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SOME PROPERTIES OF A CERTAIN SET OF INTERPOLATING POLYNOMIALS

BY
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1. **Introduction.** A Lidstone series provides a (formal) two-point expansion of a given function $f(x)$ in terms of its derivatives of even order at the nodes 0 and 1 and takes the form

$$f(x) = f(1)\Lambda_0(x)+f(0)\Lambda_0(1-x)+f''(1)\Lambda_1(x)+f''(0)\Lambda_1(1-x)+\cdots$$

where $\Lambda_n(x)$ is a polynomial of degree $2n+1$ defined by the generating function

$$\frac{\sinh xt}{\sinh t} = \sum_{n=0}^{\infty} \Lambda_n(x)t^{2n} \tag{1.1}$$

The Lidstone polynomials $\{\Lambda_n(x)\}_{n=0}^{\infty}$ have been studied extensively (see e.g. [9], [10], [11]) and their interpolatory properties are well known. In 1932, J. M. Whittaker showed the relationship between the Lidstone polynomials and the classical Bernoulli polynomials $B_n(x)$. In fact, Whittaker [10], proved that

$$\Lambda_n(x) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+x}{2}\right) \qquad n = 0, 1, \ldots \tag{1.2}$$

During an investigation of a class of infinite interpolation problems with periodic conditions defined on the nodes -1, 0 and 1 [4] the polynomial set $\{Q_{4n}(x)\}_{n=0}^{\infty}$ defined by the simple generating function

$$\frac{\cosh xt+\cos xt}{\cosh t+\cos t} = \sum_{n=0}^{\infty} \frac{Q_{4n}(x)t^{4n}}{(4n)} \tag{1.3}$$

exhibited some interesting properties in addition to the anticipated interpolating properties. This led to further investigations which have yielded a particularly interesting relationship between the polynomial set $\{Q_{4n}(x)\}$ and the Euler polynomials, stated precisely in Theorem 2.1.

Of additional interest is the fact that the normalized polynomial set $\{Q_{4n}^*(x)\}_{n=0}^{\infty}$ i.e. where

$$Q_{4n}^*(x) = \frac{Q_{4n}(x)}{(4n)!} \tag{1.4}$$

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is a generalized Appell set. Such polynomial sets which have been investigated by Osegov [6], and Al-Salaam and Verma [1] can be classified in the following way. Let r be a positive integer. A polynomial set $\{P_n(x)\}$ is in $S^{(r)}$ if there is an operator $J(D)$ of the form $J(D)=\sum_{k=0}^{\infty} a_k D^{k+r}$ $(a_0\neq 0)$ where $a_k(k\geq 0)$ is independent of x and D is the differential operator, such that

$$J(D)P_n(x) = P_{n-r}(x) \qquad (n = r, r+1, \ldots). \tag{1.5}$$

It is easily seen that the normalized polynomial set $\{Q_{4n}^*(x)\}$ belongs to the class $S^{(4)}$.

NOTE. We do not use the normalized polynomial set throughout the paper as the results are simpler stated in terms of the polynomials $\{Q_{4n}(x)\}$.

In §2 we develop the relationship between the polynomial set $\{Q_{4n}(x)\}_{n=0}^{\infty}$ and the Euler polynomials. Defining a sequence of numbers $\{Q_{4n}\}_{n=0}^{\infty}$ by setting $Q_{4n}=Q_{4n}(0)$, we give, in §3, some properties of this sequence including an asymptotic estimate for $|Q_{4n}|^{1/n}$. We also obtain a new result on divisibility of certain finite sums of products of Euler numbers. In §4, some of the properties of the polynomials $Q_{4n}(x)$ are discussed, including Theorem 4.1 which gives a zero-free interval on the real and imaginary axis.

2. The Polynomials $Q_{4n}(x)$ and the Euler Polynomials.

LEMMA 2.1. *$Q_0(x)\equiv 1$, and for $n=1, 2, \ldots$ $Q_{4n}(x)$ is a monic polynomial of degree $4n$ given by*

$$x^{4n} = \sum_{k=0}^{4n} \binom{4n}{4k} Q_{4k}(x). \tag{2.1}$$

Furthermore, for $n=0, 1, 2, \ldots$ we have the "difference equation"

$$Q_{4n}(x+1)+Q_{4n}(x-1)+Q_{4n}(x+i)+Q_{4n}(x-i) = 4x^{4n}. \tag{2.2}$$

Proof. From (1.3) we have

$$\sum_{n=0}^{\infty} \frac{t^{4n}}{(4n)!} \sum_{k=0}^{\infty} \frac{Q_{4k}(x)t^{4k}}{(4k)!} = \sum_{n=0}^{\infty} \frac{x^{4n}t^{4n}}{(4n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{Q_{4k}(x)t^{4k}}{(4k)!} \frac{t^{4n-4k}}{(4n-4k)!} = \sum_{n=0}^{\infty} \frac{x^{4n}t^{4n}}{(4n)!} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{4n}{4k} Q_{4k}(x) \right] \frac{t^{4n}}{(4n)!} = \sum_{n=0}^{\infty} \frac{x^{4n}t^{4n}}{(4n)!}.$$

Equating the coefficients of t^{4n} yields (2.1). (2.2) follows easily from (2.1) or (1.3). □

Let $E_n(x)$, $n=0, 1, \ldots$, denote the Euler polynomial of degree n defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n. \tag{2.3}$$

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We shall make use of the following well-known properties of the Euler polynomials. The nth Euler number, E_n, is defined by (2.6)

$$E_n(x)+E_n(1+x) = 2x^n, \qquad n \geq 0 \tag{2.4}$$

$$E_n(1-x) = (-1)^n E_n(x), \qquad n \geq 0 \tag{2.5}$$

$$E_n = 2^n E_n(\tfrac{1}{2}), \qquad n \geq 0. \tag{2.6}$$

The first relationship between the two sets of polynomials is given by

LEMMA 2.2. *For $n \geq 0$, we have*

$$E_{4n}(x)+E_{4n}(1+x) = 2\sum_{k=0}^{n}\binom{4n}{4k} Q_{4k}(x).$$

Proof. Immediate from (2.1) and (2.4). □

We now obtain a representation theorem for the polynomials $Q_{4n}(x)$ in terms of the Euler polynomials.

THEOREM 2.1. *For $n \geq 0$ we have*

$$Q_{4n}(x) = (-4)^n \sum_{k=0}^{2n}\binom{4n}{2k}(-1)^k E_{2k}\left(\frac{1+x}{2}\right) E_{4n-2k}\left(\frac{1+x}{2}\right). \tag{2.7}$$

Proof. From (2.3) we have

$$\frac{2e^{xt}}{e^t+1} = \frac{2e^{(x-1/2)t}}{e^{t/2}+e^{-t/2}} = \sum_{n=0}^{\infty}\frac{E_n(x)t^n}{n!}. \tag{2.8}$$

Replacing x by $1-x$ and adding yields

$$\frac{2\cosh(x-1/2)t}{\cosh t/2} = \sum_{n=0}^{\infty}[E_n(x)+E_n(1-x)]\frac{t^n}{n!}. \tag{2.9}$$

Now replacing t by $2t$ and $2x-1$ by x in (2.9) and using (2.5) we have

$$\frac{\cosh xt}{\cosh t} = \sum_{n=0}^{\infty} E_{2n}\left(\frac{1+x}{2}\right)\frac{(2t)^{2n}}{(2n)!}. \tag{2.10}$$

Replacing t by it in (2.10) yields

$$\frac{\cos xt}{\cos t} = \sum_{n=0}^{\infty} E_{2n}\left(\frac{1+x}{2}\right)\frac{(-1)^n(2t)^{2n}}{(2n)!}. \tag{2.11}$$

Finally, replacing t by $((1+i)/2)t$ and $((1-i)/2)t$ respectively in (2.11) and multiplying, we have

$$\frac{\cos\left(\frac{1+i}{2}\right)xt\,\cos\left(\frac{1-i}{2}\right)xt}{\cos\left(\frac{1+i}{2}\right)t\,\cos\left(\frac{1-i}{2}\right)t} = \sum_{n=0}^{\infty}(-4)^n\sum_{k=0}^{2n}\left[(-1)^k\binom{4n}{2k}E_{2k}\left(\frac{1+x}{2}\right)E_{4n-2k}\left(\frac{1+x}{2}\right)\right]\frac{t^{4n}}{(4n)!}. \tag{2.12}$$

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Using the identity $\cosh t+\cos t=2\cos((1+i)/2)t\cos((1-i)/2)t$ in (2.12) gives (2.7).

The first few polynomials $Q_{4n}(x)$ are:

$$Q_0(x) = 1$$

$$Q_4(x) = x^4-1$$

$$Q_8(x) = (x^4-1)(x^4-69)$$

$$Q_{12}(x) = (x^4-1)(x^8-494x^4+33{,}661)$$

$$Q_{16}(x) = (x^4-1)(x^{12}-1819x^8+886{,}211x^4-60{,}376{,}809).$$

3. **The Numbers $\{Q_{4n}\}_{n=0}^{\infty}$ and the Euler Numbers.** L. Carlitz (see [2], [3]) and other authors have considered the properties of the set of numbers $\{S_{2n}\}$ defined by the generating function

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n}\frac{x^{2n}}{(2n)}.$$

In particular, Carlitz showed that

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}E_{2k} = S_{2n} = 2^nS'_{2n} \tag{3.1}$$

where S'_{2n} is odd. In (3.1) each term in the sum is positive. The next lemma shows that a similar divisibility property holds for a special sum of products of Euler numbers in which the terms alternate in sign.

LEMMA 3.1. *For $n\geq 1$, we have*

$$\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^kE_{2k}E_{4n-2k} = (-4)^nQ_{4n} \tag{3.2}$$

where Q_{4n} is odd.

Proof. If we set $x=0$ in (2.7) and define

$$Q_{4n} = Q_{4n}(0), \qquad n\geq 0 \tag{3.3}$$

then, using (2.6) and simplifying, we get

$$Q_{4n} = (-4)^{-n}\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^kE_{2k}E_{4n-2k}. \tag{3.4}$$

To prove the lemma we need only show that Q_{4n} is odd. We have $Q_0=1$, and setting $x=0$ in (2.1) yields

$$\sum_{k=0}^{n}\binom{4n}{4k}Q_{4k} = 0. \tag{3.5}$$

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Thus, $Q_4=-1$, so assume Q_{4n-4} is odd. Then from (3.5) we have

$$Q_{4n} = -1-\binom{4n}{4}(Q_4+Q_{4n-4})-\sum_{k=2}^{n-2}\binom{4n}{4k}Q_{4k}. \tag{3.6}$$

Since, under the inductive assumption, the second and third terms in the right-hand member of (3.6) are even, Q_{4n} must necessarily be odd. □

The first seven numbers Q_{4n} are listed below

$$Q_0 = 1,\quad Q_4 = -1,\quad Q_8 = 69,\quad Q_{12} = -33{,}661,\quad Q_{16} = 60{,}376{,}809,$$
$$Q_{20} = -245{,}454{,}050{,}521,\quad Q_{24} = 3{,}019{,}098{,}162{,}602{,}349.$$

Symbolically, we can write

$$(Q+1)^{4n}+(Q-1)^{4n}+(Q+i)^{4n}+(Q-i)^{4n} = \begin{cases}4, & n=0\\ 0, & n>0\end{cases}$$

where Q^j is replaced by Q_j after multiplying out, and $Q_j=0, j\not\equiv 0 \pmod 4$.

THEOREM 3.1. *The numbers Q_{4n} have the property*

$$(-1)^n Q_{4n} > 0, \qquad n \geq 0. \tag{3.7}$$

Proof. If we set

$$Q^*_{4n} = \frac{Q_{4n}}{(4n)!} \tag{3.8}$$

then, using (3.5) we get

$$\sum_{k=0}^{n}\binom{4n}{4k}(4k)!\, Q_{4k} = 0. \tag{3.9}$$

Thus, $Q^*_0=1$, $Q^*_4=-\frac{1}{24}$. If we show that the sign of Q^*_{4n-4} determines the sign of the sum

$$\sum_{k=0}^{n-1}\binom{4n}{4k}(4k)!\, Q^*_{4k} \tag{3.10}$$

then, since the left-hand member of (3.5) is equal to zero, the numbers Q^*_{4n-4} and Q^*_{4n} must have opposite sign. Therefore, we will show by induction that, for $n\geq 2$

$$\frac{|Q^*_{4n-4}|}{4!} > 69\sum_{j=2}^{n}\frac{|Q^*_{4n-4j}|}{(4j)!}\cdot \tag{3.11}$$

(Note: 69 is the best constant in the sense that it cannot be replaced by any larger integer.)

Since,

$$\frac{1}{(4!)^2} = \frac{|Q^*_4|}{4!} > \frac{69\,|Q^*_0|}{8!} = \frac{69}{8!}$$

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(3.11) is true for $n=2$. Now assume inequality (3.11) is true for $n=k(\geq 2)$. That is,

$$\frac{|Q^*_{4k-4}|}{4!} > 69\sum_{j=2}^{k}\frac{|Q^*_{k-4j}|}{(4j)}. \tag{3.12}$$

We wish to prove that

$$\frac{|Q^*_{4k}|}{4!} > 69\sum_{j=1}^{k}\frac{|Q^*_{4k-4j}|}{(4j+4)}. \tag{3.13}$$

From (3.9) we have

$$|{}^*_{4k} > \frac{|Q^*_{4k-4}|}{4!} - \sum_{j=2}^{k}\frac{|Q^*_{4k-4j}|}{(4j)!}$$

and, using (3.12) yields

$$\frac{|Q^*_{4k}|}{4!} > \frac{68}{69}\frac{|Q^*_{4k-4}|}{(4!)^2} > 69\frac{|Q^*_{4k-4}|}{8!} + \frac{|Q^*_{4k-4}|}{8!}. \tag{3.14}$$

To complete the proof, we need only show that

$$\frac{|Q^*_{4k-4}|}{8!} > 69\sum_{j=2}^{k}\frac{|Q^*_{4k-4j}|}{(4j+4)!}. \tag{3.15}$$

Now, (3.15) will be verified if term-by-term comparison with (3.12) yields the inequalities

$$\frac{|Q^*_{4k-4j}|}{(4j+4)!} + \frac{4!}{8!}\frac{|Q^*_{4k-4j}|}{(4j)!}, \qquad (j = 2, 3, \ldots, k). \tag{3.16}$$

Inequalities (3.16) are equivalent to

$$\frac{1}{(4j+4)(4j+3)(4j+2)(4j+1)} < \frac{1}{8\cdot 7\cdot 6\cdot 5} \tag{3.17}$$

and (3.17) holds for $j=2, 3, \ldots, k$ as required. Therefore, inequality (3.15) holds. Substituting (3.15) into (3.14) gives (3.11) for $n=k+1$ and the proof by induction is complete. □

A useful result to roughly determine the size of $|Q_{4n}|$ is given by

LEMMA 3.2. *For $n\geq 1$, we have*

$$\frac{68}{69}\binom{4n}{4}|Q_{4n-4}| < |Q_{4n}| < \frac{70}{69}\binom{4n}{4}|Q_{4n-4}|. \tag{3.18}$$

Proof. From (3.14) we have

$$|Q^*_{4n}| > \frac{68}{69}\left(\frac{1}{4!}\right)|Q^*_{4n-4}|. \tag{3.19}$$

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Using (3.9) and (3.11)

$$|Q^*_{4n}| < \frac{|Q^*_{4n-4}|}{4!} + \sum_{j=2}^{n} \frac{|Q^*_{4n-4j}|}{(4j)!} < \frac{|Q^*_{4n-4}|}{4!} + \left(\frac{1}{69}\right)\frac{|Q^*_{4n-4}|}{4!}. \tag{3.20}$$

Combining inequalities (3.19) and (3.20) we have

$$\left(\frac{68}{69}\right)\frac{|Q^*_{4n-4}|}{4!} < |Q^*_{4n}| < \left(\frac{70}{69}\right)\frac{|Q^*_{4n-4}|}{4!}$$

which, by (3.8) is equivalent to (3.18). □

Repeated application of inequality (3.18) provides an asymptotic estimate for $|Q_{4n}|^{1/n}$. Roughly speaking $|Q_{4n}|^{1/n} \sim \alpha n^4$ where $\alpha \approx 0.193$.

COROLLARY 3.1. *For the numbers* $\{Q_{4n}\}_{n=0}^{\infty}$ *defined by* (3.3) *we have*

$$0.192535 < \frac{|Q_{4n}|^{1/2}}{n^4(8\pi n)^{1/2n}} < 0.198198 \qquad (n \to \infty.) \tag{3.21}$$

Proof. Applying inequality (3.18) n times we have

$$\left(\frac{68}{69}\right)^n \frac{(4n)!}{(4!)^n} < |Q_{4n}| < \left(\frac{70}{69}\right)^n \frac{(4n)!}{(4!)^n} \qquad (n \to \infty). \tag{3.22}$$

Using Stirling's formula in (3.22) yields

$$\frac{2176}{207e^4} < \frac{|Q_{4n}|^{1/n}}{n^4(8\pi n)^{1/2n}} < \frac{2240}{207e^4} \qquad (n \to \infty). \tag{3.23}$$

Approximating the right and left-hand parts of inequality (3.23) to six significant figures we get (3.21). □

REMARK. If we set $T_n = n^{-4}(8\pi n)^{-1/2n}\,|Q_{4n}|^{1/n}$ we have

$$\begin{aligned} T_1 &= 0.1994711402 & T_4 &= 0.1935040648 \\ T_2 &= 0.1949786679 & T_5 &= 0.1932707501 \\ T_3 &= 0.1939383963 & T_6 &= 0.1931265334. \end{aligned}$$

4. **Some properties of the Polynomial Set** $\{Q_{4n}(x)\}_{n=0}^{\infty}$. Since the polynomials $Q_{4n}(x)$ are polynomials in x^4, determining the roots for $x>0$, yields all real and pure imaginary roots of $Q_{4n}(x)$. The first result is given by

THEOREM 4.1. *The only zeros of the polynomial* $Q_{4n}(x)$, $(n \geq 1)$, *in* $[-1, 1]$ *are at the endpoints* $x = \pm 1$.

Proof. The polynomials $Q_{4n}(x)$ $(n \geq 0)$ are defined by the generating function

(1.3). Differentiating successively with respect to x we have

(4.1)
$$\sum_{n=1}^{\infty}\frac{Q'_{4n}(x)t^{4n}}{(4n)!}=\frac{t(\sinh xt-\sin xt)}{\cosh t+\cos t}$$
$$\sum_{n=1}^{\infty}\frac{Q''_{4n}(x)t^{4n}}{(4n)!}=\frac{t^2(\cosh xt-\cos xt)}{\cosh t+\cos t}$$
$$\sum_{n=1}^{\infty}\frac{Q'''_{4n}(x)t^{4n}}{(4n)!}=\frac{t^3(\sinh xt+\sin xt)}{\cosh t+\cos t}$$
$$\sum_{n=1}^{\infty}\frac{Q^{(4)}_{4n}(x)t^{4n}}{(4n)!}=\frac{t^4(\cosh t+\cos t)}{\cosh t+\cos t}=\sum_{n=1}^{\infty}\frac{Q_{4n-4}(x)t^{4n}}{(4n-4)!}.$$

Thus, if we set

$$(4n)_4=(4n)(4n-1)(4n-2)(4n-3)$$

we have

$$Q^{(4)}_{4n}(x)=(4n)_4Q_{4n-4}(x). \tag{4.2}$$

Setting $x=0$ in (4.1) yields

$$Q'_{4n}(0)=Q''_{4n}(0)=Q_{4n}(0)=0, \qquad n\geq 1. \tag{4.3}$$

By (3.7) and (4.2), $Q^{(4)}_{4n}(0)=(4n)_4Q_{4n-4}\neq 0$.

Since $Q_4(x)=x^4-1$, $Q_8(x)=(x^4-1)(x^4-69)$, the theorem is true for $n=1$ and $n=2$. Assume it is true for $n-1$ where n is even. Since $Q_{4n}(x)$ is symmetric in x, we consider only the interval [0, 1]. Suppose $Q_{4n}(x_1)=0$ where $0<x_1<1$. Since $Q_{4n}(1)=0$, $n\geq 1$, applying Rolle's Theorem we get $Q'_{4n}(x_0)=0$ for some x_0 such that $x_1<x_0<1$. From (4.3), $Q'_{4n}(x)$ has a zero of order three at $x=0$ and, since n is assumed even, $Q^{(4)}_{4n}(0)=(4n)_4Q_{4n-4}<0$ so $Q_{4n}(x)$ has a (positive) maximum at $x=0$. Therefore, by our assumption, $Q'_{4n}(x)$ has at least four zeros in the interval [0, 1]. Applying Rolle's Theorem three times, $Q'_{4n}(x)=(4n)_4Q_{4n-4}(x)$ has at least one zero in the interval (0, 1) which contradicts our inductive assumption. The case when n is odd is treated similarly. □

Since $Q_{4n}(x)=(x^4-1)P_n(x)$, $n\geq 1$ where $P_n(x)$ is a polynomial of degree $4n-4$, it is obvious that $Q_{4n}(3)$ is divisible by $80=5\cdot 4^2$ for $n\geq 1$. Lemma 4.1 proves a much stronger result, namely that $Q_{4n}(3)\equiv 0 \pmod{5\cdot 4^{n+1}}$ and that $n+1$ is the highest power of 4 contained in $Q_{4n}(3)$.

LEMMA 4.1. *For* $n\geq 1$, $Q_{4n}(3)\equiv 0 \pmod{4^{n+1}}$. *In fact*

$$Q_{4n}(3)=4^{n+1}[4^n+(-1)^{n+1}]. \tag{4.4}$$

Proof. Setting $x=3$ in (2.7) yields

$$Q_{4n}(3)=(-4)^n\sum_{k=0}^{2n}\binom{4n}{2k}(-1)^kE_{2k}(2)E_{4n-2k}(2).$$

Since $E_0(x)\equiv 1$, and $E_n(0)=0(n\geq 1)$, setting $x=1$ in (2.4) yields $E_n(2)=2(n\geq 1)$ Thus we have

$$Q_{4n}(3) = (-4)^n \sum_{k=1}^{2n-1} (-1)^k \binom{4n}{2k} (2)(2) + 2E_0(2)E_{4n}(2)$$
$$= (-1)^n 4^{n+1}\left[1+\sum_{k=1}^{2n-1}(-1)^k\binom{4n}{2k}\right]$$
$$= (-1)^n 4^{n+1}\sum_{k=0}^{2n-1}(-1)^k\binom{4n}{2k}.$$

Now, since

$$\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k} = \tfrac{1}{2}[(1+i)^{4n}+(1-i)^{4n}] = (-4)^n \tag{4.6}$$

we have

$$Q_{4n}(3) = (-1)^n 4^{n+1}[(-4)^n - 1]$$

which is equivalent to (4.5).

Some of the results in §2 appeared in the author's Doctoral Dissertation at the University of Alberta.

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