Proceedings of the Edinburgh Mathematical Society (2014) **57**, 253–267
DOI:10.1017/S0013091513000953

ELLIPTIC FIBRATIONS ON K3 SURFACES

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Dedicated to my old friend and colleague Slava Shokurov on the occasion of his 60th birthday

Abstract This paper consists mainly of a review and applications of our old results relating to the title. We discuss how many elliptic fibrations and elliptic fibrations with infinite automorphism groups (or Mordell–Weil groups) an algebraic K3 surface over an algebraically closed field can have. As examples of applications of the same ideas, we also consider K3 surfaces with exotic structures: with a finite number of non-singular rational curves, with a finite number of Enriques involutions, and with naturally arithmetic automorphism groups.

Keywords: K3 surface; elliptic fibration; Picard lattice; automorphism group

2010 *Mathematics subject classification:* Primary 14J28; 14J27; 14J50; 14C22

1. Introduction

This paper consists mainly of a review and applications of previous results of ours relating to elliptic fibrations on K3 surfaces over algebraically closed fields (see [**4**–**11**], the most important of which are our papers [**5**, **7**, **10**, **11**]).

This was the subject of our talk at the Oberwolfach workshop 'Higher dimensional elliptic fibrations' in October 2010. Elliptic fibrations are especially interesting for Fano and Calabi–Yau manifolds. Thus, it is interesting to study these fibrations in the case of K3 surfaces that are two-dimensional Calabi–Yau manifolds.

We consider algebraic K3 surfaces X over arbitrary algebraically closed fields k.

In § 2, we discuss basic results by Piatetsky-Shapiro and Shafarevich [**12**]. In particular, we discuss when a K3 surface X has an elliptic fibration.

In § 3, we discuss when a K3 surface X has an elliptic fibration with infinite automorphism group (or the Mordell–Weil group) (see [**7**]).

In § 4, we discuss our general results from [**7**, **10**, **11**] on the existence of non-zero exceptional elements of the Picard lattice with respect to the automorphism group of a K3 surface. Here, an element x of the Picard lattice S_X is called *exceptional* with respect to the automorphism group $\operatorname{Aut} X$ if its orbit $\operatorname{Aut} X(x)$ in S_X is finite. These results will provide the main tools for further applications.

In § 5 (see also § 4), we discuss how many elliptic fibrations and elliptic fibrations with infinite automorphism groups a K3 surface can have. In particular, for the Picard number $\rho(X) \geqslant 3$, we show that a K3 surface X has an infinite number of elliptic fibrations, and an infinite number of elliptic fibrations with infinite automorphism groups if it has one of them and the Picard lattice S_X is different from a finite number of exceptional Picard lattices S_X. This is mainly related to our results in [**5**, **7**, **10**, **11**].

As examples of applications of the same ideas, in § 6, we consider K3 surfaces with exotic structures: with a finite number of non-singular rational curves, with a finite number of Enriques involutions, and with naturally arithmetic automorphism groups.

2. Results by Piatetsky-Shapiro and Shafarevich on the existence of elliptic fibrations on K3 surfaces

We recall that an algebraic K3 surface X is a non-singular projective algebraic surface over an algebraically closed field k such that the canonical class $K_X = 0$ and the irregularity $q(X) = \dim H^1(X, \mathcal{O}_X) = 0$.

Furthermore, in this section, X is an algebraic K3 surface over an algebraically closed field. We denote by S_X the Picard lattice of X. It is well known that S_X is a hyperbolic (i.e. of signature $(1, \rho(X)-1)$) even integral lattice of rank $\rho(X)$, where $\rho(X) = \operatorname{rk} S_X$ is the Picard number of X. It can be an arbitrary even hyperbolic lattice of rank $\rho(X) \leqslant 22$, and it is an important invariant of X. Lattices of this kind are the primary focus of this paper.

According to Piatetsky-Shapiro and Shafarevich [**12**], elliptic fibrations on X are in one-to-one correspondence with primitive isotropic nef elements $c \in S_X$. That is, $c \neq 0$, $c^2 = 0$, $c/n \in S_X$ only for integers $n = \pm 1$, $c \cdot D \geqslant 0$ for any effective divisor D on X. For such $c \in S_X$, the complete linear system $|c|$ is one dimensional without base points, and it gives an elliptic fibration $|c| \colon X \to \mathbb{P}^1$, that is, the general fibre is an elliptic curve (for $\operatorname{char} k = 2$ or 3 it can be quasi-elliptic; see [**13**]).

The following facts were also observed in [**12**]. By the Riemann–Roch theorem for surfaces, any irreducible curve D on X with negative square has $D^2 = -2$, and it is then rational non-singular; hence $\mathbb{P}^1$. It follows that the nef cone $\mathrm{NEF}(X) \subset V^+(X) \subset S_X \otimes \mathbb{R}$ (or $\mathcal{M}(X) = \mathrm{NEF}(X)/\mathbb{R}^+ \subset \mathcal{L}(S_X) = V^+(X)/\mathbb{R}^+$) is a fundamental chamber for the reflection group $W^{(2)}(S_X) \subset O(S_X)$ generated by the 2-reflections $s_\delta \colon x \to x + (x \cdot \delta)\delta$ in elements $\delta \in S_X$ with $\delta^2 = -2$. Moreover, classes of non-singular rational curves on X are in one-to-one correspondence with elements $\delta \in S_X$, with $\delta^2 = -2$, which are perpendicular to codimension 1 faces of $\mathcal{M}(X)$ and directed outwards (see [**13**, § 3]). We denote this set by $P(\mathcal{M}(X))$. Here, $\mathbb{R}^+$ is the set of all positive real numbers, $V^+(X)$ is a half-cone of the cone $V(S_X)$ of elements of $S_X \otimes \mathbb{R}$ with positive square, and $\mathcal{L}(S_X)$ is the hyperbolic space related to S_X or X. We define by

$$A(\mathcal{M}(X)) = \{\phi \in O(S_X) \mid \phi(V^+(X)) = V^+(X) \text{ and } \phi(\mathcal{M}(X)) = \mathcal{M}(X)\}$$

the symmetry group of $\mathcal{M}(X)$; then, $\{\pm 1\} W^{(2)}(S_X) \rtimes A(\mathcal{M}(X)) = O(S_X)$ is the semi-direct product. By the theory of arithmetic groups (or integral quadratic forms theory), the fundamental domain for $O(S_X)$ is the same as the fundamental domain for $A(\mathcal{M}(X))$ in $\mathcal{M}(X)$. In particular, this fundamental domain is a finite rational polyhedron.

It follows that *there exist only a finite number of elliptic pencils on X up to the action of $A(\mathcal{M}(X))$*. Similarly, *there exist only a finite number of non-singular rational curves on X up to the action of $A(\mathcal{M}(X))$*. Moreover, for any isotropic element $c' \in S_X$, there exists $w \in W^{(2)}(S_X)$ such that $\pm w(c')$ is nef. Thus, *X has an elliptic fibration if and only if the Picard lattice S_X represents 0: there exists $0 \neq x \in S_X$ with $x^2 = 0$*. In particular, this is valid if $\rho(X) \geqslant 5$.

The fundamental result of [**12**], which follows from the Global Torelli theorem for K3 surfaces (also proved in [**12**]), is that the action of $\operatorname{Aut} X$ in S_X has only a finite kernel (see also [**13**] if $\operatorname{char} k > 0$), and for $\operatorname{char} k = 0$ it gives a finite index subgroup in $A(\mathcal{M}(X))$. In particular, for $\operatorname{char} k = 0$, up to finite groups, we have the natural isomorphisms of groups

$$\operatorname{Aut} X \approx A(\mathcal{M}(X)) \cong O^+(S_X)/W^{(2)}(S_X),$$

where $O^+(S_X) = \{\phi \in O(S_X) \mid \phi(V^+(X)) = V^+(X)\}$ is the subgroup of $O(S_X)$ of index 2. It follows that, *for $\operatorname{char} k = 0$, a K3 surface X has only a finite number of elliptic fibrations and non-singular rational curves up to the action of the automorphism group* $\operatorname{Aut} X$. This is the same as for all nef elements $h \in S_X$ with a fixed positive square $h^2 > 0$.

3. The existence of elliptic fibrations with infinite automorphism groups on K3 surfaces

Furthermore, X is a K3 surface over an algebraically closed field k.

Let $c \in S_X$ be a primitive isotropic nef element. By the theory of elliptic surfaces, see, for example, [**14**, Chapter VII] (or by the Global Torelli theorem for K3 surfaces, if $\operatorname{char} k = 0$), the group $\operatorname{Aut}(c)$ of automorphisms of the elliptic fibration $|c| \colon X \to \mathbb{P}^1$ is, up to finite index, the abelian group $\mathbb{Z}^{r(c)}$, where

$$r(c) = \operatorname{rk} c^{\perp} - \operatorname{rk}(c^{\perp})^{(2)}. \tag{3.1}$$

Here $c^{\perp}$ is the orthogonal complement to c in S_X (obviously, $\operatorname{rk} c^{\perp} = \rho(X) - 1$), and the sublattice $(c^{\perp})^{(2)} \subset c^{\perp}$ is generated by c and by all elements with square (-2) in $c^{\perp}$. Equivalently, $(c^{\perp})^{(2)}$ is generated by all irreducible components of fibres of $|c| \colon X \to \mathbb{P}^1$. In particular, $\operatorname{Aut}(c)$ is finite if and only if either $\rho(X) = 2$, or $c^{\perp}$ is generated by c and by elements with square (-2), up to finite index. Up to finite index, $\operatorname{Aut}(c)$ is the same as the Mordell–Weil group of the elliptic fibration c when we consider only automorphisms from $\operatorname{Aut}(c)$ that act trivially on the base $\mathbb{P}^1$.

We then ask the following interesting question.

When does X have elliptic fibrations with infinite automorphism groups?

This is important, for example, for studying the dynamics of $\operatorname{Aut} X$ (see, for example, [**1**]) and the arithmetic of X.

The main obstruction to the existence of the fibrations in question is the finiteness of the automorphism group $\mathrm{Aut}\, X$ of X. Indeed, if $\mathrm{Aut}\, X$ is finite, then the automorphism groups of all elliptic fibrations c on X are also finite, since $\mathrm{Aut}(c) \subset \mathrm{Aut}\, X$.

Surprisingly, for $\rho(X) \geqslant 6$, this obstruction is sufficient and necessary according to [**7**], and is valid for k of any characteristic. These results can be formulated for arbitrary hyperbolic lattices S if one fixes a fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ for $W^{(2)}(S)$ and considers the fundamental primitive isotropic elements $c \in S$, that is, $\mathbb{R}^+ c \in \bar{\mathcal{M}}$. Instead of $\mathrm{Aut}\, X$, one should consider the symmetry group $A(\mathcal{M}) \subset O^+(S)$ or $O^+(S)/W^{(2)}(S)$.

By (3.1), all elliptic fibrations on X have finite automorphism groups if and only if the hyperbolic lattice $S = S_X$ satisfies the property

$$\mathrm{rk}(c^\perp) = \mathrm{rk}(c^\perp)^{(2)} \quad \text{for any isotropic } c \in S. \tag{3.2}$$

We have the following results from [**7**].

Theorem 3.1. *Let S be an even hyperbolic lattice of rank $\rho = \mathrm{rk}\, S \geqslant 6$ (respectively, X is a K3 surface over an algebraically closed field, and $\rho(X) \geqslant 6$). The following three conditions are then equivalent.*

(a) *S satisfies (3.2) (respectively, the automorphism groups of all elliptic fibrations on X are finite).*

(b) *The group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite (respectively, $\mathrm{Aut}\, X$ is finite).*

(c) *The lattice S belongs to the finite list of even hyperbolic lattices below, found in [**7**] (respectively, $S = S_X$ is one of the lattices from the list).*

The list of lattices found in [**7**] is the following. (We use notation from [**4, 7**], which is now standard: $\oplus$ is the orthogonal sum of lattices; U is the even unimodular lattice of signature $(1,1)$; A_n, D_m and E_k are negative definite root lattices corresponding to the root systems $\mathbb{A}_n$, $\mathbb{D}_m$ and $\mathbb{E}_k$, respectively; $S(\lambda)$ is obtained from a lattice S by multiplication of its form by $\lambda \in \mathbb{Z}$; and $\langle A\rangle$ is a lattice with the matrix A in some basis.)

*The list of all even hyperbolic lattices S with $[O(S)\colon W^{(2)}(S)] < \infty$ and $\mathrm{rk}\, S \geqslant 6$ (see [**7**]) is*

$$\begin{aligned}
S = {} & U \oplus 2E_8 \oplus A_1; U \oplus 2E_8; U \oplus E_8 \oplus E_7; U \oplus E_8 \oplus D_6; U \oplus E_8 \oplus D_4 \oplus A_1;\\
& U \oplus E_8 \oplus D_4, U \oplus D_8 \oplus D_4, U \oplus E_8 \oplus 4A_1;\\
& U \oplus E_8 \oplus 3A_1, U \oplus D_8 \oplus 3A_1, U \oplus A_3 \oplus E_8;\\
& U \oplus E_8 \oplus 2A_1, U \oplus D_8 \oplus 2A_1, U \oplus D_4 \oplus D_4 \oplus 2A_1, U \oplus A_2 \oplus E_8;\\
& U \oplus E_8 \oplus A_1, U \oplus D_8 \oplus A_1, U \oplus D_4 \oplus D_4 \oplus A_1, U \oplus D_4 \oplus 5A_1;\\
& U \oplus E_8, U \oplus D_8, U \oplus E_7 \oplus A_1, U \oplus D_4 \oplus D_4,\\
& \quad U \oplus D_6 \oplus 2A_1, U(2) \oplus D_4 \oplus D_4, U \oplus D_4 \oplus 4A_1, U \oplus 8A_1, U \oplus A_2 \oplus E_6;\\
& U \oplus E_7, U \oplus D_6 \oplus A_1, U \oplus D_4 \oplus 3A_1, U \oplus 7A_1, U(2) \oplus 7A_1,\\
& \quad U \oplus A_7, U \oplus A_3 \oplus D_4, U \oplus A_2 \oplus D_5, U \oplus D_7, U \oplus A_1 \oplus E_6;
\end{aligned}$$

$U\oplus D_6, U\oplus D_4\oplus 2A_1, U\oplus 6A_1, U(2)\oplus 6A_1, U\oplus 3A_2, U\oplus 2A_3,$
$U\oplus A_2\oplus A_4, U\oplus A_1\oplus A_5, U\oplus A_6, U\oplus A_2\oplus D_4, U\oplus A_1\oplus D_5, U\oplus E_6;$
$U\oplus D_4\oplus A_1, U\oplus 5A_1, U(2)\oplus 5A_1, U\oplus A_1\oplus 2A_2,$
$U\oplus 2A_1\oplus A_3, U\oplus A_2\oplus A_3, U\oplus A_1\oplus A_4, U\oplus A_5, U\oplus D_5;$
$U\oplus D_4, U(2)\oplus D_4, U\oplus 4A_1, U(2)\oplus 4A_1, U\oplus 2A_1\oplus A_2,$
$U\oplus 2A_2, U\oplus A_1\oplus A_3, U\oplus A_4, U(4)\oplus D_4, U(3)\oplus 2A_2.$

Thus, a K3 surface X over an algebraically closed field, and with $\rho(X)\geqslant 6$, has an elliptic fibration with infinite automorphism group if and only if its Picard lattice S_X is different from each lattice of this finite list. If the Picard lattice S_X of X is one of the lattices from the list, then not only are the automorphism groups of all elliptic fibrations on X finite, but the full automorphism group $\operatorname{Aut} X$ is also finite.

If $\operatorname{rk} S=5$, then a similar theorem is valid if one excludes two infinite series of even hyperbolic lattices (see [**7**]).

Theorem 3.2. *Let S be an even hyperbolic lattice of rank $\operatorname{rk} S=5$, let S be different from the lattices $\langle 2^m\rangle\oplus D_4$, $m\geqslant 5$, and $\langle 2\cdot 3^{2n-1}\rangle\oplus 2A_2$, $n\geqslant 2$ (respectively, a K3 surface X over an algebraically closed field has $\rho(X)=5$ and S_X different from the lattices of these two series).*

The following three conditions are then equivalent.

(a) *S satisfies (3.2) (respectively, automorphism groups of all elliptic fibrations on X are finite).*

(b) *The group $A(\mathcal{M})\cong O^+(S)/W^{(2)}(S)$ is finite (respectively, $\operatorname{Aut} X$ is finite).*

(c) *The lattice S belongs to the finite list of even hyperbolic lattices of rank 5 below, found in [**7**] (respectively, S_X is one of the lattices from this list).*

If S is one of the lattices $\langle 2^m\rangle\oplus D_4$, $m\geqslant 5$, and $\langle 2\cdot 3^{2n-1}\rangle\oplus 2A_2$, $n\geqslant 2$, then S satisfies (3.2), but the group $A(\mathcal{M})\cong O^+(S)/W^{(2)}(S)$ is infinite (equivalently, if S_X is one of the lattices from these two series, then all elliptic fibrations on X have finite automorphism groups, but $\operatorname{Aut} X$ is infinite if $\operatorname{char} k=0$).

The list of lattices of rank 5 found in [**7**] is as follows.

*The list of all even hyperbolic lattices S with $[O(S)\colon W^{(2)}(S)]<\infty$ and $\operatorname{rk} S=5$ (see [**7**]) is*

$$S=U\oplus 3A_1, U(2)\oplus 3A_1, U\oplus A_1\oplus A_2, U\oplus A_3, U(4)\oplus 3A_1, \langle 2^k\rangle\oplus D_4,\quad k=2,3,4, \langle 6\rangle\oplus 2A_2.$$

Thus, a K3 surface X, with $\rho(X)=5$ and any $\operatorname{char} k$, has elliptic fibrations with infinite automorphism groups if and only if its Picard lattice S_X is different from each lattice of this finite list and from the lattices of the two infinite series in Theorem 3.2. If the Picard lattice of X is one of the lattices from the finite list, then not only are the automorphism groups of all elliptic fibrations on X finite, but the full automorphism group $\operatorname{Aut} X$ is also finite. If the Picard lattice of X is one of the lattices from the two infinite series of lattices of Theorem 3.2, then the automorphism groups of all elliptic fibrations on X are finite, but $\operatorname{Aut} X$ is infinite if $\operatorname{char} k = 0$ (if $\operatorname{char} k > 0$, it is not known).

If the Picard number $\rho(X) = 4$ or 3, no results, similar to those of Theorems 3.1 and 3.2, are known, except the results that we cite at the end of this section.

If $\rho(X) = 2$, then the automorphism groups of all elliptic fibrations on X are evidently finite. If $\rho(X) = 1$, then X has no elliptic fibrations.

In particular, Theorems 3.1 and 3.2 describe all even hyperbolic lattices S having the finite group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ (they are called elliptically 2-reflective) of rank $\rho = \operatorname{rk} S \geqslant 5$. A similar finite description of elliptically 2-reflective even hyperbolic lattices was obtained for $\rho = 4$ (14 lattices) in [**18**] (see also [**9**]), and for $\rho = 3$ (26 lattices) in [**8**]. (As a correction to the list of lattices in [**8**], the lattices $S'_{6,1,2}$ and $S_{6,1,1}$ are isomorphic.) Finiteness was also generalized to arbitrary arithmetic hyperbolic reflection groups and the corresponding reflective hyperbolic lattices over rings of integers of totally real algebraic number fields (see [**5**, **6**, **9**, **16**] and also [**17**]).

4. Elliptic fibrations with infinite automorphism groups and exceptional elements in Picard lattices for K3 surfaces

In what follows, X is a K3 surface over an algebraically closed field.

We consider the following general notion. For a hyperbolic lattice S and a subgroup $G \subset O(S)$, we call $x \in S$ exceptional with respect to G if its stabilizer subgroup G_x has finite index in G; equivalently, the orbit $G(x)$ is finite. All exceptional elements with respect to G define the exceptional sublattice $E \subset S$ with respect to G. Since S is hyperbolic, logically the following four cases are possible.

(i) *Elliptic type of* G. The exceptional sublattice E for G is hyperbolic. Obviously, G is then finite and $E = S$. Then E is called *hyperbolic*.

(ii) *Parabolic type of* G. The exceptional sublattice E for G is semi-negative definite and has a one-dimensional kernel. Then E is called *parabolic*.

(iii) *Hyperbolic type of* G. The exceptional sublattice E for G is negative definite. Then E is called *elliptic*.

(iv) *General hyperbolic type of* G. The exceptional sublattice E for G is 0.

Replacing G by the action of $\operatorname{Aut} X$ in S_X, we obtain the following main definition. An element of the Picard lattice $x \in S_X$ is called *exceptional (with respect to* $\operatorname{Aut} X$*)* if its stabilizer subgroup $(\operatorname{Aut} X)_x$ has finite index in $\operatorname{Aut} X$; equivalently, the orbit $(\operatorname{Aut} X)(x)$ of x is finite.

All exceptional elements of S_X define a primitive sublattice $E(S_X)$. We call it *the exceptional sublattice of the Picard lattice* (for $\operatorname{Aut} X$). This sublattice was introduced in [**7**] (therein, it was denoted by $R(S_X)$), and the results that we discuss below were mentioned, and in fact proved, in [**7**, **10**, **11**] (see [**11**, § 3]). We simply give more details below.

We assume that X has at least one elliptic fibration c with infinite automorphism group. We then have the following statement, where for a sublattice $F \subset S_X$ we denote by $F_{\mathrm{pr}} \subset S_X$ the primitive sublattice $F_{\mathrm{pr}} = S_X \cap (F \otimes \mathbb{Q}) \subset S_X \otimes \mathbb{Q}$ generated by F.

Theorem 4.1. *Let X be a K3 surface over an algebraically closed field that has at least one elliptic fibration with infinite automorphism group.*

The exceptional sublattice $E(S_X)$ is then equal to

$$E(S_X) = \bigcap_c (c^{\perp})^{(2)}_{\mathrm{pr}}, \tag{4.1}$$

where c runs through all elliptic fibrations on X with infinite automorphism groups (or the Mordell–Weil groups).

In particular, two exceptional sublattices of S_X, for $\operatorname{Aut} X$ and for the subgroup of $\operatorname{Aut} X$ generated by the Mordell–Weil groups of all elliptic fibrations with infinite automorphism groups on X, coincide.

Proof. Simple calculations, using the theory of elliptic surfaces (see [**14**, Chapter VII]), show (see [**7**]) that exceptional elements for $(\operatorname{Aut} X)_c$ (equivalently, for the Mordell–Weil group of the elliptic fibration $|c|$) in S_X define the sublattice $(c^{\perp})^{(2)}_{\mathrm{pr}}$. It follows that $E(S_X) \subset \operatorname{Ell}(S_X)$, where $\operatorname{Ell}(S_X)$ is the right-hand side of (4.1).

Since X has at least one elliptic fibration with infinite automorphism group and S_X is hyperbolic, $\operatorname{Ell}(S_X)$ is either semi-negative definite with a one-dimensional kernel $\mathbb{Z}c$ (that is, $\operatorname{Ell}(S_X)$ is parabolic), when X has only one elliptic fibration c with infinite automorphism group, or $\operatorname{Ell}(S_X)$ is negative definite (that is, $\operatorname{Ell}(S_X)$ is elliptic), when X has more than one elliptic fibration with infinite automorphism groups. In both cases, $\operatorname{Aut} X$ gives the finite action on $\operatorname{Ell}(S_X)$. It follows that $\operatorname{Ell}(S_X) \subset E(S_X)$. Thus, $E(S_X) = \operatorname{Ell}(S_X)$.

This completes the proof. □

As above, for an abstract hyperbolic lattice S (replacing S_X), a fundamental chamber $\mathcal{M} \subset \mathcal{L}(S)$ for the reflection group $W^{(2)}(S)$ (replacing $\mathcal{M}(X) = \operatorname{NEF}(X)/\mathbb{R}^+ \subset \mathcal{L}(S_X)$), and for the group $A(\mathcal{M})$ of symmetries of $\mathcal{M}$ (replacing $\operatorname{Aut} X$), we can similarly consider the exceptional elements $x \in S$ for $A(\mathcal{M})$ and the sublattice $E(S) \subset S$ of all exceptional elements for $A(\mathcal{M})$. For a fundamental primitive isotropic element $c \in S$ for $\mathcal{M}$ (replacing an elliptic fibration of X), we can similarly consider the stabilizer subgroup $A(\mathcal{M})_c \subset A(\mathcal{M})$ (replacing the automorphism group $\operatorname{Aut}(c)$ of the elliptic fibration c on X). As for K3 surfaces, we have isomorphism up to finite groups:

$$A(\mathcal{M})_c \approx \mathbb{Z}^{r(c)}, \quad r(c) = \operatorname{rk} c^{\perp} - \operatorname{rk}(c^{\perp})^{(2)}. \tag{4.2}$$

Using (4.2), exactly the same considerations as for Theorem 4.1 give a similar result for arbitrary hyperbolic lattices.

Theorem 4.2. *Let S be a hyperbolic lattice over $\mathbb{Z}$, let $\mathcal{M} \subset \mathcal{L}(S)$ be a fundamental chamber for $W^{(2)}(S)$ and let $A(\mathcal{M}) \subset O^+(S)$ be its symmetry group. We assume that S has at least one fundamental primitive isotropic element c with infinite stabilizer subgroup $A(\mathcal{M})_c$.*

The exceptional sublattice $E(S)$ for $A(\mathcal{M})$ is then equal to

$$E(S) = \bigcap_c (c^\perp)^{(2)}_{\mathrm{pr}}, \tag{4.3}$$

where c runs through all fundamental primitive isotropic elements c for $\mathcal{M}$ with infinite stabilizer subgroups $A(\mathcal{M})_c$.

For the K3 surfaces X and S_X, we take $\mathcal{M} = \mathcal{M}(X) = \mathrm{NEF}(X)/\mathbb{R}^+$. By [**12**], the fundamental primitive isotropic elements for $\mathcal{M}$ and the elliptic fibrations on X give the same set. The right-hand sides of (4.1) and (4.3) give the same result. Thus, we obtain the following result, which shows that the calculations of exceptional sublattices of S_X for the geometric group $\operatorname{Aut} X$ and for the lattice-theoretic group $A(\mathcal{M}(X))$ give the same result.

Theorem 4.3. *Let X be a K3 surface over an algebraically closed field, having at least one elliptic fibration with infinite automorphism group.*

The exceptional sublattices $E(S_X) \subset S_X$ for $\operatorname{Aut} X$ and for $A(\mathcal{M}(X))$ are then equal.

We have the following general result obtained in [**7**, **8**, **10**, **11**, **18**].

Theorem 4.4. *For each fixed $\rho \geqslant 3$, the number of hyperbolic lattices S of rank ρ having non-zero exceptional sublattices $E(S) \subset S$ for $A(\mathcal{M})$ is finite.*

Proof. For hyperbolic $E(S)$ (equivalently, when $A(\mathcal{M})$ is finite), finiteness was proved in [**7**, **8**, **18**] (discussed in § 3). The full list of such hyperbolic lattices S is known.

For parabolic $E(S)$, finiteness was proved in [**10**], but the list of such hyperbolic lattices S is not known.

For elliptic $E(S) \neq \{0\}$, finiteness was proved in [**11**], but the list of such hyperbolic lattices S is not known. □

Since $\operatorname{rk} S_X \leqslant 22$ for K3 surfaces, using Theorem 4.4, we can introduce the following finite set of even hyperbolic lattices S of $3 \leqslant \operatorname{rk} S \leqslant 22$.

Definition 4.5. $\mathcal{S}$EK3 is the set of all even hyperbolic lattices S such that $\operatorname{rk} S \geqslant 3$, $E(S) \neq 0$ for $A(\mathcal{M})$, where $\mathcal{M} \subset \mathcal{L}(S)$ is a fundamental chamber for $W^{(2)}(S)$, and S is isomorphic to the Picard lattice of some K3 surface X over an algebraically closed field. By Theorem 4.4, the set $\mathcal{S}$EK3 is finite.

We denote by $\mathcal{S}\mathrm{EK3}_\mathrm{e}$, $\mathcal{S}\mathrm{EK3}_\mathrm{p}$ and $\mathcal{S}\mathrm{EK3}_\mathrm{h}$ subsets of $\mathcal{S}$EK3 corresponding to $A(\mathcal{M})$ of elliptic type (i.e. finite), parabolic type and hyperbolic type (equivalently, $E(S) = S$, $E(S)$ is semi-negative definite and has a one-dimensional kernel, $E(S) \neq 0$ is negative definite), respectively.

Combining Theorems 4.1–4.4, we obtain the main results.

Theorem 4.6. *Let X be a K3 surface over an algebraically closed field, let $\rho(X) \geqslant 3$ and let X have an elliptic fibration with infinite automorphism group. Assume that S_X is different from lattices belonging to the finite set $\mathcal{S}$EK3.*

The exceptional sublattice $E(S_X) \subset S_X$ for $\operatorname{Aut} X$ is then equal to 0.

Moreover, the exceptional sublattice $E(S_X) \subset S_X$ is equal to 0 *for the subgroup of* $\operatorname{Aut} X$ *generated by automorphism groups of all elliptic fibrations on X with infinite automorphism groups (or by their Mordell–Weil groups).*

Moreover, we have the equality

$$\bigcap_c (c^\perp)^{(2)}_{\mathrm{pr}} = E(S_X) = \{0\}, \tag{4.4}$$

where c runs through all elliptic fibrations on X with infinite automorphism groups.

This theorem shows that, except for a finite number of Picard lattices from $\mathcal{S}$EK3, a K3 surface X has many elliptic fibrations with infinite automorphism groups if it has one of them: (4.4) quantifies exactly how many. We also discuss directly the number of elliptic fibrations in the next section.

It would be interesting to find the finite set of Picard lattices $\mathcal{S}$EK3 of K3 surfaces. Only its subset $\mathcal{S}\mathrm{EK3}_{\mathrm{e}}$ is known.

For $\rho(X) = 1$, the exceptional sublattice is equal to S_X. For $\rho(X) = 2$, the exceptional sublattice is equal to S_X if X has an elliptic fibration. Indeed, in both these cases, $\operatorname{Aut} X$ is finite since $O(S_X)$ is finite (this was observed in [**12**]). Thus, only the case of $\rho(X) \geqslant 3$, which we considered above, is interesting.

5. The number of elliptic fibrations and elliptic fibrations with infinite automorphism groups on K3 surfaces

Using Theorem 4.6, we obtain the following results, which show that, for $\rho(X) \geqslant 3$, the K3 surface X has an infinite number of elliptic fibrations, and an infinite number of elliptic fibrations with infinite automorphism groups if it has one of them, if S_X is different from a finite number of exceptional Picard lattices.

Theorem 5.1. *Let X be a K3 surface over an algebraically closed field, let $\rho(X) \geqslant 3$ and let X have at least one elliptic fibration.*

Then, X has an infinite number of elliptic fibrations if S_X is different from the finite set of Picard lattices S_X when the number of elliptic fibrations is finite. This holds for the following cases.

- $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{e}}$*; in particular,* $\operatorname{Aut} X$ *is finite.*
- $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{p}}$ *and X has only one elliptic fibration; in particular, X has one elliptic fibration with an infinite automorphism group, and no other elliptic fibrations.*

Proof. We assume that X has a finite number of elliptic fibrations. Then, $\mathcal{M}(X)$ has only a finite number of fundamental primitive isotropic elements, which are all exceptional for $A(\mathcal{M}(X))$. It follows that $E(S_X) \neq 0$ and $S_X \in \mathcal{S}$EK3. We consider two cases.

Case 1. We assume that all elliptic fibrations on X have finite automorphism groups (equivalently, fundamental primitive isotropic elements c for $\mathcal{M}(X)$ have finite stabilizer subgroups $A(\mathcal{M}(X))_c$). Then, $A(\mathcal{M}(X))$ is finite and $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{e}}$. Conversely, if $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{e}}$, then $\mathcal{M}(X)$ is a fundamental chamber for the arithmetic group $W^{(2)}(S_X)$ in $\mathcal{L}(S_X)$, and it has only a finite number of fundamental primitive isotropic elements. Thus, X has only a finite number of elliptic fibrations.

Case 2. We assume that X has an elliptic fibration c with the infinite automorphism group $\mathrm{Aut}(c)$. Since the number of elliptic fibrations on X is finite, all of them are exceptional for $\mathrm{Aut}\, X$, and $E(S_X)$ is not 0. Since $\mathrm{Aut}\, X$ is infinite, $E(S_X)$ cannot be hyperbolic. Since elliptic fibrations give isotropic elements, $E(S_X)$ cannot be elliptic (i.e. negative definite). Thus, $E(S_X)$ is parabolic and $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{p}}$. By Theorem 4.1, we have that

$$E(S_X) = \bigcap_c (c^\perp)^{(2)}_{\mathrm{pr}},$$

where c runs through all elliptic fibrations on X with infinite automorphism groups. Since $E(S_X)$ is parabolic, it follows that X has only one elliptic fibration c with infinite automorphism group and $\mathrm{Aut}\, X = \mathrm{Aut}(c)$. Since all elliptic fibrations on X are exceptional for $\mathrm{Aut}\, X$, they must also have infinite automorphism groups, and they must be equal to c. Thus, X has only one elliptic fibration c.

Conversely, we assume that $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{p}}$ and that X has only one elliptic fibration. Then, $E(S_X) \neq S_X$ and $\mathrm{Aut}\, X$ is infinite. Since X has only one elliptic fibration c, $\mathrm{Aut}\, X = \mathrm{Aut}(c)$ is its automorphism group, which is infinite.

This completes the proof. □

Theorem 5.2. *Let X be a K3 surface over an algebraically closed field, let $\rho(X) \geqslant 3$ and let X have at least one elliptic fibration with infinite automorphism group.*

Then, X has an infinite number of elliptic fibrations with infinite automorphism groups if S_X is different from the finite set of Picard lattices S_X when the number of elliptic fibrations on X with infinite automorphism groups is finite:

- $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{p}}$*; in particular, X has only one elliptic fibration with infinite automorphism group.*

Proof. If the number of elliptic fibrations on X with infinite automorphism groups is finite, all of them are exceptional for $\mathrm{Aut}\, X$, and $E(S_X)$ is not trivial. Then, $S_X \in \mathcal{S}\mathrm{EK3}$, which is finite by Theorem 4.6. Since each of these elliptic fibrations is exceptional for $\mathrm{Aut}\, X$ and corresponds to an isotropic element, $E(S_X)$ cannot be elliptic (that is, negative definite). Since $\mathrm{Aut}\, X$ is infinite, $E(S_X)$ cannot be hyperbolic either. Thus, $E(S_X)$ is parabolic and has a one-dimensional kernel. By Theorem 4.1,

$$E(S_X) = \bigcap_c (c^\perp)^{(2)}_{\mathrm{pr}},$$

where c runs through all elliptic fibrations on X with infinite automorphism groups. Since $E(S_X)$ is parabolic, it follows that X has only one elliptic fibration c with infinite automorphism group.

Conversely, if X has only one elliptic fibration c with infinite automorphism group, then $E(S_X) = (c^{\perp})^{(2)}_{\mathrm{pr}}$ is parabolic and $S_X \in \mathcal{S}\mathrm{EK3_p}$.

This completes the proof. □

If $\rho(X) = 1$ or 2, then X has two elliptic fibrations or fewer, and these cases are trivial.

6. Applications to K3 surfaces with exotic structures

Here, we give some other applications of finiteness of the set of Picard lattices of K3 surfaces with the non-trivial exceptional sublattice $E(S_X)$, and elliptic fibrations with an infinite automorphism group.

6.1. K3 surfaces with a finite number of non-singular rational curves

Recently, Matsushita asked me what we can say about K3 surfaces with a finite number of non-singular rational (equivalently, irreducible) (-2)-curves. We have the following theorem.

Theorem 6.1. *A K3 surface X over an algebraically closed field has no non-singular rational curves if and only if its Picard lattice S_X has no elements with square (-2).*

If a K3 surface X over an algebraically closed field has non-singular rational curves (equivalently, its Picard lattice S_X has elements with square (-2)), then their number is finite in only the following cases (1) and (2):

(1) $\rho(X) = 2$,

(2) $\rho(X) \geqslant 3$ *and the Picard lattice S_X is elliptically 2-reflective:* $[O(S_X) : W^{(2)}(S_X)] < \infty$. *The number of elliptically 2-reflective hyperbolic lattices is finite, and they are enumerated in [**7**, **8**, **18**] (for $\rho(X) \geqslant 5$, they are listed in § 3).*

Proof. Let $S = S_X$ and let $\mathcal{M} = \mathcal{M}(X)$ be the fundamental chamber for $W^{(2)}(S)$. The non-singular rational curves on X are then in one-to-one correspondence with elements of the set $P(\mathcal{M})$ of perpendicular vectors to $\mathcal{M}$ with square (-2) and directed outwards.

If S has no elements with square (-2), then $P(\mathcal{M})$ is empty and X has no non-singular rational curves.

We assume that S has an element δ with $\delta^2 = -2$. Then, $\pm w(\delta)$ gives one of the elements of $P(\mathcal{M})$ for some $w \in W^{(2)}(S)$, the set $P(\mathcal{M})$ is not empty, and X contains a non-singular rational curve.

Since S is hyperbolic and S has elements with square (-2), $\rho(X) = \mathrm{rk}\, S \geqslant 2$.

Let $\rho(X) = 2$. Then, $\mathcal{M} = V^{+}(S)/\mathbb{R}^{+}$ is an interval, and elements of $P(\mathcal{M})$ correspond to terminals of this interval. Thus, $P(\mathcal{M})$ has not more than two elements, and the number of non-singular rational curves on X is one or two.

Let $\rho(X) \geqslant 3$, and let $P(\mathcal{M})$ be non-empty and finite. All elements of $P(\mathcal{M})$ are then exceptional for the symmetry group $A(\mathcal{M}) \subset O^{+}(S)$ of $\mathcal{M}$, and the exceptional sublattice $E(S)$ is not 0. Then, by Theorem 4.4 (from [**7**, **8**, **10**, **11**, **18**]), S is one of a finite number of hyperbolic lattices of rank less than or equal to 22.

Actually, the main idea of the proof of this theorem in [**8**, **10**, **11**] is that $P(\mathcal{M})$ has the elements $\delta_1, \dots \delta_\rho \in S$, $\rho = \operatorname{rk} S$, which generate $S \otimes \mathbb{Q}$ and $\delta_i \cdot \delta_j \leqslant 19$, $1 \leqslant i < j \leqslant \rho$. (These elements define a narrow part of $\mathcal{M}$.) It follows that $P(\mathcal{M})$ generates $S \otimes \mathbb{Q}$, the group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite since $P(\mathcal{M})$ is finite, the lattice S is elliptically 2-reflective, $[O(S)\colon W^{(2)}(S)] < \infty$, and the number of such lattices is finite.

This completes the proof. □

6.2. K3 surfaces with a finite number of Enriques involutions

Here, we restrict our study to basic fields k of $\operatorname{char} k \neq 2$.

We recall that an involution σ on a K3 surface X over an algebraically closed field k of $\operatorname{char} k \neq 2$ is called an *Enriques involution* if σ has no fixed points on X. Then $X/\{id, \sigma\}$ is an Enriques surface (see [**2**]). It is well known (see [**2**]) that σ in S_X has the eigenvalue 1 part that is isomorphic to the standard hyperbolic lattice $S_X^\sigma \cong U(2) \oplus E_8(2)$ of rank 10. A general K3 surface with an Enriques involution has $S_X = S_X^\sigma \cong U(2) \oplus E_8(2)$, and only a finite number of Enriques involutions (if $\operatorname{char} k = 0$, only one).

We have the following result.

Theorem 6.2. *Let X be a K3 surface over an algebraically closed field k of* $\operatorname{char} k \neq 2$, *and let X have an Enriques involution.*

If X has only a finite number of Enriques involutions, then either S_X is isomorphic to $U(2) \oplus E_8(2)$, or S_X belongs to the finite set $\mathcal{S}$EK3.

In particular, if S_X is different from the lattices of these two finite sets, then X has an infinite number of Enriques involutions.

Proof. Let σ be an Enriques involution on X.

Since $S_X^\sigma \cong U(2) \oplus E_8(2)$ is a sublattice of S_X, it follows that $\rho(X) \geqslant 10$.

Let $\rho(X) = 10$. Then, $S_X = S_X^\sigma$, and σ is the identity on S_X. Since $\operatorname{Aut} X$ has only a finite kernel in S_X, it follows that X has only a finite number of Enriques involutions (only one if $\operatorname{char} k = 0$).

Let $\rho(X) > 10$. For each Enriques involution σ on X, the orthogonal complement $S_\sigma = (S_X^\sigma)^\perp$ in S_X is then a non-zero negative definite sublattice of S_X, which has a finite number of automorphisms. If X has only a finite number of Enriques involutions, then all these orthogonal complements are contained in the exceptional sublattice $E(S_X)$ of S_X for $\operatorname{Aut} X$, and $E(S_X) \neq \{0\}$. Since $\rho(X) \geqslant 10 \geqslant 6$, by Theorem 3.1 either $\operatorname{Aut} X$ is finite and X has only a finite number of Enriques involutions, and $S_X \in \mathcal{S}\mathrm{EK3}_{\mathrm{e}}$, or X has an elliptic fibration with an infinite automorphism group. By Theorem 4.3, the exceptional sublattices of S_X for $\operatorname{Aut} X$ and for $A(\mathcal{M}(X))$ are the same, and $S_X \in \mathcal{S}\mathrm{EK3}$. By Theorem 4.4 (from [**7**, **8**, **10**, **11**, **18**]), the set $\mathcal{S}$EK3 is finite.

This completes the proof. □

The method of the proof is so general that, by the same considerations, one can prove similar results for other types of involutions or automorphisms on K3 surfaces, and other structures on K3 surfaces.

6.3. K3 surfaces with naturally arithmetic automorphism groups

This is related to the recent preprint by Totaro [**15**].

Definition 6.3. Let X be a K3 surface over an algebraically closed field, and let S_X be its Picard lattice.

We say that the automorphism group $\operatorname{Aut} X$ is *naturally arithmetic* if there exists a sublattice $K \subset S_X$ such that the action of $\operatorname{Aut} X$ in S_X identifies $\operatorname{Aut} X$ as a subgroup of finite index in $O(K)$. More precisely, there exists a subgroup $G \subset \operatorname{Aut} X$ of finite index such that K is G-invariant, and the natural homomorphism $G \to O(K)$ has finite kernel and cokernel.

For example, if $\operatorname{Aut} X$ is finite, then one can take $K = \{0\} \subset S_X$, and $\operatorname{Aut} X$ is naturally arithmetic. Thus, all K3 surfaces with elliptically 2-reflective Picard lattices S_X (for $\rho(X) \geqslant 5$, see the list in § 3) have naturally arithmetic automorphism groups.

We have the following result, which uses the Global Torelli theorem for K3 surfaces [**12**], and it is valid over $\mathbb{C}$ (or over an algebraically closed field k of $\operatorname{char} k = 0$).

Theorem 6.4. *Let X be a K3 surface over $\mathbb{C}$. Then $\operatorname{Aut} X$ is naturally arithmetic in only the following cases (1), (2) and (3).*

(1) *The Picard lattice S_X has no elements with square (-2).*

(2) *S_X has elements with square (-2) and $\rho(X) = 2$.*

(3) *S_X has elements with square (-2), $\rho(X) \geqslant 3$, and S_X is one of the lattices from the subset (described in the proof) of the finite set $\mathcal{S}$EK3.*

In particular, if $\rho(X) \geqslant 3$ and S_X has elements with square (-2), then $\operatorname{Aut} X$ is not naturally arithmetic, except for a finite number of Picard lattices S_X.

Proof. We identify $\operatorname{Aut} X$ with its action in S_X. By [**12**], the automorphism group $\operatorname{Aut} X$ is a subgroup of finite index in $A(\mathcal{M})$, where $\mathcal{M} = \mathcal{M}(X)$, and $O^+(S_X) = A(\mathcal{M}) \ltimes W^{(2)}(S_X)$ is the semi-direct product. We can consider $A(\mathcal{M})$ instead of $\operatorname{Aut} X$. Thus, the natural arithmeticity of $\operatorname{Aut} X$ depends on S_X only. If S_X has no elements with square (-2), then $W^{(2)}(S_X)$ is trivial, and $A(\mathcal{M})$ and $\operatorname{Aut} X$ are naturally arithmetic (one can take $K = S_X$). We obtain the case (1).

If S_X has elements with square (-2) and $\rho(X) = 2$, then $A(\mathcal{M})$ and $\operatorname{Aut} X$ are finite, and they are naturally arithmetic (one can take $K = \{0\}$). We obtain the case (2).

We assume that S_X has elements with square (-2), $\rho(X) \geqslant 3$, and $\operatorname{Aut} X$ is naturally arithmetic for some sublattice $K \subset S_X$. We show that the exceptional sublattice $E(S_X)$ for $A(\mathcal{M})$ (or $\operatorname{Aut} X$) is then not trivial.

We assume that $W^{(2)}(S_X)$ is finite. The group $W^{(2)}(S_X)$ is generated by the reflections s_δ, where $\delta \in P(\mathcal{M})$, and all such reflections are different. It follows that $P(\mathcal{M})$ is finite and non-empty (equivalently, the number of non-singular rational curves on X is finite and non-empty). By Theorem 6.1, S_X is then elliptically 2-reflective, $[O(S_X) \colon W^{(2)}(S_X)] < \infty$, the groups $A(\mathcal{M})$ and $\operatorname{Aut} X$ are finite, and the exceptional sublattice $E(S_X) = S_X$ is not trivial.

We assume that $W^{(2)}(S_X)$ is infinite. Then $W^{(2)}(S_X)$ and $O^+(S_X)$ act transitively on an infinite number of fundamental chambers for $W^{(2)}(S_X)$ in $\mathcal{L}(S_X)$. But $A(\mathcal{M})$ sends $\mathcal{M}$ to itself. Thus, $A(\mathcal{M})$ has an infinite index in $O^+(S_X)$. It follows that $K \subset S_X$ has $\operatorname{rk} K < \operatorname{rk} S_X$, and the orthogonal complement $E = K^\perp$ in S_X is not 0.

If K is negative definite, then $A(\mathcal{M})$ and $\operatorname{Aut} X$ are finite, and $E(S_X) = S_X$ is not trivial.

If K is semi-negative definite and not negative definite, then it has a one-dimensional kernel $\mathbb{Z}c$, where c is exceptional, and $E(S_X)$ is not trivial.

If K is hyperbolic, then $E = K^\perp$ is negative definite and non-zero. It gives a non-trivial sublattice in $E(S_X)$, since E has a finite automorphism group. Thus, $E(S_X)$ is not trivial.

By Theorem 4.4, the lattice S_X is one of a finite number of hyperbolic lattices S with $3 \leqslant \operatorname{rk} S \leqslant 22$ and with non-trivial exceptional sublattice $E(S)$ for $A(\mathcal{M})$. Thus, S_X belongs to the finite set $\mathcal{S}$EK3 of hyperbolic lattices.

This completes the proof. □

Because of Theorem 6.4, the following result is important. It has been known for many years, and is a corollary of results of [**4**], but it was never published.

Theorem 6.5. *Let X be a K3 surface over $\mathbb{C}$, and let $\rho(X) = \operatorname{rk} S_X \geqslant 12$.*

Then, S_X has elements with square (-2). In particular, X contains a non-singular rational curve $\mathbb{P}^1$.

Proof. The Picard lattice $S = S_X$ is a primitive sublattice of the lattice $H^2(X, \mathbb{Z}) = L$, which is an even unimodular lattice of signature $(3, 19)$. It is unique up to isomorphisms. The transcendental lattice $T = S_L^\perp$ has rank less than or equal to 10, and $\operatorname{rk} T + \operatorname{rk}(T^*/T) \leqslant 20 = \operatorname{rk} L - 2$. By [**4**, Theorem 1.14.4], the lattice T has a unique primitive embedding into L, up to isomorphisms. Thus, for any primitive embedding $T \subset L$, we have that $(T)_L^\perp \cong S = S_X$.

On the other hand, by [**4**, Theorem 1.12.2], the lattice $T \oplus \langle -2\rangle$ has a primitive embedding into L. For this primitive embedding, $T_L^\perp$ contains a sublattice $\langle -2\rangle$. Thus, $S = S_X$ also contains a primitive sublattice $\langle -2\rangle$. Equivalently, there exists $\delta \in S_X$ with $\delta^2 = -2$.

This completes the proof. □

From Theorems 6.4 and 6.5, we obtain the following.

Corollary 6.6. *Up to isomorphisms, there exist only a finite number of Picard lattices S_X of K3 surfaces over $\mathbb{C}$ such that $\operatorname{rk} S_X = \rho(X) \geqslant 12$ and $\operatorname{Aut} X$ is naturally arithmetic.*

On the contrary, by [**4**, Theorem 1.12.4], any even hyperbolic lattice S of $\operatorname{rk} S \leqslant 11$ has a primitive embedding into an even unimodular lattice of signature $(3, 19)$. Thus, by the epimorphicity of the Torelli map for K3 surfaces (see [**3**]), the lattice S is isomorphic to the Picard lattice S_X of a K3 surface X over $\mathbb{C}$. It follows that, for each $1 \leqslant \rho \leqslant 11$, there exist an infinite number of non-isomorphic Picard lattices S_X of K3 surfaces over $\mathbb{C}$ such that $\operatorname{rk} S_X = \rho$, S_X has no elements with square (-2) and, thus, $\operatorname{Aut} X$ is naturally arithmetic.

Acknowledgements. The authors thank the referee for careful reading of the paper and important remarks.

References

1. S. Cantat, Dynamique des automorphismes des surfaces K3, *Acta Math.* **187**(1) (2001), 1–57.
2. F. R. Cossec and I. V. Dolgachev, *Enriques surfaces I*, Progress in Mathematics, Volume 76 (Birkhäuser, 1989).
3. V. S. Kulikov, Degenerations of K3 surfaces and Enriques surfaces, *Math. USSR Izv.* **11**(5) (1977), 957–989.
4. V. V. Nikulin, Integral symmetric bilinear forms and some of their geometric applications, *Math. USSR Izv.* **14**(1) (1980), 103–167.
5. V. V. Nikulin, On arithmetic groups generated by reflections in Lobachevsky spaces, *Math. USSR Izv.* **16**(3) (1981), 573–601.
6. V. V. Nikulin, On the classification of arithmetic groups generated by reflections in Lobachevsky spaces, *Math. USSR Izv.* **18**(1) (1982), 99–123.
7. V. V. Nikulin, On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections, algebraic–geometric applications, *J. Sov. Math.* **22** (1983), 1401–1476.
8. V. V. Nikulin, Surfaces of type K3 with finite automorphism group and Picard group of rank 3, *Proc. Steklov Math. Inst.* **165** (1984), 113–142.
9. V. V. Nikulin, Discrete reflection groups in Lobachevsky spaces and algebraic surfaces, in *Proceedings of the International Congress of Mathematicians, Berkeley, CA*, Volume 1, pp. 654–669 (Defense Technical Information Center, Fort Belvoir, VA, 1988).
10. V. V. Nikulin, Reflection groups in Lobachevsky spaces and the denominator identity for Lorentzian Kac–Moody algebras, *Izv. Math.* **60** (1996), 305–334.
11. V. V. Nikulin, K3 surfaces with interesting groups of automorphisms, *J. Math. Sci.* **95**(1) (1999), 2028–2048.
12. I. I. Piatetsky-Shapiro and I. R. Shafarevich, A Torelli theorem for algebraic surfaces of type K3, *Math. USSR Izv.* **5**(3) (1971), 547–588.
13. A. N. Rudakov and I. R. Shafarevich, Surfaces of type K3 over fields of finite characteristic, *J. Sov. Math.* **22** (1983), 1476–1533.
14. I. R. Shafarevich, *Algebraic surfaces*, Proceedings of the Steklov Institute of Mathematics, Volume 75, pp. 3–215 (American Mathematical Society, Providence, RI, 1965).
15. B. Totaro, Algebraic surfaces and hyperbolic geometry, eprint (arXiv:1008.3825v1, 2010).
16. É. B. Vinberg, Absence of crystallographic groups of reflection in Lobachevskii spaces of large dimension, *Funct. Analysis Applic.* **15**(2) (1981), 128–130.
17. É. B. Vinberg, Hyperbolic groups of reflections, *Russ. Math. Surv.* **40**(1) (1985), 31–75.
18. É. B. Vinberg, Classification of 2-reflective hyperbolic lattices of rank 4, *Trans. Mosc. Math. Soc.* **68** (2007), 39–66.