BULL. AUSTRAL. MATH. SOC. 26C05, 30C15, 30C75
VOL. 27 (1983), 403-406.

AN INEQUALITY FOR THE DERIVATIVE OF SELF-INVERSIVE POLYNOMIALS

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In this paper it is shown that if $p(z)$ is a polynomial of degree n satisfying $p(z) \equiv z^n p(1/z)$ then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| .$$

The result is best possible.

1.

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n and $p'(z)$ its derivative. Concerning the estimate of $|p'(z)|$ on the unit disc $|z| \leq 1$, we have the following inequality, due to Bernstein [1]:

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| . \tag{1.1}$$

An inequality analogous to (1.1) for the class of polynomials having no zero in $|z| < 1$ is due to Lax [4].

If $p(z)$ has all its zeros in $|z| \leq 1$, then it was proved by Turan [5] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)| . \tag{1.2}$$

An inequality analogous to (1.2) for polynomials having all its zeros

Received 3 February 1983.

in $|z| \le K$, $K \ge 1$ has been obtained by Govil [2].

It was proposed by Professor Q.I. Rahman to study the class of polynomials satisfying $p(z) \equiv z^n p(1/z)$ and obtain inequalities corresponding to (1.1) and (1.2). The class of polynomials satisfying $p(z) \equiv z^n p(1/z)$ is interesting in view of the fact that if $p(z)$ is any polynomial of degree n , then $P(z) = z^n p(z+(1/z))$ is a polynomial of degree $2n$ satisfying the condition $P(z) \equiv z^{2n} P(1/z)$. In an attempt to solve the problem proposed by Professor Rahman, the following theorem was proved by Govil, Jain and Labelle [3].

THEOREM A. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n p(1/z)$ *and having all its zeros lying either in the right half plane or in the left half plane, then*

(1.3) $$\max_{|z|=1} |p'(z)| \le \frac{n}{\sqrt{2}} \max_{|z|=1} |p(z)|$$

and

(1.4) $$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)| .$$

Inequality (1.4) *is best possible and equality holds for the polynomial* $p(z) = (1+z)^n$ *when the zeros lie in the left half plane and for the polynomial* $p(z) = (1-z)^n$ *when the zeros lie in the right half plane.*

In this note we strengthen inequality (1.4) by proving it without the assumption that $p(z)$ has all its zeros either in the left half plane or in the right half plane. Our result is best possible. We prove

THEOREM. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n p(1/z)$, *then*

(1.5) $$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)| .$$

This result is best possible and equality holds for the polynomial

https://doi.org/10.1017/S0004972700025909 Published online by Cambridge University Press

$p(z) = (1+z^n)$.

2.

Proof. Since the polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ satisfies $p(z) \equiv z^n \overline{p(1/\bar{z})}$, we have

$$p'(z) = nz^{n-1}\overline{p(1/\bar{z})} - z^{n-2}\overline{p'(1/\bar{z})} .$$

Thus

$$|p'(e^{i\theta})| = |ne^{i\theta}\overline{p(e^{-i\theta})}-\overline{p'(e^{-i\theta})}| .$$

In particular if θ_0 , $0 \le \theta_0 < 2\pi$, is such that

$$\max_{0\le\theta<2\pi} |p(e^{i\theta})| = |p(e^{-i\theta_0})| ,$$

then

$$(2.1) \qquad \max_{0\le\theta<2\pi} |p'(e^{i\theta})| \ge |p'(e^{i\theta_0})|$$

$$= |ne^{i\theta_0}\overline{p(e^{-i\theta_0})}-\overline{p'(e^{-i\theta_0})}|$$

$$\ge n|p(e^{-i\theta_0})| - |p'(e^{-i\theta_0})|$$

$$= n \max_{0\le\theta<2\pi} |p(e^{i\theta})| - |p'(e^{-i\theta_0})| .$$

Inequality (2.1) is equivalent to

$$|p'(e^{-i\theta_0})| + \max_{0\le\theta<2\pi} |p'(e^{i\theta})| \ge n \max_{0\le\theta<2\pi} |p(e^{i\theta})|$$

which implies

$$2 \max_{0\le\theta<2\pi} |p'(e^{i\theta})| \ge n \max_{0\le\theta<2\pi} |p(e^{i\theta})| .$$

From here the result follows.

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