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DERIVATIONS FROM HEREDITARY SUBALGEBRAS OF C^*-ALGEBRAS

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Introduction. Let A be a C^*-algebra, B a C^*-subalgebra of A, $\delta:B \to A$ a derivation, i.e., a linear map with

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \text{for } a, b \in B.$$

There has been considerable interest for several years now in the question of when δ can be extended from B to a derivation of A (see, for example, [**8**], Section 4, [**1**], [**5**], [**4**], [**6**], [**9**], [**10**], [**11**]). The paper before the reader will be concerned with this extension problem when B is a hereditary C^*-subalgebra of A.

Our work takes its cue from the paper [**6**] of George Elliott. We prove in Section 2 of the present paper that derivations as described above of a unital hereditary C^*-subalgebra always extend whenever A is either simple, AW^*, separable and AF, or separable with continuous trace, thus generalizing and extending Theorem 4.5 of [**6**]. In Section 3 we solve in the negative Problem 3.4 of [**6**] by exhibiting two rather different examples of a separable, liminal, AF C^*-algebra A, a hereditary C^*-subalgebra B of A, and a derivation δ of B such that for each multiplier m of B, ad $m|_B + \delta$ does not extend to a derivation of A.

We will now fix some notation and terminology that will be useful in our work. Let A be a C^*-algebra, B a C^*-subalgebra of A. By a derivation of A, we will mean a derivation of A into itself. If $x \in A$, ad x will denote the inner derivation of A generated by x, i.e., the mapping $a \to xa - ax$, $a \in A$. A derivation δ of B into A is *inner in A* if δ extends to an inner derivation of A, and is *outer in A* if it does not. Finally, $M(A)$ will always denote the multiplier algebra of A.

2. Derivations from unital hereditary C^*-subalgebras. An hereditary C^*-subalgebra B of a C^*-algebra A is unital if and only if there exists a projection e of A for which $B = eAe$. In Theorem 4.5 of [**6**], Elliott proved that any derivation of eAe extends to a derivation of A whenever A is separable and AF, i.e., whenever A contains an ascending sequence (A_n) of finite-dimensional C^*-subalgebras with norm-dense union. In Theorem 2.2 below, we extend this result in two directions: first we relax the

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condition that the derivation of eAe have range in eAe and instead allow it to map into the algebra A itself, and secondly we show that derivations of this type extend to derivations of A if A is either simple, AW^*, or separable with continuous trace.

2.1. LEMMA. *Let A be a C^*-algebra, e a projection in A, $\delta:eAe \to A$ a derivation. Suppose every derivation of eAe is inner in eAe. Then δ is inner in A.*

Proof. One easily checks that $e\delta e:eAe \to eAe$ is a derivation of eAe. Since every derivation of eAe is inner in eAe, there exists $a \in eAe$ such that

$$e\delta e = ada|_{eAe}.$$

Let

$$\delta_1 = -ada|_{eAe} + \delta.$$

Then $e\delta_1 e = 0$. We assert that

$$\delta_1 = \mathrm{ad}\ \delta_1(e)(2e - 1)|_{eAe}.$$

The truth of this will establish the lemma.

If $x \in eAe$, then

$$\delta_1(x) = \delta_1(exe) = \delta_1(e)xe + e\delta_1(x)e + ex\delta_1(e)$$
$$= \delta_1(e)x + x\delta_1(e).$$

Since $(1 - e)x = 0$ and $x\delta_1(e)e = xe\delta_1(e)e = 0$, it follows that

$$\delta_1(x) = \delta_1(e)x + x\delta_1(e) - \delta_1(e)(1 - e)x - 2x\delta_1(e)e$$
$$= \delta_1(e)(2e - 1)x - x\delta_1(e)(2e - 1).$$

2.2. THEOREM. *Let A be a C^*-algebra, e a projection in A, $\delta:eAe \to A$ a derivation.*

(i) *If A is the direct sum of a family $\{A_\alpha\}$ of C^*-algebras such that for each α, A_α is either simple, AW^*, or has continuous trace and paracompact spectrum, then δ is inner in A.*

(ii) *If A is separable and AF, then δ extends to a derivation of A (which may be outer in $M(A)$).*

Proof. (i). It will suffice by Lemma 2.1 to show that each derivation of eAe is inner in eAe, and this will follow from Theorem 2 of **[13]** and Corollaries 8.6.10 and 8.6.11 of **[14]** once we observe that whenever a C^*-algebra is either simple, AW^*, or has continuous trace with paracompact spectrum, the same is true for each of its unital hereditary C^*-subalgebras. But this follows from Proposition 4.1.10 of **[14]** when the C^*-algebra is simple, from Proposition 1.4.8 (iii) of **[3]** when it is AW^*, and from Propositions 4.1.10 and 6.2.10 of **[14]** when it has continuous trace and paracompact spectrum (upon noticing that an open subset of a paracompact space is paracompact).

(ii). We consider the derivation $e\delta e$ of eAe and apply Theorem 4.5 of [**6**] to extend $e\delta e$ to a derivation D of A. Setting

$$\delta_1 = -D|_{eAe} + \delta,$$

we notice as in the proof of Lemma 2.1 that $e\delta_1(x)e = 0$, $x \in eAe$. We may hence use the proof of Lemma 2.1 to extend δ_1 to an inner derivation of A.

We now give an example to show that both (i) and (ii) of Theorem 2.2 can fail if eAe is replaced by a nonunital hereditary C^*-subalgebra, even when A is *UHF*.

Let A denote the *UHF* algebra obtained from the canonical anticommutation relations of mathematical physics. A is the norm closure of an ascending sequence (A_n) of C^*-subalgebras of A, all with the same unit, such that A_n is isomorphic to the algebra of complex $2^n \times 2^n$ matrices. There is hence a selection $\{e_n(i, j):i, j = 1, \dots, 2^n\}$ of matrix units for A_n, $n = 1, 2, 3, \dots$, such that A_n is embedded in A_{n+1} by the relations

$$e_n(i, j) = e_{n+1}(i, j) + e_{n+1}(i + 2^n, j + 2^n), \quad i, j, = 1, \dots, 2^n.$$

For $n = 1, 2, 3, \dots$, set

$$p_n = e_{n+1}(2^n, 2^n).$$

Then $\{p_n\}$ is a sequence of pairwise orthogonal projections in A. Set

$$p = \sum_n p_n, \quad q = \sum_n p_{2n}$$

(convergence of these sums taken σ-weakly in the enveloping von Neumann algebra A'' of A). Set $B = A \cap (pA''p)$. Then B is a hereditary C^*-subalgebra of A with open support p in A''. In Lemmas 2.3 and 2.4 below, we will show that q multiplies B and the derivation $\delta = \text{ad}\, q|_B$ of B is outer in A. Since every derivation of A is inner in A, this will give us the example that we want.

2.3. LEMMA. *q multiplies B.*

Proof. Let

$$e_k = \sum_1^k p_n.$$

We assert that

(2.1) $\quad B = \text{norm closure of } \bigcup_k e_k A e_k.$

Let B_1 denote the right-hand side of (2.1). (2.1) will be established by showing that B_1 is a hereditary C^*-subalgebra of A. Since B_1 clearly has open support p in A'', it will hence follow from the one-to-one correspondence between hereditary C^*-subalgebras of A and their open supports in A'' ([**14**], Section 3.11.10) that $B = B_1$.

B_1 is clearly a C^*-subalgebra of A. Note next that

$$(2.2)\quad \lim_k \|ae_k - a\| = \lim_k \|e_k a - a\| = 0, \quad a \in B_1.$$

Suppose $x \in A$ with $0 \leqq x \leqq a \in B_1$. Then by Proposition 1.4.5 of [**14**], there exists $u \in A$ and a number α, $0 < \alpha < \dfrac{1}{2}$, such that $x^{1/2} = ua^{\alpha}$. Thus

$$x = a^{\alpha}u^*ua^{\alpha},$$

and since $a^{\alpha} \in B_1$, it follows from (2.2) that

$$\begin{aligned}\lim_k e_k x e_k &= \lim_k e_k a^{\alpha}u^*ua^{\alpha}e_k \\ &= a^{\alpha}u^*ua^{\alpha} \\ &= x\end{aligned}$$

(limits taken in norm), whence $x \in B_1$ and B_1 is hereditary.

To now prove that q multiplies B, it suffices by (2.1) to prove that q multiplies e_kAe_k for each k. Fix k, and let $a \in e_kAe_k$. Then

$$qa = qe_ka = \left(\sum_1^{[k/2]} p_{2n}\right)a = e_k\left(\sum_1^{[k/2]} p_{2n}\right)ae_k,$$

and this is clearly in e_kAe_k. Similarly, $aq \in e_kAe_k$.

2.4. Lemma. *$\delta = \text{ad } q|_B : B \to B$ is outer in A.*

Proof. Suppose δ in inner in A. Then there exists $a \in A$ with $q - a$ in the commutant of B relative to A''.

Let H denote a separable Hilbert space with orthonormal basis $(\xi_m)_{m=1}^{\infty}$. We represent A on H as follows: fix positive integers m and n. Write $m = s \cdot 2^n + r$ uniquely with r and s integers, $s \geqq 0$, $1 \leqq r \leqq 2^n$. For each $i, j = 1, \ldots, 2^n$, set

$$e_n(i,j)\xi_m = \begin{cases} 0 & , j \neq r \\ \xi_{s\cdot 2^n + i}, & j = r. \end{cases}$$

We identify A with its image under this representation, and we identify p and q with their images in the algebra $B(H)$ of all bounded linear operators on H under the normal extension of this representation to a representation of A'' into $B(H)$.

A acts irreducibly on H and is hence dense in $B(H)$ with respect to the weak operator topology. B is therefore likewise dense in $pB(H)p$. There hence exists a scalar λ and

$$T \in (I - p)B(H)(I - p) \quad \text{with } a = \lambda p + q + T$$

(here I denotes the identity operator on H). We will prove that this is not possible.

Since $a \in A$, there is a sequence (k_n) of positive integers and elements $a_n \in A_{k_n}$ such that

$$\|a - a_n\| \to 0.$$

Since $aq = (1 + \lambda)q$ and $a(p - q) = \lambda(p - q)$, it follows that

(2.3) $\|(1 + \lambda)q - a_nq\| \to 0,$

(2.4) $\|\lambda(p - q) - a_n(p - q)\| \to 0.$

For $n = 1, 2, 3, \ldots$, we can find scalars $\lambda_{ij}^{(n)}$, $i, j = 1, \ldots, 2^{k_n}$, such that

$$a_n = \sum_{1 \leqq i,j \leqq 2^{k_n}} \lambda_{ij}^{(n)} e_{k_n}(i, j).$$

For each positive integer m, set $x_m = \xi_{2^{2m}}$. Then x_m is in the range of q for each m. For $m \geqq k_n/2$,

$$(2^{2m-k_n} - 1)2^{k_n} + 2^{k_n}$$

is the unique representation of 2^{2m} in the form $s \cdot 2^{k_n} + r$, s and r integers, $s \geqq 0$, $1 \leqq r \leqq 2^{k_n}$. Hence for $m \geqq k_n/2$,

$$[(1 + \lambda)q - a_nq]x_m = (1 + \lambda)\xi_{2^{2m}} - \sum_{i=1}^{2^{k_n}} \lambda_{i,2^{k_n}}^{(n)} \xi_{2^{2m}-2^{k_n}+i},$$

and so for each n,

$$|1 + \lambda - \lambda_{2^{k_n},2^{k_n}}^{(n)}| \leqq \|[(1 + \lambda)q - a_nq]x_m\|$$
$$\leqq \|(1 + \lambda)q - a_nq\|,$$

whence by (2.3),

(2.5) $\lambda_{2^{k_n},2^{k_n}}^{(n)} \to 1 + \lambda.$

On the other hand, setting $y_m = \xi_{2^m}$, m an odd positive integer, and noting that y_m is in the range of $p - q$ for each such m, we obtain by a similar computation

$$|\lambda - \lambda_{2^{k_n},2^{k_n}}^{(n)}| \leqq \|\lambda(p - q) - a_n(p - q)\|.$$

We conclude by (2.4) that

$$\lambda^{(n)}_{2^{k_n},2^{k_n}} \to \lambda,$$

which contradicts (2.5).

3. Extending derivations from hereditary subalgebras modulo multiplier derivations. After seeing the results and example of the preceding section, one is led naturally to inquire if some weaker extension process holds for derivations of hereditary subalgebras, at least in the separable, AF case. A desirable candidate for this process would be extensions modulo multiplier derivations. By this we mean the following: given a C^*-subalgebra B of a C^*-algebra A and a derivation δ of B, does there exist a multiplier m of B such that ad $m|_B + \delta$ extends to a derivation of A? When A is separable and AF and B is a closed, two-sided ideal of A, Elliott answered this affirmatively with Theorem 3.3 of [**6**], and asked in Problem 3.4 of [**6**] if the same answer obtained when B is merely assumed to be hereditary in A. In this section we give two examples which answer Elliott's question negatively. Our results can be described more concisely if we call an hereditary C^*-subalgebra B of a C^*-algebra A *bad* if it has a derivation δ such that for each multiplier m of B, ad $m|_B + \delta$ does not extend to a derivation of A, and we will say that B is *good* if it is not bad.

In order to eliminate some notational clutter in what follows, we note here that all subscripts and superscripts will assume only positive integral values.

The following proposition will be useful later; it describes the structure of hereditary subalgebras of AF algebras.

3.1. PROPOSITION. *Let A be a separable AF C^*-algebra.*

(i) *If B is an hereditary C^*-subalgebra of A, then B is AF, and if (B_n) is an ascending sequence of finite-dimensional C^*-subalgebras of B whose union is norm dense in B, then there is an ascending sequence (A_n) of finite-dimensional C^*-subalgebras of A whose union is norm dense in A, and for which $B \cap A_n = B_n$, $n \geqq 1$.*

(ii) *If (A_n) is an ascending sequence of finite-dimensional C^*-subalgebras of A with norm-dense union and if B_n is an hereditary C^*-subalgebra of A_n with $B_n \subseteq B_{n+1}$, $n \geqq 1$, then the norm closure of $\cup_n B_n$ is an hereditary C^*-subalgebra B of A.*

Proof. (i) This is Theorem 3.1 and Remark 3.2 of [**7**].

(ii) For each n, there is a projection $e_n \in A_n$ with $B_n = e_n A_n e_n$. The argument of Lemma 2.3 now shows that B is hereditary in A.

By Elliott's results, a bad hereditary subalgebra of an AF algebra must be neither unital nor an ideal. The examples we give show that among such subalgebras, bad ones exist in the presence of what we will call "multiple" and "joint" coverings.

To explain what we mean by this, let A be an AF algebra with hereditary C^*-subalgebra B. By Proposition 3.1 (i), we may choose an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A which generate A and for which $(B \cap A_n = B_n)$ generates B. B_n is a unital hereditary C^*-subalgebra of A_n, and so $B_n = e_nA_ne_n$ with e_n the unit of B_n, $n \geqq 1$. Suppose that for each n,

$$A_n = \bigoplus_{i=1}^{k_n} A_i^{(n)}, \quad B_n = \bigoplus_{i=1}^{k_n} B_i^{(n)}$$

are the Wedderburn decompositions of A_n and B_n (we suppose here that $B_i^{(n)} \subseteq A_i^{(n)}$, $1 \leqq i \leqq k_n$, and notice that some direct summands of B_n could thus be (0)). Let $f_i^{(n)}$ and $e_i^{(n)}$ denote the units of $A_i^{(n)}$ and $B_i^{(n)}$, respectively, so that

$$B_i^{(n)} = e_i^{(n)}A_i^{(n)}e_i^{(n)}, \quad 1 \leqq i \leqq k_n,$$

$$e_n = \bigoplus_i e_i^{(n)},$$

and the unit f_n of A_n is

$$\bigoplus_i f_i^{(n)}, \quad n \geqq 1.$$

We say that $e_n - e_m$ *is multiply covered* by A_m, $m \leqq n - 1$, if there is an index i, $1 \leqq i \leqq k_n$ such that

(a) $q_i = e_i^{(n)}(1 - e_m) \neq 0$, and

(b) there is an index j, $1 \leqq j \leqq k_m$, with $e_j^{(m)} \neq 0$, such that $q_i \leqq f_j^{(m)}$ and $A_j^{(m)}$ is partially embedded in $A_i^{(n)}$ with multiplicity at least 2.

We say that $e_n - e_m$ *is jointly covered by* A_m, $m \leqq n - 1$, if there is an index i, $1 \leqq i \leqq k_n$ such that (a) holds and for which

(c) there exist distinct indices $j_1, \ldots, j_p$ between 1 and k_m with $e_{j_k}^{(m)} \neq 0$, $k = 1, \ldots, p$, such that q_i from (a) is majorized by

$$\bigoplus_{k=1}^{p} f_{j_k}^{(m)},$$

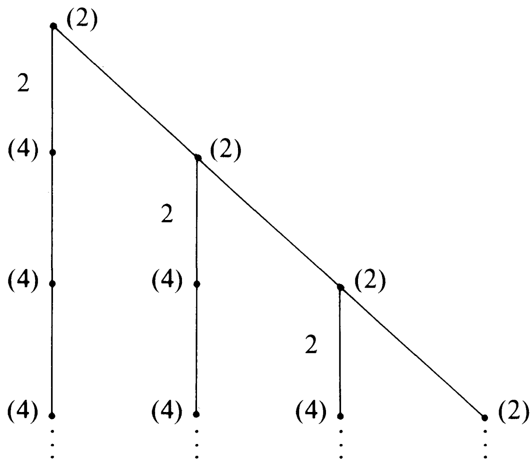
and the minimal p for which this obtains is at least 2.

Our first example will show that multiple coverings at infinitely many levels can produce bad hereditary subalgebras. This example is generated by an ascending sequence (A_n) of finite-dimensional C^*-subalgebras with Wedderburn decompositions

$$A_n = \bigoplus_{i=1}^{n} A_i^{(n)},$$

$$\dim A_i^{(n)} = 4^2,\ 1 \leqq i \leqq n - 1,\ \dim A_n^{(n)} = 2^2,$$

and Bratteli diagram (in the notation of **[12]**, Section 2)

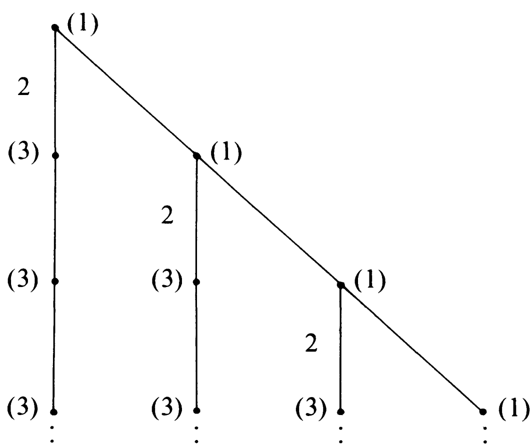


The bad hereditary subalgebra B of A is generated by an analogous sequence (B_n) with Wedderburn decompositions

$$B_n = \bigoplus_{i=1}^{n} B_i^{(n)},$$

$$\dim B_i^{(n)} = 3^2,\ 1 \leqq i \leqq n - 1,\ \dim B_n^{(n)} = 1,$$

and Bratteli diagram



Thus there are no joint coverings at any level, but for $n \geqq 2$, in the notation introduced before, we have

$$e_{n-1}^{(n)}(1 - e_{n-1}) = e_n - e_{n-1} \neq 0,$$

$$e_{n-1}^{(n-1)} \neq 0,$$

$$e_n - e_{n-1} \leqq f_{n-1}^{(n-1)},$$

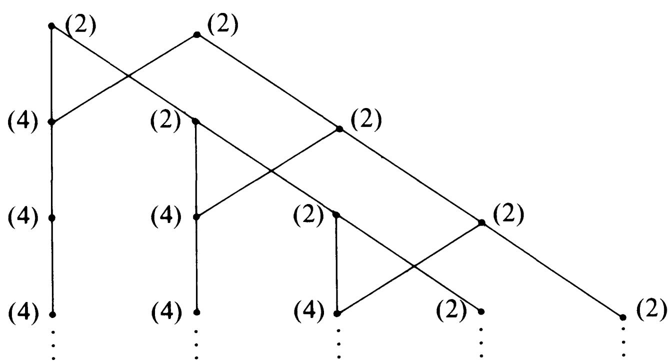
and $A_{n-1}^{(n-1)}$ is partially embedded in $A_{n-1}^{(n)}$ with multiplicity 2. Hence $e_n - e_{n-1}$ is multiply covered by A_{n-1}, $n \geqq 2$.

Our second example will show that joint coverings at infinitely many levels may produce bad subalgebras. This example is generated by an ascending sequence (A_n) of finite-dimensional C^*-subalgebras with Wedderburn decompositions

$$A_n = \bigoplus_{i=1}^{n+2} A_i^{(n)},$$

$$\dim A_i^{(n)} = 4^2,\ 1 \leqq i \leqq n,\ \dim A_{n+1}^{(n)} = \dim A_{n+2}^{(n)} = 2^2,$$

and Bratteli diagram



The bad hereditary subalgebra B of A is generated by an analogous sequence (B_n) with Wedderburn decompositions

$$B_n = \bigoplus_{i=1}^{n+2} B_i^{(n)},$$

$$\dim B_i^{(n)} = 3^2,\ 1 \leqq i \leqq n,\ \dim B_{n+1}^{(n)} = \dim B_{n+2}^{(n)} = 1,$$

and Bratteli diagram

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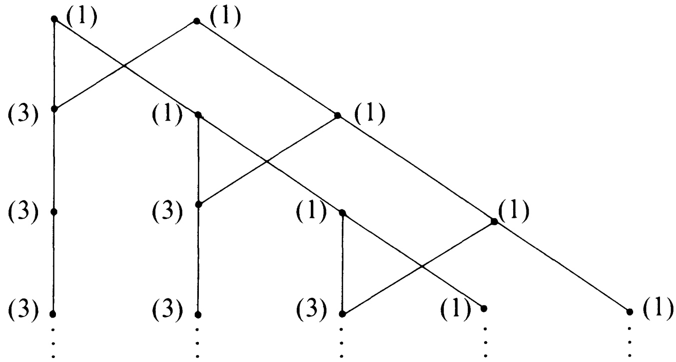

Thus all partial embeddings are of multiplicity 1 and there are hence no multiple coverings at any level, but for $n \geqq 2$,

$$e_n^{(n)}(1 - e_{n-1}) = e_n - e_{n-1} \neq 0,$$
$$e_n^{(n-1)} \neq 0 \neq e_{n+1}^{(n-1)},$$
$$e_n - e_{n-1} \leqq f_n^{(n-1)} \oplus f_{n+1}^{(n-1)},$$

and $e_n - e_{n-1}$ is majorized by a sum of no fewer units from A_{n-1}. Thus $e_n - e_{n-1}$ is jointly covered by A_{n-1}, $n \geqq 2$.

Other examples with different obstructions to the type of extensions under consideration may be possible. In any case, the examples we give and the positive results for ideals and unital hereditary subalgebras seem to indicate that a fairly complete description of good hereditary subalgebras of AF algebras could be extremely complicated. We turn now to a detailed analysis of our counterexamples.

Let M_k denote the algebra of $k \times k$ matrices with entries in the complex numbers **C**. We let R_k denote the W^*-algebra of all norm-bounded sequences of elements of M_k, equipped with pointwise operations and the supremum norm. If $(x_n) \in R_k$, we denote x_n by $(x_{ij}(n))_{1 \leqq i,j \leqq k}$. Suppose D is a C^*-subalgebra of R_k. Since R_k is a W^*-algebra, we may suppose by Proposition 2.4 of [**2**] that the multiplier algebra $M(D)$ of D is contained in R_k. Suppose $\delta:D \rightarrow D$ is a derivation. Then δ extends to a derivation of the σ-weak closure of D in R_k, and so by Corollary 8.6.6 of [**14**], there exists $x \in R_k$ with $\delta = \text{ad}\, x|_D$. Both of these facts will be useful in what follows.

Example 1. Let A denote the separable C^*-subalgebra of R_4 consisting of all elements of R_4 which converge in norm to a matrix of the form

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{pmatrix}$$

Then A is liminal and AF. Let B denote the C^*-subalgebra of A consisting of all elements

$$x = ((x_{ij}(n))_{1 \leqq, i, j \leqq 4})$$

of A with

$$x_{ij}(n) = 0 \quad \text{if } \max\{i, j\} = 4, \quad n \geqq 1.$$

It follows straightforwardly from Proposition 3.1 (ii) that B is hereditary in A. We will prove

3.2. THEOREM. *B is a bad subalgebra of A.*

The proof will unfold in a series of claims and their demonstrations.

Claim 1. Let δ be a derivation of A. Then there exists $x = (x_n) \in R_4$ with $\delta(a) = (\text{ad } x)(a)$, $a \in A$, and

$$\lim_n (x_{13}(n) - x_{24}(n)) = 0. \tag{3.1}$$

The existence of x follows from the comments which precede this example. Since δ maps A into A, we have from the definition of A that

$$\lim_n \left(\sum_k x_{1k}(n) a_{k4}(n) - \sum_k a_{1k}(n) x_{k4}(n) \right) = 0$$

for each $a = (a_{ij}(n)) \in A$, and evaluating this at the element $a = (a_n)$ of A defined by

$$a_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

yields (3.1).

Claim 2. If $m = (m_n) \in R_4$ is a multiplier of B, then

$$\lim_n m_{13}(n) = 0. \tag{3.2}$$

Since $mb \in B$ for each $b \in B$, we have

$$\lim_n \sum_k m_{1k}(n) b_{k3}(n) = 0,$$

and evaluating this at the element $b = (b_n) \in B$ defined by

$$b_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

yields (3.2).

Claim 3. Suppose $x = (x_n) \in R_4$ is such that $(\mathrm{ad}\ x)(A) \subseteq A$ and $(\mathrm{ad}\ x)(B) \subseteq B$. Then

(3.3) $\lim_n x_{13}(n) = 0.$

Since $(\mathrm{ad}\ x)(B) \subseteq B$, it follows easily from the definition of B that $x_{24}(n) = 0$, $n \geqq 1$. (3.3) hence follows from (3.1).

Now, let (δ_n) be a bounded sequence of scalars which does not converge to 0. Let $y = (y_n)$ be the element of R_4 defined by

$$y_n = \begin{pmatrix} 0 & 0 & \delta_n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

and set $\delta(b) = (\mathrm{ad}\ y)(b)$, $b \in B$. Since

$$(\mathrm{ad}\ y_n)(b_n) = \begin{pmatrix} \delta_n b_{31}(n) & \delta_n b_{32}(n) & \delta_n(b_{33}(n) - b_{11}(n)) & 0 \\ 0 & 0 & -\delta_n b_{21}(n) & 0 \\ 0 & 0 & -\delta_n b_{31}(n) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

it follows that δ is a derivation of B.

Claim 4. There is no multiplier m of B such that $\mathrm{ad}\ m|_B + \delta$ extends to a derivation of A.

Suppose such an $m = (m_n) \in M(B)$ exists. Then there exists $x = (x_n) \in R_4$ with $(\mathrm{ad}\ x)(A) \subseteq A$ and $z = (z_n)$ in the commutant of B relative to R_4 such that $m + y = x + z$. Thus $m + y - z = x$ is an element of R_4 which satisfies the conditions of Claim 3. Hence by (3.3),

(3.4) $\lim_n x_{13}(n) = 0.$

Since each z_n is a diagonal matrix,

$$x_{13}(n) = m_{13}(n) + \delta_n, \quad n \geqq 1.$$

By (3.2),

$$\lim_n m_{13}(n) = 0,$$

and so we conclude by (3.4) that

$$\lim_n \delta_n = 0,$$

contrary to the choice of (δ_n).

Example 2. Here A is the separable C^*-subalgebra of R_4 consisting of all elements of R_4 which converge in norm to a matrix of the form

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \varphi & \chi \\ 0 & 0 & \psi & \omega \end{pmatrix}$$

A is liminal and AF. We now begin to construct the bad hereditary subalgebra B of A that we want.

Let $\{e_{ij}:1 \leqq i, j \leqq 4\}$ denote the standard matrix units in M_4. Set

$$w_{11} = e_{11}, w_{12} = e_{13}, w_{13} = \frac{1}{\sqrt{2}}(e_{12} + e_{14}),$$

$$w_{21} = e_{31}, w_{22} = e_{33}, w_{23} = \frac{1}{\sqrt{2}}(e_{32} + e_{34}),$$

$$w_{31} = \frac{1}{\sqrt{2}}(e_{21} + e_{41}), w_{32} = \frac{1}{\sqrt{2}}(e_{23} + e_{43}),$$

$$w_{33} = \frac{1}{2}(e_{22} + e_{24} + e_{42} + e_{44}),$$

and let

$$W = \text{linear span in } M_4 \text{ of } \{w_{ij}:1 \leqq i, j, \leqq 3\}.$$

3.3. LEMMA. (i) $\{w_{ij}:1 \leqq i, j \leqq 3\}$ *is a* 3×3 *system of matrix units in* M_4.

(ii) *W is an hereditary C^*-subalgebra of M_4 isomorphic to M_3.*

(iii) *The commutant of W in M_4 is*

$$\left\{\lambda \sum_{i=1}^{3} w_{ii} + \frac{\mu}{2}(e_{22} + e_{44} - e_{24} - e_{42}):\lambda, \mu \in \mathbf{C}\right\}.$$

Proof. Let ξ_i denote the standard vector basis of $\mathbf{C}^4$, and set

$$\eta_1 = \xi_1, \eta_2 = \xi_3, \eta_3 = \frac{1}{\sqrt{2}}(\xi_2 + \xi_4), \eta_4 = \frac{1}{\sqrt{2}}(\xi_2 - \xi_4).$$

Then $\{\eta_i:1 \leqq i \leqq 4\}$ is an orthonormal basis of $\mathbf{C}^4$. Let u denote the unitary matrix defined by $u\eta_i = \xi_i$, $1 \leqq i \leqq 4$. Thus

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

Then $w_{ij} = u^*e_{ij}u$, $1 \leqq i, j \leqq 3$, and (i) and (ii) hold. As for (iii), simply notice that the commutant of $\{e_{ij}:1 \leqq i, j \leqq 3\}$ in M_4 is

$$\left\{\lambda \sum_{i=1}^{3} e_{ii} + \mu e_{44}:\lambda, \mu \in \mathbf{C}\right\},$$

and so the commutant of W in M_4 is

$$\left\{\lambda \sum_{i=1}^{3} w_{ii} + \mu u^* e_{44} u:\lambda, \mu \in \mathbf{C}\right\}$$

while

$$u^* e_{44} u = \frac{1}{2}(e_{22} + e_{44} - e_{24} - e_{42}).$$

Now, let B denote the set of all elements (x_n) of R_4 with

$$x_n = \sum_{1 \leqq i,j \leqq 3} x_{ij}(n) w_{ij},$$

where $(x_{ij}(n))_n$ converges for $1 \leqq i, j, \leqq 3$ and

$$\lim_n x_{ij}(n) = 0 \quad \text{if } (i, j) \neq (1, 1) \text{ or } (2, 2).$$

3.4. LEMMA. *Let C denote the C^*-subalgebra of R_3 consisting of all sequences which converge in norm to a matrix of the form*

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then B is an hereditary C^*-subalgebra of A isomorphic to C.

Proof. It is easy to check that the mapping

$$\varphi:((x_{ij}(n))_{1 \leqq i,j \leqq 3}) \rightarrow \left(\sum_{1 \leqq i,j \leqq 3} x_{ij}(n) w_{ij} \right)$$

defines a bijection of C onto B which is isometric and preserves all the linear and algebraic structure, and so B is a C^*-subalgebra of R_4 isomorphic to C. Now B consists of all sequences of 4×4 matrices of the form

$$\begin{pmatrix} x_{11}(n) & \frac{1}{\sqrt{2}}x_{13}(n) & x_{12}(n) & \frac{1}{\sqrt{2}}x_{13}(n) \\ \frac{1}{\sqrt{2}}x_{31}(n) & \frac{1}{2}x_{33}(n) & \frac{1}{\sqrt{2}}x_{32}(n) & \frac{1}{2}x_{33}(n) \\ x_{21}(n) & \frac{1}{\sqrt{2}}x_{23}(n) & x_{22}(n) & \frac{1}{\sqrt{2}}x_{23}(n) \\ \frac{1}{\sqrt{2}}x_{31}(n) & \frac{1}{2}x_{33}(n) & \frac{1}{\sqrt{2}}x_{32}(n) & \frac{1}{2}x_{33}(n) \end{pmatrix}$$

with $(x_{ij}(n))$ satisfying the specified conditions, and it follows easily that $B \subseteq A$. The fact that B is hereditary in A follows from Proposition 3.1 (ii) and Lemma 3.3 (ii).

We will now prove

3.5. THEOREM. *B is a bad subalgebra of A.*

As in Example 1, the proof will proceed via a series of claims.

Claim 5. Let $m = (m_n) \in R_3$ be a multiplier of C. Then $(m_{11}(n))_n$ and $(m_{22}(n))_n$ are both convergent.

Since $mx \in C$ for each $x \in C$, we have by the definition of C that

$$\left(\sum_k m_{1k}(n)x_{k1}(n)\right)_n \quad \text{and} \quad \left(\sum_k m_{2k}(n)x_{k2}(n)\right)_n$$

converge for each $x = (x_{ij}(n)) \in C$. Evaluating these sequences at the element (x_n) of C defined by

$$x_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

yields $(m_{11}(n))_n$ and $(m_{22}(n))_n$.

Claim 6. Let δ be a derivation of A. Then there exists $x = (x_n) \in R_4$ such that $\delta = \text{ad}\, x|_A$, and $(x_{24}(n))$, $(x_{11}(n) - x_{22}(n))$, and $(x_{33}(n) - x_{44}(n))$ are all convergent.

The existence of x follows from the remarks which immediately precede Example 1. Since $(\text{ad}\, x)(A) \subseteq A$, it follows from the definition of A that

$$\left(\sum_k x_{1k}(n)a_{k4}(n) - \sum_k a_{1k}(n)x_{k4}(n)\right)_n, \tag{3.5}$$

$$\left(\sum_k x_{1k}(n)a_{k2}(n) - \sum_k a_{1k}(n)x_{k2}(n)\right)_n, \tag{3.6}$$

$$\left(\sum_k x_{3k}(n)a_{k4}(n) - \sum_k a_{3k}(n)x_{k4}(n)\right)_n, \tag{3.7}$$

are all convergent for each $a = ((a_{ij}(n))) \in A$. Evaluating (3.6) and (3.7) at the element (a_n) of A defined by

$$a_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

yields $(x_{11}(n) - x_{22}(n))$ and $(x_{33}(n) - x_{44}(n))$, respectively. Evaluating (3.5) at (a_n) and $(b_n) \in A$ defined by

$$b_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1,$$

and averaging the sequences which result yields $(-x_{24}(n))$. The claim follows.

Now, let (δ_n) be a bounded sequence of scalars which does not converge. Let $x = (x_n) \in R_3$ be defined by

$$x_n = \begin{pmatrix} \delta_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1.$$

One easily checks that $d = \text{ad } x|_C$ is a derivation of C, and thus if we set $\delta = \varphi d \varphi^{-1}$, where φ is the isomorphism of C onto B defined in the proof of Lemma 3.4, then δ is a derivation of B. We note that $\delta = \text{ad } y|_B$, where $y = (y_n) \in R_4$ is defined by

$$y_n = \begin{pmatrix} \delta_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad n \geqq 1.$$

Claim 7. There is no multiplier m of B for which ad $m|_B + \delta$ extends to a derivation of A.

Suppose such an $m = (m_n) \in M(B)$ does exist. Then there exists $x = (x_n) \in R_4$ with $(\text{ad } x)(A) \subseteq A$ and $z = (z_n)$ in the commutant of B relative to R_4 such that $m + y = x + z$. The isomorphism φ between C and B extends in the expected way to an isomorphism of $M(C)$ onto $M(B)$ ([**2**], Proposition 2.4), and so by Claim 5,

$$m_n = \sum_{1 \leqq i,j \leqq 3} m_{ij}(n) w_{ij}, \quad n \geqq 1,$$

with $(m_{11}(n))$ and $(m_{22}(n))$ both convergent. By Lemma 3.3 (iii), there are scalars λ_n and μ_n such that for $n \geqq 1$,

$$x_n = \begin{pmatrix} \delta_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$+\begin{pmatrix} m_{11}(n) & \frac{1}{\sqrt{2}}m_{13}(n) & m_{12}(n) & \frac{1}{\sqrt{2}}m_{13}(n) \\ \frac{1}{\sqrt{2}}m_{31}(n) & \frac{1}{2}m_{33}(n) & \frac{1}{\sqrt{2}}m_{32}(n) & \frac{1}{2}m_{33}(n) \\ m_{21}(n) & \frac{1}{\sqrt{2}}m_{23}(n) & m_{22}(n) & \frac{1}{\sqrt{2}}m_{23}(n) \\ \frac{1}{\sqrt{2}}m_{31}(n) & \frac{1}{2}m_{33}(n) & \frac{1}{\sqrt{2}}m_{32}(n) & \frac{1}{2}m_{33}(n) \end{pmatrix}$$

$$+\begin{pmatrix} \lambda_n & 0 & 0 & 0 \\ 0 & \frac{\lambda_n+\mu_n}{2} & 0 & \frac{\lambda_n-\mu_n}{2} \\ 0 & 0 & \lambda_n & 0 \\ 0 & \frac{\lambda_n-\mu_n}{2} & 0 & \frac{\lambda_n+\mu_n}{2} \end{pmatrix}.$$

Thus

$$x_{24}(n) = \frac{1}{2}m_{33}(n) + \frac{1}{2}(\lambda_n - \mu_n),$$

and so

$$\begin{aligned} x_{11}(n) - x_{22}(n) &= \delta_n + m_{11}(n) + \lambda_n - \frac{1}{2}m_{33}(n) - \frac{1}{2}(\lambda_n + \mu_n) \\ &= \delta_n + m_{11}(n) + \lambda_n - \mu_n - x_{24}(n). \end{aligned}$$

By Claims 5 and 6, $(x_{11}(n) - x_{22}(n))$, $(m_{11}(n))$, and $(x_{24}(n))$ are all convergent, whence $(\delta_n + \lambda_n - \mu_n)$ is convergent. But we also have

$$\begin{aligned} x_{33}(n) - x_{44}(n) &= m_{22}(n) + \lambda_n - \frac{1}{2}m_{33}(n) - \frac{1}{2}(\lambda_n + \mu_n) \\ &= m_{22}(n) + \lambda_n - \mu_n - x_{24}(n). \end{aligned}$$

Again by Claims 5 and 6, $(x_{33}(n) - x_{44}(n))$, $(m_{22}(n))$, and $(x_{24}(n))$ are convergent, and so therefore is $(\lambda_n - \mu_n)$, whence (δ_n) converges, contrary to its choice.

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Added in proof. In a paper to appear in the Journal of the London Mathematical Society, we have obtained a theorem which gives some more affirmative solutions to Elliott's extension problem. Our result asserts that if B is an hereditary subalgebra of an AF-algebra A, then, roughly speaking, derivations of B extend modulo multiplier derivations to derivations of A whenever B does not possess obstructions similar to those appearing in Examples 1 and 2.

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