THE GENERAL STATIC SPHERICALLY SYMMETRIC SOLUTION OF THE "WEAK" UNIFIED FIELD EQUATIONS

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1. Introduction. In 1947 Einstein and Strauss **(2)** proposed a unified field theory based on a four-dimensional manifold characterized by a non-symmetric tensor g_{ij} and a non-symmetric connection $L_i{}^j{}_k$, where

(1) $$L_i \equiv L_{\underset{\vee}{i}}{}^j{}_{\underset{\vee}{j}} = 0.$$

Using a variational principle in which g_{ij} and $L_j{}^i{}_k$ are independently varied, the above authors obtain the equivalent of the following field equations:*

(2a) $$g_{ij,k} - g_{rj}L_i{}^r{}_k - g_{ir}L_k{}^r{}_j = 0,$$

(2b) $$W_{\underline{ij}} = 0,\ W_{\underset{\vee}{ij},k} + W_{\underset{\vee}{ki},j} + W_{\underset{\vee}{jk},i} = 0.$$

In these equations a comma denotes partial differentiation with respect to the co-ordinates of the manifold, W_{ij} is the Ricci tensor formed from $L_j{}^i{}_k$, and the notation

$$Q_{\underline{ij}} = \tfrac{1}{2}(Q_{ij} + Q_{ji}),\ Q_{\underset{\vee}{ij}} = \tfrac{1}{2}(Q_{ij} - Q_{ji})$$

for the symmetric and skew-symmetric parts of geometric objects Q is employed.

The equations (2a) may be solved explicitly for the parameters $L_j{}^i{}_k$ in terms of the g_{ij} and their derivatives (see, for example, **5**, ch. III), provided that an algebraic function of the g_{ij} does not vanish. To complete the solution of the field equations it is then only necessary to solve equations (2b) for the g_{ij}. If one assumes that the "source" of gravitational and electromagnetic effects produces a spherically symmetric field, Papapetrou **(4)** has shown that the tensor g_{ij} is given by a matrix of the form

(3) $$\begin{pmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & u\sin\theta & 0 \\ 0 & -u\sin\theta & -\beta\sin^2\theta & 0 \\ -w & 0 & 0 & \sigma \end{pmatrix},$$

in polar co-ordinates $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, $x_4 = t$, where α, β, σ, u, w are functions of r and t. From this, the connection parameters and hence the tensor

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*See Tonnelat **(5**, p. 31**)**. Equations (2a) here are equivalent to the first part of equations IIa (cf. **5**, ch. I, §5).

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W_{ij} may be calculated. It is found that the field equations (2b) reduce to (see **5**, p. 75)

$$(4) \qquad W_{11} = 0,\ W_{22} = 0,\ W_{44} = 0,\ W_{\underline{14}} = 0;\ W_{\underline{23}} = -c \sin \theta,$$

where c is a real constant of integration. The condition (1) is equivalent to

$$(5) \qquad \frac{w^2}{\alpha\sigma} = \frac{k^2}{k^2 + \beta^2 + u^2},$$

where, again, k is a real constant.

In the static case (α, β, σ, u, w are functions of r only), Papapetrou **(4)** has solved these equations on the assumption that $u = 0$. Also Wyman **(6)** has found the general solutions when $w = 0$. Both these solutions have been criticized by Wyman on various physical grounds. Tonnelat has pointed out the fact that if "current" is defined in a natural way, the assumption $u = 0$ seems to correspond to the existence of isolated magnetic poles. On the other hand, Wyman shows that if the boundary conditions

$$\alpha \to 1,\ \sigma \to 1,\ \beta \to \infty,\ u \to 0,\ w \to 0 \text{ as } r \to \infty,$$

suggested by general relativity are applied to his solutions, it follows necessarily that $u = 0$. Both Tonnelat and Wyman express the hope that more satisfactory conclusions could be drawn in the case when $uw \neq 0$. Accordingly it is felt that the general solution, given below, should be of some interest as a test of the validity of the hypotheses of the unified theory.

2. The field equations. It is convenient to use (5) to write the explicit form of the field equations (4) in terms of the following expressions:

$$(2.1) \qquad \text{(a)} \quad U \equiv 1 - \frac{w^2}{\alpha\sigma} = \frac{\rho^2}{k^2 + \rho^2};\ \rho^2 = u^2 + \beta^2;$$

$$\text{(b)} \quad A = \ln \rho,\ B = \tan^{-1}\left(\frac{\beta}{u}\right),\ A' = \frac{\partial A}{\partial r}, \text{ etc.}$$

In the static case then, the equation $W_{\underline{14}} = 0$ is an identity while $W_{22} = 0$, $W_{\check{23}} + c \sin \theta = 0$, $W_{11} = 0$, $W_{44} = 0$ become (see **5**, p. 73)

$$(2.2) \quad \text{(a)} \quad 0 = 1 + \left(\frac{uB' - \beta A'}{2\alpha}\right)' + B'\left(\frac{\beta B' - uA'}{2\alpha}\right) + \tfrac{1}{2}\left(\frac{uB' - \beta A'}{2\alpha}\right) \times (\ln \alpha\sigma U)'$$

$$\text{(b)} \quad 0 = c + \left(\frac{\beta B' + uA'}{2\alpha}\right)' - B'\left(\frac{uB' - \beta A'}{2\alpha}\right) + \tfrac{1}{2}\left(\frac{\beta B' + uA'}{2\alpha}\right) \times (\ln \alpha\sigma U)'$$

$$\text{(c)} \quad 0 = -A'' + \tfrac{1}{2}(\ln \alpha)'A' - \tfrac{1}{2}[(A')^2 + (B')^2] - \tfrac{1}{2}(\ln \sigma U)'' + \tfrac{1}{4}(\ln \sigma U)'\left(\ln \frac{\alpha}{\sigma U}\right)'$$

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$$\text{(d)}\quad 0 = (1-U)[(A')^2 + (B')^2] + \tfrac{1}{2}(\ln\sigma)'\left(\ln\frac{\sigma\rho^2}{\alpha}\right)' + \tfrac{1}{2}(\ln U)'\left(\ln\frac{\sigma\rho^4}{\alpha^2}\right)' + \tfrac{1}{2}\frac{(2U-1)}{1-U}(\ln U)'^2 + (\ln\sigma U^2)'',$$

respectively. If we multiply (2.2a) by u, (2.2b) by β and add, we obtain, after some simplification,

$$0 = (u + c\beta) + \left(\frac{\rho^2 B'}{2\alpha}\right)' - \frac{\rho^2}{2\alpha}A'B' + \frac{\rho^2}{4\alpha}(\ln\alpha\sigma U)'B'. \tag{2.3}$$

A similar calculation yields

$$0 = (cu - \beta) + \left(\frac{\rho^2 A'}{2\alpha}\right)' - \frac{\rho^2}{2\alpha}A'^2 + \frac{\rho^2}{4\alpha}(\ln\alpha\sigma U)'A'. \tag{2.3$'$}$$

If we introduce* a complex variable q defined by $u + i\beta = e^q$, it follows easily that $q = A + iB$ and consequently the two equations (2.3), (2.3)$'$ may be combined in a single complex condition,

$$0 = (c+i)e^q + \left(\frac{\rho^2 q'}{2\alpha}\right)' + \frac{\rho^2}{2\alpha}\cdot\frac{1}{2}\left(\ln\left(\frac{(\alpha\sigma U)}{\rho^2}\right)\right)'q', \tag{2.4}$$

where the definition of A from (2.1b) has been used. The form of this relation suggests that the field equations would be more simply expressed in terms of the variables

$$x = \frac{\rho^2}{\alpha}, \qquad y = \sigma U \tag{2.5}$$

rather than α, σ themselves. With these definitions, equation (2.4) becomes

$$0 = q'' + \tfrac{1}{2}(\ln xy)'q' + 2(c+i)\frac{e^q}{x}, \tag{2.6}$$

after division by x. Note that, for a physically meaningful solution, neither α nor β may vanish.

Returning now to the remaining equations of (2.2) and using (2.1b) (2.5), we find

$$\text{(2.7)}\quad \text{(a)}\quad 0 = -A'' + \tfrac{1}{2}[2A' - (\ln x)']A' - \tfrac{1}{2}(A'^2 + B'^2) - \tfrac{1}{2}(\ln y)'' + \tfrac{1}{4}(\ln y)'[2A' - (\ln x)' - (\ln y)'].$$

$$\text{(b)}\quad 0 = (1-U)(A'^2 + B'^2) + \tfrac{1}{2}(\ln(yU^{-1}))'\cdot(\ln(xyU^{-1}))' + \tfrac{1}{2}(\ln U)'(\ln(x^2yU^{-1}))' + \frac{1}{2}\frac{(2U-1)}{1-U}(\ln U)'^2 + (\ln y)'' + (\ln U)''.$$

*This is a generalization of the technique devised by Wyman in (6).

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From (2.1a), it follows that

$$(\ln U)' = 2A' - \frac{2\rho\rho'}{k^2 + \rho^2} = 2(1 - U)A'; \ (\ln U)'' = 2(1 - U)[A'' - 2UA'^2].$$

If we substitute these expressions into equations (2.7) and collect terms we obtain

$$\text{(2.8)} \quad \text{(a)} \ 0 = \left\{2A'' - A'^2 + A'\left(\ln\frac{x}{y}\right)' + B'^2\right\} + \{(\ln y)'' + \tfrac{1}{2}(\ln y)'(\ln xy)'\}$$

$$\text{(b)} \ 0 = (1 - U)\left\{2A'' - A'^2 + A'\left(\ln\frac{x}{y}\right)' + B'^2\right\} + \{(\ln y)'' + \tfrac{1}{2}(\ln y)'(\ln xy)'\}.$$

Since $U \neq 0$, it follows that

$$\text{(2.9)} \quad \text{(a)} \quad 2A'' - A'^2 + A'\left(\ln\frac{x}{y}\right)' + B'^2 = 0$$

$$\text{(b)} \quad (\ln y)'' + \tfrac{1}{2}(\ln y)'(\ln xy)' = 0.$$

From (2.9b) we deduce immediately that

$$\text{(2.10)} \qquad (y')^2 = \lambda\frac{y}{x},$$

where λ is a real constant. We must now distinguish two cases according as $\lambda \neq 0$ or $\lambda = 0$.

3. Case (i) $\lambda \neq 0$. In this case (2.10) may be used to eliminate x from the remaining field equations (2.6) and (2.9a). If y is taken as a new independent variable, these become

$$\text{(3.1)} \qquad 0 = \frac{d^2q}{dy^2} + \frac{1}{y}\frac{dq}{dy} + \frac{2(c+i)}{\lambda}\frac{e^q}{y},$$

$$\text{(3.2)} \qquad 0 = 2\frac{d^2A}{dy^2} - \left(\frac{dA}{dy}\right)^2 + \left(\frac{dB}{dy}\right)^2.$$

The substitutions

$$\text{(3.3)} \qquad q = p - z, \qquad z = \ln y$$

transform (3.1) into

$$\text{(3.4)} \qquad 0 = \frac{d^2p}{dz^2} + \frac{2(c+i)}{\lambda}e^p,$$

of which a first integral is

$$\text{(3.5)} \qquad c_1 = \left(\frac{dp}{dz}\right)^2 + \frac{4(c+i)}{\lambda}e^p,$$

where c_1 is a complex constant.

From (3.4) and (3.5) it follows that

$$0 = 2\frac{d^2p}{dz^2} - \left(\frac{dp}{dz}\right)^2 + c_1.$$

The real part of this equation expressed in terms of q and y, by (3.3) is equivalent to (3.2) provided

$$R_e(c_1) = 1. \tag{3.6}$$

Therefore the solution of the field equations is reduced to the solution of (3.5) subject to (3.6). The general solution of (3.5) is easily obtained in the form

$$e^p = \frac{c_1\lambda(i-c)}{4(c^2+1)}\sinh^{-2}\left[\frac{\sqrt{c_1}}{2}(z-a)\right], \tag{3.7}$$

where a is a complex constant. Using (3.3), (2.5), (2.1a), and the fact that $e^q = u + i\beta$, we therefore have the general solution

$$u + i\beta = \frac{\lambda c_1}{4}\frac{(i-c)}{(1+c^2)y}\sinh^{-2}\left[\frac{\sqrt{c_1}}{2}(\ln y - a)\right] \tag{3.8}$$

$$\alpha = \frac{(u^2+\beta^2)(y')^2}{\lambda y}, \quad \sigma = \frac{k^2+u^2+\beta^2}{u^2+\beta^2}\cdot y, \quad w = \frac{ky'}{\sqrt{\lambda}},$$

where y is an arbitrary function of r, a is a complex constant, $c_1 = 1 + ic_0$, and c_0, c, $\lambda \neq 0$, k are real constants.

4. Case (ii) $\lambda = 0$. It follows from (2.10) that y is a constant in this case. Thus y does not occur in either equation (2.6) or (2.9a). If these equations are expressed in terms of the independent variable

$$z = \int^r x^{-\frac{1}{2}}\,dr, \tag{4.1}$$

they become

$$0 = \frac{d^2q}{dz^2} + 2(c+i)e^q, \tag{4.2}$$

$$0 = 2\frac{d^2A}{dz^2} - \left(\frac{dA}{dz}\right)^2 + \left(\frac{dB}{dz}\right)^2. \tag{4.3}$$

As before, a first integral of (4.2) is

$$c_1 = \left(\frac{dq}{dz}\right)^2 + 4(c+i)e^q, \tag{4.4}$$

where c_1 is a complex constant. From this and (4.2) we obtain

$$0 = 2\frac{d^2q}{dz^2} - \left(\frac{dq}{dz}\right)^2 + c_1$$

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and the real part of this is identical to (4.3) provided

$$\text{(4.5)} \qquad \operatorname{Re}(c_1) = 0.$$

Equation (4.4) is solved in the same way as (3.5) if $c_1 \neq 0$ and consequently the general solution of the field equations in this case is

$$\text{(4.6)} \quad \begin{cases} u + i\beta = \dfrac{c_1(i-c)}{4(1+c^2)} \sinh^{-2}\left[\dfrac{\sqrt{c_1}}{2}(z-a)\right], \\ \alpha = (u^2+\beta^2)(z')^2, \quad \sigma = \dfrac{(k^2+u^2+\beta^2)}{u^2+\beta^2} y, \quad w = k\sqrt{y}\, z', \end{cases}$$

where z is an arbitrary function of r, a is a complex constant, $c_1 = ic_0$ and $0 \neq c_0$, c, y, k are real constants. On the other hand, if $c_1 = 0$, the solution is

$$\text{(4.7)} \quad \begin{cases} u + i\beta = \dfrac{i-c}{(1+c^2)}(z-a)^{-2}, \\ \alpha = (u^2+\beta^2)(z')^2, \quad \sigma = \dfrac{(k^2+u^2+\beta^2)}{u^2+\beta^2} y, \quad w = k\sqrt{y}\, z', \end{cases}$$

where z, a, c, y, and k have the same significance as above.

5. Remarks. The solution of Papapetrou is obtained from (3.8) by putting $y = 1 - 2m/r$, $a = 0$, $c_1 = 1$, $c = 0$, $\lambda = 4m^2$, which yields

$$\text{(5.1)} \quad u + i\beta = ir^2, \quad \alpha = \frac{1}{1-2m/r}, \quad \sigma = \left(1+\frac{k^2}{r^4}\right)\left(1-\frac{2m}{r}\right), \quad w = \frac{k}{r^2}.$$

The solutions of Wyman are obtained from (3.8), (4.6), and (4.7) by putting $k = 0$, adjusting the complex constant a and, when y is constant, replacing it by 1.

It is also clear that Bonnor's (real) solutions of the so-called "strong" field equations (which are the same as (4) with the added conditions $W_{14} = 0$ and $c = 0$) are special cases of (4.6) and (4.7) (see **1** or **5**, p. 85).

If we try to find non-static solutions of the field equations (4) in which u and β are functions of r only, it follows from the explicit form of the equation $W_{14} = 0$, that $(\ln \alpha)_4 = 0$ so that α must be a function of r only (cf. Mavridès **3**) and consequently the field equations reduce to the form (2.2), where σ and w may depend on t. Since w is defined by (2.1a) in terms of the other field functions and since σ occurs in (2.2) only in the form $(\ln \sigma)'$, it follows that we may derive non-static solutions from static ones by multiplying the static σ by an arbitrary function of t (and making the corresponding adjustment in the static w). Thus, for example, (3.8) yields

$$\text{(5.2)} \quad u + i\beta = \frac{\lambda c_1(i-c)}{4(1+c^2)y} \sinh^{-2}\left[\frac{\sqrt{c_1}}{2}(\ln y - a)\right]$$

$$\alpha = \frac{(u^2+\beta^2)(y')^2}{\lambda y}, \quad \sigma = \frac{k^2+u^2+\beta^2}{u^2+\beta^2} y y_0, \quad w = \frac{k}{\sqrt{\lambda}} \sqrt{y_0}\, y',$$

where $y = y(r)$, $y_0 = y_0(t)$ are arbitrary functions and the other symbols have the same significance as in (3.8).

6. Boundary conditions. Consider now the "strong" boundary conditions as $r \to \infty$:

$$\alpha \to 1,\ \beta \to r^2,\ \sigma \to 1,\ u \to 0,\ w \to 0. \tag{6.1}$$

In the case of (3.8), the condition on σ implies that $y \to 1$ as $r \to \infty$. Therefore the complex constant a must vanish if $u + i\beta$ is to become unbounded. Expanding the expression for $u + i\beta$ in powers of $\epsilon = \ln y$ and using $a = 0$, $c_1 = 1 + ic_0$ we obtain

$$u + i\beta = \frac{\lambda(i-c)}{1+c^2}\frac{1}{\epsilon^2}\left[1 - \epsilon + (5 - ic_0)\frac{\epsilon^2}{12} + 0(\epsilon^3)\right],$$

whence

$$u = \frac{\lambda}{1+c^2}\cdot\frac{1}{\epsilon^2}\left[-c + c\epsilon + \frac{(c_0 - 5c)}{12}\epsilon^2 + 0(\epsilon^3)\right],$$

$$\beta = \frac{\lambda}{1+c^2}\cdot\frac{1}{\epsilon^2}\left[1 - \epsilon + \frac{(5 + cc_0)}{12}\epsilon^2 + 0(\epsilon^3)\right].$$

Now the boundary condition on β implies that $\epsilon = 0(1/r)$ and consequently the condition on u can be satisfied only if $c = c_0 = 0$. But then $c_1 = 1$ and by (3.8) we have $u + i\beta = i\lambda(y-1)^{-2}$ and hence u is identically zero. The other boundary conditions are fulfilled if y is any function such that

$$y = 1 \pm \frac{\sqrt{\lambda}}{r} + 0\left(\frac{1}{r^2}\right).$$

In the cases (4.6) and (4.7) the boundary conditions imply that $y = 1$ and that z has the form $z = z_0 \pm 1/r + 0(1/r^2)$, where z_0 is a constant. In order that $u + i\beta$ become unbounded as $r \to \infty$ it follows that $a = z_0$. Putting $z - z_0 = \epsilon$ and considering $u + i\beta$ as given by (4.6) ($c_1 = ic_0$), we find

$$u = \frac{1}{(1+c^2)\epsilon^2}\left[-c + \frac{c_0}{12}\epsilon^2 + \dots\right], \tag{6.3}$$

$$\beta = \frac{1}{(1+c^2)\epsilon^2}\left[1 + \frac{cc_0}{12}\epsilon^2 + \dots\right].$$

Since $\epsilon = 0(1/r)$ and $c_0 \neq 0$, it follows that the boundary condition on u cannot be fulfilled. Finally, considering $u + i\beta$ as given by (4.7), we have

$$u = \frac{-c}{1+c^2}\frac{1}{\epsilon^2},\quad \beta = \frac{1}{1+c^2}\frac{1}{\epsilon^2} \qquad (\epsilon = z - a = z - z_0). \tag{6.4}$$

In order to satisfy the boundary condition on u, then, we must take $c = 0$ and thus u is identically zero.

Summarizing the above results, we note that the only solutions for which the strong boundary conditions hold are those for which u is identically zero. This does not, however, imply that Papapetrou's solution is the only one satisfying these conditions since we have not assumed that $\beta \equiv r^2$.

Weaker boundary conditions than those of (6.1) have been adopted by Bonnor **(1)**, namely,

$$\alpha \to 1, \quad \beta \to r^2, \quad \sigma \to 1, \quad \frac{u}{r^2} \to 0, \quad w \to 0 \qquad \text{as } r \to \infty. \tag{6.5}$$

It is clear from (6.2), (6.3), and (6.4) that these conditions may be satisfied simply by taking $c = 0$ and choosing y (or z) as indicated above. For such solutions u need not be identically zero.

7. Wyman's construction. Wyman **(6)** has pointed out the fact that there is no compelling reason to choose $g_{\underline{ij}}$ as the metric tensor of the underlying space-time manifold. In fact he has shown that, if the metric tensor is defined in a certain way in terms of g_{ij} and $L_j{}^i{}_k$, the Papapetrou solution of the field equations leads to the Schwartzschild metric. With this definition, then, the metric tensor does not involve the constant k, which presumably is related to the charge of the "source." This state of affairs is clearly not very satisfactory. It might have been expected that the general solution of the field equations would not produce such a difficulty; however, we will now show that any solution of the field equations (static and spherically symmetric), which satisfies the strong boundary conditions, yields a metric tensor, under Wyman's definition, which does not involve the constant k.

Consider, first, Wyman's definition* of the metric tensor, which we will denote by a_{ij}:

$$\begin{cases} a_{ij} = g_{\underline{ij}} - q_i q_j\,; \quad q_i = g_{i\check{j}} g^{\underline{jk}} u_k Q^{-\frac{1}{2}}\,; \quad Q = 1 + \tfrac{1}{2} g_{i\check{j}} g^{\check{j}i} \\ u_k = h_k(-g^{\underline{ij}} h_i h_j)^{-\frac{1}{2}}, \quad h_i = g_{\underline{jk}} g^{\check{mj}} L^{\;k}_{\check{mi}}, \end{cases} \tag{7.1}$$

where g^{ij} is, as usual defined by $g^{ij} g_{jk} = \delta_k{}^i$. We calculate a_{ij} in the static spherically-symmetric case. It will be sufficient to note that the form of $L_m{}^k{}_i$ in this case is such that h_i has only one non-vanishing component, namely h_1. Thus $u_k = \pm(-g^{11})^{-\frac{1}{2}} \delta_{k1}$. A straightforward calculation of Q from (3) yields

$$Q = 1 + \tfrac{1}{2} \cdot 2(g_{14} g^{41} + g_{23} g^{32}) = 1 + \frac{w^2}{\alpha\sigma - w^2} - \frac{u^2}{\rho^2} = \frac{\beta^2 + k^2}{\rho^2},$$

where, in the last step, we have used (5) and the definition of ρ from (2.1a). We therefore have

$$q_i = \pm\, \delta_{i4} g_{41} (-g^{11})^{\frac{1}{2}} Q^{-\frac{1}{2}}$$

and finally, then,

*For notational convenience we have revised the original definition slightly.

$$(7.1)' \quad \begin{cases} a_{ij} = g_{\underline{ij}} \qquad (i, j) \neq (4, 4), \\ a_{44} = \sigma - q_4^{\,2} = \sigma + (g_{41})^2 g^{11} Q^{-1} \\ \qquad = \sigma\left[1 - \dfrac{w^2}{\alpha\sigma - w^2}\dfrac{\rho^2}{k^2 + \beta^2}\right] = \sigma\dfrac{\beta^2}{k^2 + \beta^2}. \end{cases}$$

In **6** we saw that all solutions of the field equations which satisfy the strong boundary conditions have $u = 0$. Thus, for all such solutions,

$$\sigma = \frac{k^2 + \beta^2}{\beta^2} y$$

(cf. (3.8), (4.7)) and consequently $a_{44} = y$. Since, in no solution, does $y(r)$, α or β involve k, this is the result stated above.

In conclusion, then, it would appear that the criticism of Wyman is just as pertinent for the general solutions of the static spherically symmetric field equations as it is for the special case he considered. Physical criteria (such as those suggested by Tonnelat (**5**, ch. VI) leading to a more complete determination of the metric tensor are essential to the further development of the unified theory.

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