.1 follows from the standard continuation principle, which can be stated as follows.

It is well known [5] that there is a unique strong solution $\tilde{u} \in L^\infty(0, T_0; H^1(\mathbb{R}^3)) \cap L^2(0, T_0; H^2(\mathbb{R}^3))$ to (1.1), for some $0 < T_0$, for any given $u_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Since u is a Leray-Hopf weak solution which satisfies the energy inequality, we have according to the uniqueness result, $u \equiv \tilde{u}$ on $[0, T_0)$. By the *a priori* estimate (2.3) and the standard continuation argument, the local strong solution u can be extended to time T. So we have proved u actually is a strong solution on $[0, T]$. This completes the proof of Theorem 1.1.

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