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Automorphisms of the semigroup of all onto mappings of a set

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The semigroup of all onto mappings of a set to itself and the semigroup of all one-to-one mappings of a set to itself are shown to have the property that every automorphism is inner.

1. Introduction

Let X be a non-empty set. Let G denote the group of *permutations* of X and E, M, and F the semigroups of *onto* mappings, *one-to-one* mappings and *all* mappings from X to itself respectively. Throughout, the operation on G, E, M, and F will be mapping composition. Finally, let R denote the semigroup of all *binary relations* on X, the composition operation given by

$$f \circ g = \{(x, y) \in X \times X : (x, z) \in f \text{ and } (z, y) \in g \text{ for some } z \in X\}$$

for elements f and g in R.

An automorphism ϕ of a group or semigroup (S) of mappings or relations is said to be *inner* if there exists a permutation h of X such that

$$(*) \qquad f\phi = h^{-1}fh \quad \text{for every } f \text{ in } S .$$

(Functions juxtaposed imply composition.)

It is well known that for finite sets other than those with six elements $(|X| \neq 6)$, G has the property that every automorphism is inner. Schreier and Ulam in 1937 [3] extended this to infinite sets, while Schreier [2] showed that every automorphism of F is inner for any set

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X . More recently Magill proved that every automorphism of R is inner [1]. The purpose of this note is to show that the semigroups E and M ($|X| \neq 6$) also have this property.

The core of the proofs for G, F , and R is the following: an automorphism ϕ is shown to preserve a subset of the group or semigroup, allowing a natural definition of a permutation h of X with the property (*). In the case of G the set is the conjugacy class of a transposition, for F it is the unique minimal ideal of constant functions, while for R it is the set of constant relations with domain X . Here our technique is different. We observe that ϕ has the form (*) on G and show that the form extends to E (M) using the composition properties of transpositions and arbitrary onto (one-to-one) mappings.

The following notions will be useful. If $a \in X$ is the only element in X carried to af by $f \in F$ we say f *is one-to-one at* a . That is, $(af)f^{-1} = \{a\}$. Let M_f denote the set of all such points for the mapping f . If $af^{-1} = S$ consists of more than one point we call S a *condensation set* of f and a a *condensation point* of f .

2. Automorphisms

We proceed to the proof of the main theorem.

THEOREM 1. *Every automorphism* ϕ *of* E *is inner, for* $|X| \neq 6$.

Proof. If X is finite, $E = G$, so the result follows from well known group theory. For infinite X the proof is in five steps.

1. *There exists a permutation* h *of* X *such that* $f\phi = h^{-1}fh$ *for every* f *in* G .

Since $G\phi = G$, ϕ restricted to G is an automorphism of G . The result of Schreier and Ulam [3] guarantees the existence of a permutation h of X such that $f\phi = h^{-1}fh$ for every f in G .

The next result shows that condensation sets are preserved.

2. S *is a condensation set for* f *in* E *if and only if* Sh *is a condensation set for* $f\phi$.

Take ah and bh in Sh . Now $f = (a, b)f$, where (a, b) in G

is the transposition reversing elements a and b in S. Hence

$$f\phi = \big((a, b)f\big)\phi = (a, b)\phi(f\phi) = (ah, bh)(f\phi) ,$$

using

$$(a, b)\phi = h^{-1}(a, b)h = (ah, bh) .$$

Thus $ah(f\phi) = bh(f\phi)$, so $f\phi$ is constant on Sh.

A result true for ϕ is true also for ϕ^{-1}, so if $f\phi$ is constant on Sh, $(f\phi)\phi^{-1} = f$ is constant on $(Sh)h^{-1} = S$.

3. *For all* a *in* M_f, f *in* E, $ah(f\phi) = afh$.

We show this for those f in E which are one-to-one at three or more points. Maps which are one-to-one at two points or one point can be expressed as a composition of two such maps and the result will follow.

Suppose a and b are in M_f and $a \neq b$. Now

$$f = (a, b)f(af, bf) ,$$

so

$$f\phi = (ah, bh)f\phi(afh, bfh) .$$

Suppose $ah(f\phi) = x \neq afh$ or bfh. Then

$$\begin{aligned} bh(f\phi) &= bh(ah, bh)f\phi(afh, bfh) \\ &= x , \end{aligned}$$

also.

Since $a \neq b$, $ah \neq bh$ so $f\phi$ is not one-to-one at ah, contradicting step two. So $ah(f\phi) = afh$ or bfh. Applying the same argument to a and c in X where $a \neq c \neq b$ gives $ah(f\phi) = afh$.

Suppose now that f is one-to-one at only a and b in X. Take a condensation set S of f and suppose $Sf = z$. Define f_1 and f_2 in E as follows,

$$xf_1 = \begin{cases} xf & \text{if } x \in S \cup \{a, b\} , \\ xg & \text{if } x \in X \backslash (S \cup \{a, b\}) , \end{cases}$$

where g is a one-to-one correspondence between $X \backslash (S \cup \{a, b\})$ and

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$X\backslash\{af, bf, z\}$. The former set is of the same cardinality as the latter since f is onto. Let $xf_2 = yf$ when $x = yf_1$.

Then f_1 and f_2 are in E , $f_1f_2 = f$ and both f_1 and f_2 are one-to-one at three points or more. Consequently

$$ah(f\phi) = ah(f_1\phi)(f_2\phi) = afh(f_2\phi) = afh ,$$

$$bh(f\phi) = bfh .$$

A similar construction shows that if f is one-to-one at only one point, the result holds.

4. *If* f *in* E *has precisely one condensation set* S *then* $f\phi = h^{-1}fh$.

We have only to show that the single condensation point of $f\phi$ is Sfh . From step three it follows that $f\phi$ affords a one-to-one and onto correspondence between $(X\backslash S)h$ and $(X\backslash Sf)h$. But since $f\phi$ is onto,

$$\begin{aligned} Sh(f\phi) &= X\backslash(X\backslash Sf)h \\ &= X\backslash(X\backslash Sfh) \\ &= Sfh . \end{aligned}$$

5. *For every* f *in* E , $f\phi = h^{-1}fh$.

We must show that if $a \in S$, a condensation set of f , then $ah(f\phi) = afh$. We do this by writing f as a composition of a map f_1 in E with single condensation set S and a map f_2 in E which is one-to-one at af_1 . Specifically, let

$$xf_1 = \begin{cases} af & \text{if } x \in S , \\ xk & \text{if } x \in X\backslash S , \end{cases}$$

where k is a one-to-one correspondence from $X\backslash S$ onto $X\backslash\{af\}$. Let $xf_2 = yf$ when $x = yf_1$. Note that f_1 and f_2 are in E , $f_1f_2 = f$, and that f_2 is one-to-one at af_1 . As before

$$ah(f\phi) = ah(f_1\phi)(f_2\phi) = afh ,$$

completing the proof.

With only small modifications of steps one and two we have the next theorem.

THEOREM 2. *Every automorphism ϕ of M is inner, for $|X| \neq 6$.*

3. Automorphism groups

Let A_S be the automorphism group of the semigroup (or group) S. We have the following relationship between A_S and G.

THEOREM 3. *For $S = R, F, E, M$ or G,*

$$A_S \cong G$$

except for $S = G$ $(= E = M)$ when $|X| = 2$ or 6.

Proof. The map $G \to A_S$ (for S any of R, F, E, M, or G) which takes h in G to the automorphism which carries f in S to $h^{-1}fh$, is always a homomorphism. It is one-to-one precisely when $gf = fg$ for all f in S implies g in G is the identity mapping on X. This is so in all cases except for $S = G$ and $|X| \leq 2$. When $|X| = 2$, $A_G \ncong G$. The homomorphism is onto precisely when every automorphism of S is inner. This is so in all cases except for $S = G$ and $|X| = 6$, and in this case $A_G \ncong G$. To complete the proof we may check that when $|X| = 1$, A_R, A_F, and G are the trivial group.

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