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ON ADDITIVITY OF CENTRALISERS

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Let R be a ring and let M be a bimodule over R. We consider the question of when a map $\varphi : R \to M$ such that $\varphi(ab) = \varphi(a)b$ for all $a, b \in R$ is additive.

1. INTRODUCTION AND PRELIMINARIES

Let R be a ring (not necessarily with an identity element) and let M be a bimodule over R. A *left centraliser* φ is a map $\varphi : R \to M$ such that $\varphi(ab) = \varphi(a)b$ for all $a, b \in R$. The notion of a right centraliser is defined analogously. We consider the question of when a left centraliser is additive.

The systematic study of centralisers was initiated by Johnson in [**4**]. Among many results presented in this pioneering paper, we emphasize automatic linearity of a left centraliser $\varphi : \mathcal{K}(X) \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the algebra of all compact operators on a Banach space X. Furthermore, in [**8**] Saworotnow and Giellis proved that each left centraliser $\varphi : A \to A$ on a semisimple complemented algebra A is linear. Thus, the aim of our paper is to generalise these results in the setting of rings. In particular, we shall see that every left centraliser $\varphi : R \to R$ is automatically additive if R is either a prime ring with a nonzero idempotent or a semiprime ring whose socle is essential. We were also motivated by similar results on additivity of isomorphisms [**5, 6, 7**] and derivations [**2**] on rings.

Let $\varphi : R \to M$ be a left centraliser. First, note that φ is additive if R has an identity element. Next, we set some notation that will be used in the sequel. Let M' be the set $\{m \in M \mid mZ(R) = 0\}$, where $Z(R)$ denotes the centre of R. Note that M' is a submodule of M. It follows easily that $\varphi(a+b) - \varphi(a) - \varphi(b) \in M'$ for all $a, b \in R$. Hence, φ is additive if $M' = 0$.

Assume that there exists a nontrivial idempotent $e_1 \in R$ (that is, $e_1^2 = e_1$ and $e_1 \neq 0, 1$). Let us remark here that for any $x \in M \cup R$ we shall write $x(1 - e_1)$ instead of $x - xe_1$ and $(1 - e_1)x$ instead of $x - e_1x$. By e_2 we denote $1 - e_1$. We set $R_{ij} = e_iRe_j$ and $M_{ij} = e_iMe_j$, $i, j \in \{1, 2\}$. Thus, R can be written in its Peirce decomposition as $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. Analogously, $M = M_{11} \oplus M_{12} \oplus M_{21} \oplus M_{22}$.

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According to the Peirce decomposition of M we have

$$\varphi(x) = \varphi_{11}(x) + \varphi_{12}(x) + \varphi_{21}(x) + \varphi_{22}(x)$$

for each $x \in R$, where $\varphi_{ij} : R \to M_{ij}$ denotes the map defined by $\varphi_{ij}(x) = e_i\varphi(x)e_j$, $i, j \in \{1,2\}$. Let $x = x_{11} + x_{12} + x_{21} + x_{22}$ and $y = y_{11} + y_{12} + y_{21} + y_{22}$ be arbitrary elements of R (by x_{ij} and y_{ij} we denote elements of R_{ij}). Then the identity $\varphi(xy) = \varphi(x)y$ yields

$$\varphi_{11}(xy) = \varphi_{11}(x)y_{11} + \varphi_{12}(x)y_{21}, \tag{1}$$

$$\varphi_{12}(xy) = \varphi_{11}(x)y_{12} + \varphi_{12}(x)y_{22}, \tag{2}$$

$$\varphi_{21}(xy) = \varphi_{21}(x)y_{11} + \varphi_{22}(x)y_{21}, \tag{3}$$

$$\varphi_{22}(xy) = \varphi_{21}(x)y_{12} + \varphi_{22}(x)y_{22}. \tag{4}$$

Note that

$$xy = (x_{11}y_{11} + x_{12}y_{21}) + (x_{11}y_{12} + x_{12}y_{22}) + (x_{21}y_{11} + x_{22}y_{21}) + (x_{21}y_{12} + x_{22}y_{22}).$$

2. The main results

Lemma 1. *Let R be a ring and let M be a bimodule over R. Further, let $e_1 \in R$ be a nontrivial idempotent such that for any $m \in M'$ the following holds:*

(A1) $e_1me_1Re_2 = 0$ *implies* $e_1me_1 = 0$,

(A2) $e_1me_2Re_1 = 0$ *implies* $e_1me_2 = 0$,

(A3) $e_1me_2Re_2 = 0$ *implies* $e_1me_2 = 0$.

Then for any left centraliser $\varphi : R \to M$ the maps φ_{11} and φ_{12} are additive.

Proof: First, let us prove that φ_{11} is additive on $R_{11} \oplus R_{12} \oplus R_{22}$ and that φ_{12} is additive on $R_{11} \oplus R_{12} \oplus R_{21}$. Obviously,

$$\begin{aligned}\varphi_{11}(x_{11} + x_{12} + x_{21} + x_{22}) &= e_1\varphi(x_{11} + x_{12} + x_{21} + x_{22})e_1 \\ &= e_1\varphi\big((x_{11} + x_{12} + x_{21} + x_{22})e_1\big)e_1 \\ &= \varphi_{11}(x_{11} + x_{21})\end{aligned} \tag{5}$$

for all $x_{ij} \in R_{ij}$. In particular,

$$\varphi_{11}(x_{11} + x_{12} + x_{22}) = \varphi_{11}(x_{11}) = \varphi_{11}(e_1)x_{11}$$

for all $x_{ij} \in R_{ij}$, which means that φ_{11} is additive on $R_{11} \oplus R_{12} \oplus R_{22}$. On the other hand, one can easily verify that

$$\big(\varphi_{12}(x_{11} + x_{12} + x_{21} + x_{22}) - \varphi_{12}(x_{12} + x_{22})\big)R = 0$$

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for all $x_{ij} \in R_{ij}$. In particular, $\varphi_{12}(x_{11}+x_{12}+x_{21}+x_{22}) - \varphi_{12}(x_{12}+x_{22}) \in M'$ and

$$\big(\varphi_{12}(x_{11}+x_{12}+x_{21}+x_{22}) - \varphi_{12}(x_{12}+x_{22})\big)R_{22} = 0$$

for all $x_{ij} \in R_{ij}$. Thus, we may apply assumption (A3), which yields that

$$\varphi_{12}(x_{11}+x_{12}+x_{21}+x_{22}) = \varphi_{12}(x_{12}+x_{22}) \tag{6}$$

for all $x_{ij} \in R_{ij}$. Consequently,

$$\varphi_{12}(x_{11}+x_{12}+x_{21}) = \varphi_{12}(x_{12}) = \varphi_{11}(e_1)x_{12}$$

for all $x_{ij} \in R_{ij}$. Thus, φ_{12} is additive on $R_{11} \oplus R_{12} \oplus R_{21}$.

Our next aim is to prove that φ_{11} is additive on R_{21} and that φ_{12} is additive on R_{22}. Using (5) and (6) we may rewrite (1) and (2) as

$$\varphi_{11}\big((x_{11}y_{11}+x_{12}y_{21}) + (x_{21}y_{11}+x_{22}y_{21})\big) = \varphi_{11}(x_{11}+x_{21})y_{11} + \varphi_{12}(x_{12}+x_{22})y_{21}, \tag{7}$$

and

$$\varphi_{12}\big((x_{11}y_{12}+x_{12}y_{22}) + (x_{21}y_{12}+x_{22}y_{22})\big) = \varphi_{11}(x_{11}+x_{21})y_{12} + \varphi_{12}(x_{12}+x_{22})y_{22} \tag{8}$$

for all $x_{ij}, y_{ij} \in R_{ij}$. Setting $x_{11} = x_{12} = 0$ in (7) we obtain

$$\varphi_{11}(x_{21}y_{11}+x_{22}y_{21}) = \varphi_{11}(x_{21})y_{11} + \varphi_{12}(x_{22})y_{21}, \tag{9}$$

which in particular implies that

$$\varphi_{11}(x_{21}y_{11}) = \varphi_{11}(x_{21})y_{11} \quad \text{and} \quad \varphi_{11}(x_{22}y_{21}) = \varphi_{12}(x_{22})y_{21} \tag{10}$$

for all $x_{ij}, y_{ij} \in R_{ij}$. Thus, (9) can also be written as

$$\varphi_{11}(x_{21}y_{11}+x_{22}y_{21}) = \varphi_{11}(x_{21}y_{11}) + \varphi_{11}(x_{22}y_{21}) \tag{11}$$

for all $x_{ij}, y_{ij} \in R_{ij}$. Replacing y_{11} by $x_{12}y_{21}$ and x_{22} by $z_{21}x_{12}$ in (11) we get

$$\varphi_{11}(x_{21}x_{12}y_{21} + z_{21}x_{12}y_{21}) = \varphi_{11}(x_{21}x_{12}y_{21}) + \varphi_{11}(z_{21}x_{12}y_{21}),$$

which according to (10) implies

$$\varphi_{11}(x_{21}+z_{21})x_{12}y_{21} = \varphi_{11}(x_{21})x_{12}y_{21} + \varphi_{11}(z_{21})x_{12}y_{21}$$

for all $x_{12} \in R_{12}$, $x_{21}, y_{21}, z_{21} \in R_{21}$. Therefore,

$$\big(\varphi_{11}(x_{21}+z_{21}) - \varphi_{11}(x_{21}) - \varphi_{11}(z_{21})\big)R_{12}R_{21} = 0$$

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for all $x_{21}, z_{21} \in R_{21}$. Using assumptions (A2) and (A1) we see that φ_{11} is additive on R_{21}, indeed. Now it follows from (10) that

$$\begin{aligned}\varphi_{12}(x_{22}+y_{22})y_{21} &= \varphi_{11}(x_{22}y_{21}+y_{22}y_{21}) \\ &= \varphi_{11}(x_{22}y_{21})+\varphi_{11}(y_{22}y_{21}) \\ &= \big(\varphi_{12}(x_{22})+\varphi_{12}(y_{22})\big)y_{21}\end{aligned}$$

and hence $\big(\varphi_{12}(x_{22}+y_{22})-\varphi_{12}(x_{22})-\varphi_{12}(y_{22})\big)R_{21}=0$ for all $x_{22}, y_{22} \in R_{22}$. Again, using assumption (A2) we see that φ_{12} is additive on R_{22}.

We are now ready to prove that φ_{11} and φ_{12} are additive on R. Note that according to the conclusions derived above it only remains to prove that $\varphi_{11}(x_{11}+x_{21})=\varphi_{11}(x_{11}) + \varphi_{11}(x_{21})$ and $\varphi_{12}(x_{12}+x_{22})=\varphi_{12}(x_{12})+\varphi_{12}(x_{22})$ for all $x_{ij} \in R_{ij}$. Setting first $y_{12}=0$ and then $y_{22}=0$ in (8), we get, respectively,

$$\varphi_{12}(x_{12}y_{22}+x_{22}y_{22})=\varphi_{12}(x_{12}+x_{22})y_{22}$$

and

$$\varphi_{12}(x_{11}y_{12}+x_{21}y_{12})=\varphi_{11}(x_{11}+x_{21})y_{12} \tag{12}$$

for all $x_{ij}, y_{ij} \in R_{ij}$. Thus, putting $x_{11}=e_1$, $x_{12}=x_{21}=0$, $y_{12}=z_{12}y_{22}$ in (8) it follows that

$$\varphi_{12}(z_{12}y_{22}+x_{22}y_{22})=\varphi_{11}(e_1)z_{12}y_{22}+\varphi_{12}(x_{22})y_{22}$$

and so

$$\varphi_{12}(z_{12}+x_{22})y_{22}=\varphi_{12}(z_{12})y_{22}+\varphi_{12}(x_{22})y_{22}$$

for all $z_{12} \in R_{12}$ and $x_{22}, y_{22} \in R_{22}$. Hence,

$$\big(\varphi_{12}(x_{12}+x_{22})-\varphi_{12}(x_{12})-\varphi_{12}(x_{22})\big)R_{22}=0,$$

which according to assumption (A3) implies $\varphi_{12}(x_{12}+x_{22})=\varphi_{12}(x_{12})+\varphi_{12}(x_{22})$ for all $x_{12} \in R_{12}$ and $x_{22} \in R_{22}$. Consequently, using (12) it follows that

$$\begin{aligned}\varphi_{11}(x_{11}+x_{21})y_{12} &= \varphi_{12}(x_{11}y_{12}+x_{21}y_{12}) \\ &= \varphi_{12}(x_{11}y_{12})+\varphi_{12}(x_{21}y_{12}) \\ &= \varphi_{11}(x_{11})y_{12}+\varphi_{11}(x_{21})y_{12}\end{aligned}$$

for all $x_{11} \in R_{11}$, $x_{21} \in R_{21}$ and $y_{12} \in R_{12}$. Thus,

$$\big(\varphi_{11}(x_{11}+x_{21})-\varphi_{11}(x_{11})-\varphi_{11}(x_{21})\big)R_{12}=0$$

and so assumption (A1) yields $\varphi_{11}(x_{11}+x_{21})=\varphi_{11}(x_{11})+\varphi_{11}(x_{21})$ for all $x_{11} \in R_{11}$ and $x_{21} \in R_{21}$. We have therefore proved that φ_{11} and φ_{12} are additive. ∎

In an analogous manner, using (3) and (4), one can obtain the following lemma.

Lemma 2. *Let R be a ring and let M be a bimodule over R. Further, let $e_1 \in R$ be a nontrivial idempotent such that for any $m \in M'$ the following holds:*

(A4) $e_2me_1Re_2 = 0$ *implies* $e_2me_1 = 0$,

(A5) $e_2me_2Re_1 = 0$ *implies* $e_2me_2 = 0$,

(A6) $e_2me_2Re_2 = 0$ *implies* $e_2me_2 = 0$.

Then for any left centraliser $\varphi : R \to M$ the maps φ_{21} and φ_{22} are additive.

Since $\varphi = \varphi_{11} + \varphi_{12} + \varphi_{21} + \varphi_{22}$, Lemma 1 and Lemma 2 imply our main result:

Theorem 3. *Let R be a ring and let M be a bimodule over R. Further, let $e_1 \in R$ be a nontrivial idempotent such that for any $m \in M'$* (A1)–(A6) hold. Then any left centraliser $\varphi : R \to M$ is additive.

Remark 4. Let A be an algebra over a field $\mathbb{F}$ and let M be a bimodule over A equipped with the structure of a vector space over $\mathbb{F}$ such that $(\lambda m)a = m(\lambda a)$ for all $\lambda \in \mathbb{F}$, $m \in M$, $a \in A$. If $mA = 0$ (where $m \in M$) implies $m = 0$, then any left centraliser $\varphi : A \to M$ is homogeneous. In particular, if there exists a nontrivial idempotent $e_1 \in A$ such that for any $m \in M'$ (A1)-(A6) hold, then any left centraliser $\varphi : A \to M$ is linear.

3. Applications

Using our main results we shall be able to prove automatic additivity of left centralisers on a certain class of semiprime rings (Corollaries 5 and 6). These results will be further applied to more concrete examples.

First, let us recall some preliminaries. A left ideal of a ring R is called *minimal* if it is nonzero and does not properly contain any nonzero left ideal of R. Let L be a minimal left ideal of R. If $a \in R$ and $La \neq 0$, then L and La are isomorphic as left R-modules, which shows that La is also a minimal left ideal of R. Consequently, the sum of all minimal left ideals of R, which is called the *left socle* of R, is an ideal of R. Analogously we introduce the right socle of R which in general does not coincide with the left socle. Recall that a left ideal L of R is said to be *dense* if given any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $rr_1 \neq 0$ and $rr_2 \in L$. One defines a dense right ideal in an analogous fashion. Let us also mention that an ideal I of R is called *essential* if for every nonzero ideal J of R we have $I \cap J \neq 0$.

Henceforth we shall assume that R is a semiprime ring. We say that a nonzero idempotent $e \in R$ is *minimal* if eRe is a divison ring. It turns out that a left ideal L of R is minimal if and only if $L = Re$ for some minimal idempotent $e \in R$ (see [**1**, Proposition 4.3.3]). Since the same holds for minimal right ideals we see that for any idempotent $e \in R$, Re is a minimal left ideal of R if and only if eR is a minimal right ideal of R. This further implies that the left socle of R coincides with the right socle of R. We call this ideal the *socle* of R and denote it by $\mathrm{soc}(R)$. If R has no minimal one-sided ideals, we define $\mathrm{soc}(R) = 0$. Let I be an ideal of R. Recall that the left, the right and the

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two-sided annihilator of I coincide. Hence, we call this ideal the annihilator of I. It is straightforward to verify that I is essential if and only if its annihilator is zero or if and only if I is a dense left (right) ideal.

We refer the reader to the book [**1**] for an account on the theory of various rings of quotients. Let us just recall here that any semiprime ring R can be considered as a subring of both its *symmetric Martindale ring of quotients* $Q_s = Q_s(R)$, and its *maximal left ring of quotients* $Q_{ml} = Q_{ml}(R)$. Both of these rings have an identity element, they are semiprime (or prime if R is prime), and $R \subseteq Q_s \subseteq Q_{ml}$. By C we denote the centre of Q_{ml}, which is called the *extended centroid* of R. It turns out that C is a field if and only if R is prime. Moreover, C coincides with the centre of Q_s. Thus, Q_s and Q_{ml} can also be considered as algebras over C. It turns out that for any $q \in Q_{ml}$, $qRq = 0$ implies $q = 0$. Namely, assume that $qRq = 0$ and $q \neq 0$. Then there exists $x \in R$ such that $0 \neq xq \in R$ (see [**1**, Proposition 2.1.7]). Therefore, $0 \neq (xq)R(xq) \subseteq xqRq$ and hence $qRq \neq 0$, a contradiction.

Corollary 5. *Let R be a semiprime ring containing a nontrivial idempotent e. Suppose that for any $a \in Q'_{ml}$ the following holds:*

(i) $eaeR(1-e) = 0$ *implies* $eae = 0$,

(ii) $(1-e)a(1-e)Re = 0$ *implies* $(1-e)a(1-e) = 0$.

Then any left centraliser $\varphi : R \to Q_{ml}$ is additive.

Proof: We set $e_1 = e$ and $e_2 = 1 - e$. Let $a \in Q'_{ml}$ be such that $e_i a e_j R e_k = 0$ for some $i, j, k \in \{1, 2\}$. If $i = k$, then $(e_i a e_j)R(e_i a e_j) = (e_i a e_j R e_i) a e_j = 0$ which implies $e_i a e_j = 0$. Next, suppose $j = k$. According to [**1**, Proposition 2.1.7 (ii)] there exists a dense left ideal L of R such that $Le_i a \subseteq R$. Hence $(Le_i a e_j)R(Le_i a e_j) \subseteq Le_i a e_j R e_j = 0$, and so $Le_i a e_j = 0$. This implies $e_i a e_j = 0$ (see [**1**, Proposition 2.1.7 (iii)]). Finally, if $i \neq k$ and $j \neq k$ we have $(i, j, k) \in \{(1,1,2), (2,2,1)\}$. Hence, assumptions (i) and (ii) imply $e_i a e_j = 0$. Therefore, (A1)-(A6) hold and so Theorem 3 can be applied to obtain additivity of φ. ∎

Corollary 6. *Let R be a semiprime ring with an essential socle. Then any left centraliser $\varphi : R \to Q_{ml}$ is additive.*

Proof: Let $e \in R$ be an arbitrary minimal idempotent. Without loss of generality we may assume that e is not an identity element. We claim that

$$e\big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) = 0$$

for all $x, y \in R$. Suppose that $e \in Z(R)$. Then $e\big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) = 0$, since $\varphi(x+y) - \varphi(x) - \varphi(y) \in Q'_{ml}$. Thus, we may assume that $e = e_1 \notin Z(R)$. Let $a \in Q'_{ml}$. As in the proof of Corollary 5 we see that $e_1 a e_2 R e_1 = 0$ implies $e_1 a e_2 = 0$, and that $e_1 a e_2 R e_2 = 0$ implies $e_1 a e_2 = 0$. Let us prove that $e_1 a e_1 R e_2 = 0$ implies $e_1 a e_1 = 0$. Suppose that $e_1 a e_1 R e_2 = 0$ and $e_1 a e_1 \neq 0$. Then $(e_2 R e_1 a e_1) R (e_2 R e_1 a e_1) = 0$ and so

$e_2Re_1ae_1 = 0$. We can take a dense left ideal L of R such that $0 \neq Le_1ae_1 \subsetneq R$. Therefore, $0 \neq Le_1ae_1 \subseteq Re_1$. Since Re_1 is a minimal left ideal of R it follows that $Le_1ae_1 = Re_1$. Consequently, $e_2Re_1 = e_2Le_1ae_1 = 0$. Hence $xe_1 = e_1xe_1$ for all $x \in R$. Moreover, $(e_1Re_2)R(e_1Re_2) = 0$ and so $e_1Re_2 = 0$. This implies $e_1x = e_1xe_1$ for all $x \in R$. Thus, $e_1x = xe_1$ for all $x \in R$, which contradicts the assumption that $e_1 \notin Z(R)$. We have just seen that all assumptions of Lemma 1 are satisfied. Thus, applying Lemma 1 it follows that φ_{11} and φ_{12} are additive. Hence

$$\begin{aligned} e\big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) &= \varphi_{11}(x+y) - \varphi_{11}(x) - \varphi_{11}(y) \\ &\quad + \varphi_{12}(x+y) - \varphi_{12}(x) - \varphi_{12}(y) \\ &= 0 \end{aligned}$$

for all $x, y \in R$. Thus, according to the definition of the socle we have

$$\operatorname{soc}(R)\big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) = 0$$

for all $x, y \in R$. Since $\operatorname{soc}(R)$ is an essential ideal it follows that φ is additive. □

Let A be a semisimple complemented algebra and let us denote by S_0 the annihilator of $\operatorname{soc}(A)$. If S_0 is nonzero, then [**8**, p. 143, Corollary] implies the existence of a nonzero idempotent (more precisely, a primitive left projection) $e \in S_0$. Since the smallest closed ideal of A containing e is also a minimal closed ideal of A, we can refer to [**8**, Lemma 1] to conclude that eAe is a division ring. Hence e is minimal and so $e \in \operatorname{soc}(A)$ as well. This implies $e = e^2 \in \operatorname{soc}(A)S_0 = 0$, which is a contradiction. Thus, $\operatorname{soc}(A)$ is an essential ideal and so Corollary 6 and Remark 4 imply the result of Saworotnow and Giellis [**8**] saying that each left centraliser $\varphi : A \to A$ is linear.

Further, we consider the case when R is a prime ring. In this case for any $q, q' \in Q_{ml}$, $qRq' = 0$ implies $q = 0$ or $q' = 0$. Namely, assume that $qRq' = 0$ and $q, q' \neq 0$. Then there exist $x, y \in R$ such that $0 \neq xq, yq' \in R$ (see [**1**, Proposition 2.1.7]). Therefore, $0 \neq (xq)R(yq') \subseteq xqRq'$, a contradiction. Thus, the following result follows immediately from Corollary 5.

Corollary 7. *Let R be a prime ring containing a nonzero idempotent. Then any left centraliser $\varphi : R \to Q_{ml}$ is additive.*

Remark 8. Let R be a prime ring with a nonzero centre. Then any left centraliser $\varphi : R \to Q_{ml}$ is additive. Namely, since the extended centroid C of R is a field and since $0 \neq Z(R) \subseteq C$ it follows that $Q'_{ml} = 0$. Thus, according to the argument in the first section of the paper we see that φ is additive, indeed.

Corollary 9. *Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a real or a complex Banach space X. Let $A \subseteq \mathcal{B}(X)$ be a standard operator algebra (that is a subalgebra of $\mathcal{B}(X)$ containing the ideal of all finite rank operators). If $\varphi : A \to \mathcal{B}(X)$ is a left centraliser, then φ is linear.*

PROOF: By $\mathcal{F}(X)$ we denote the ideal of all finite rank operators of $\mathcal{B}(X)$. According to [**3**, p. 78, Example 5] and [**1**, Theorem 4.3.8] it follows that A is primitive, $\mathcal{F}(X) = \text{soc}(A)$, and $\mathcal{B}(X) = Q_s(A)$. Thus, Corollary 7 yields the additivity of φ. Further using Remark 4 we see that φ is linear. ∎

Note that Corollary 9 generalises Johnson's result [**4**, p. 313, Corollary] on automatic linearity of left centralisers of $\mathcal{K}(X)$.

We end this paper with an example of a left centraliser which is not additive.

EXAMPLE 10. Let $\mathcal{A} = \mathbb{F}\langle X, Y\rangle$ be the free algebra in noncommuting indeterminates X and Y over a field $\mathbb{F}$. Let $\mathcal{A}_1$ be a subalgebra of $\mathcal{A}$ generated by X and Y, that is, $\mathcal{A}_1 = X\mathcal{A} + Y\mathcal{A}$. Note that $\mathcal{A}_1$ is a domain having a zero centre. Thus, $\mathcal{A}_1$ has no nonzero idempotents. We define $\varphi : \mathcal{A}_1 \to \mathcal{A}_1$ by

$$\varphi(p) = \begin{cases} p & \text{if } p \in X\mathcal{A} \\ 0 & \text{if } p \notin X\mathcal{A} \end{cases}.$$

It is straightforward to see that φ is a well defined left centraliser. However, φ is not additive. Namely, $\varphi(X+Y) \neq \varphi(X) + \varphi(Y)$.

REMARK 11. The analogous results hold for right centralisers as well.

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