A CYCLIC INVOLUTION OF PERIOD ELEVEN

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In two earlier papers* the writer discussed involutions of periods five and seven on certain cubic surfaces in S_3. In this paper, a quartic surface containing a cyclic involution of period eleven is considered.

The surface

$$F_4(x_1,x_2,x_3,x_4) \equiv ax_2x_3^3 + bx_1x_2x_4^2 + cx_1x_3^2x_4 + dx_2^2x_3x_4 = 0$$

is invariant under the cyclic collineation T of period eleven,

$$x'_1:x'_2:x'_3:x'_4 = x_1:Ex_2:E^2x_3:E^3x_4 \qquad (E^{11} = 1).$$

Points $P_1(1,0,0,0)$, $P_2(0,1,0,0)$, $P_3(0,0,1,0)$, and $P_4(0,0,0,1)$ are all invariant under T and lie on the surface F_4. This fact may be stated in the following theorem.

THEOREM 1. *Each vertex of the tetrahedron of reference not only lies on the surface but is a point of coincidence.*

By rewriting F_4 in the order

$$ax_2x_3^3 + x_4(bx_1x_2x_4 + cx_1x_3^2 + dx_2^2x_3) = 0$$

it is easily seen that the line P_1P_2 ($x_3 = x_4 = 0$) lies on the surface. However, only the two points P_1 and P_2 of the line are invariant under T. In similar manner P_1P_4, P_1P_3, P_2P_4, and P_3P_4 lie on F_4 with only two invariant points on each line. The line P_2P_3 does not lie on the surface. A second theorem has been proved.

THEOREM 2. *This surface includes all the six edges of the tetrahedron of reference, except P_2P_3.*

It is true that P_3 is simple on F_4 while P_2 and P_4 are double, and P_1 is triple. In this paper only point P_3 will be investigated in detail.

Consider a curve C, not transformed into itself by T, and passing through P_3. Take the plane $x_4 + Kx_1 = 0$ of the pencil passing through P_2 and P_3, tangent to C. This plane is transformed into $E^3x_4 + Kx_1 = 0$ or $x_4 + KE^8x_1 = 0$ by T and hence is non-invariant. The curve cut out on F_4 by $x_4+Kx_1=0$ is therefore non-invariant. The common tangent to the two curves is not

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transformed into itself. Thus the two curves do not touch each other at P_3. Now, since C was a variable curve through P_3 satisfying the non-invariant property, it follows that P_3 is an imperfect coincidence point. In similar manner it can be shown that P_1, P_2, and P_4 are also imperfect coincidence points. The following theorem has just been proved.

THEOREM 3. *The I_{11} belonging to F_4 in S_3 has four imperfect points of coincidence.*

Consider the complete system of curves $|A|$ cut out on F_4 by all surfaces of order eleven. Its dimension is 243, its genus is 243, and the number of variable intersections of two members of the system is 484. A curve A of this system is not in general transformed into itself by T. There are, however, eleven partial systems $|A_i|$ in $|A|$ which are transformed into themselves. By use of $|A_1|$ we find

$$
\begin{aligned}
&a_1x_1^{11} + a_2x_2^{11} + a_3x_3^{11} + a_4x_4^{11} + a_5x_1^7x_3x_4^3 + a_6x_1^6x_2^2x_4^3 + a_7x_1^6x_2x_3^2x_4^2 \\
&+ a_8x_1^5x_2^3x_3x_4^2 + a_9x_1^4x_2^5x_4^2 + a_{10}x_1^6x_3^4x_4 + a_{11}x_1^5x_2^2x_3^3x_4 + a_{12}x_1^4x_2^4x_3^2x_4 \\
&+ a_{13}x_1^3x_2^6x_3x_4 + a_{14}x_1^2x_2^8x_4 + a_{15}x_1^5x_2x_3^5 + a_{16}x_1^4x_2^3x_3^4 + a_{17}x_1^3x_2^5x_3^3 \\
&+ a_{18}x_1^2x_2^7x_3^2 + a_{19}x_1x_2^9x_3 + a_{20}x_1^3x_2x_4^7 + a_{21}x_1^3x_3^2x_4^6 + a_{22}x_1^2x_2^2x_3x_4^6 \\
&+ a_{23}x_1x_2^4x_4^6 + a_{24}x_1^2x_2x_3^3x_4^5 + a_{25}x_1x_2^3x_3^2x_4^5 + a_{26}x_2^5x_3x_4^5 + a_{27}x_1^2x_3^5x_4^4 \\
&+ a_{28}x_1x_2^2x_3^4x_4^4 + a_{29}x_2^4x_3^3x_4^4 + a_{30}x_1x_2x_3^6x_4^3 + a_{31}x_2^3x_3^5x_4^3 + a_{32}x_1x_3^8x_4^2 \\
&+ a_{33}x_2^2x_3^7x_4^2 = 0.
\end{aligned}
$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of thirty-two dimensions. We obtain a surface φ, of order 44, as the image of I_{11}. The equations of the transformation for mapping I_{11} upon φ in S_{32} are

$$
\begin{array}{lll}
\rho X_1 = x_1^{11} & \rho X_{12} = x_1^4x_2^4x_3^2x_4 & \rho X_{23} = x_1x_2^4x_4^6 \\
\rho X_2 = x_2^{11} & \rho X_{13} = x_1^3x_2^6x_3x_4 & \rho X_{24} = x_1^2x_2x_3^3x_4^5 \\
\rho X_3 = x_3^{11} & \rho X_{14} = x_1^2x_2^8x_4 & \rho X_{25} = x_1x_2^3x_3^2x_4^5 \\
\rho X_4 = x_4^{11} & \rho X_{15} = x_1^5x_2x_3^5 & \rho X_{26} = x_2^5x_3x_4^5 \\
\rho X_5 = x_1^7x_3x_4^3 & \rho X_{16} = x_1^4x_2^3x_3^4 & \rho X_{27} = x_1^2x_3^5x_4^4 \\
\rho X_6 = x_1^6x_2^2x_4^3 & \rho X_{17} = x_1^3x_2^5x_3^3 & \rho X_{28} = x_1x_2^2x_3^4x_4^4 \\
\rho X_7 = x_1^6x_2x_3^2x_4^2 & \rho X_{18} = x_1^2x_2^7x_3^2 & \rho X_{29} = x_2^4x_3^3x_4^4 \\
\rho X_8 = x_1^5x_2^3x_3x_4^2 & \rho X_{19} = x_1x_2^9x_3 & \rho X_{30} = x_1x_2x_3^6x_4^3 \\
\rho X_9 = x_1^4x_2^5x_4^2 & \rho X_{20} = x_1^3x_2x_4^7 & \rho X_{31} = x_2^3x_3^5x_4^3 \\
\rho X_{10} = x_1^6x_3^4x_4 & \rho X_{21} = x_1^3x_3^2x_4^6 & \rho X_{32} = x_1x_3^8x_4^2 \\
\rho X_{11} = x_1^5x_2^2x_3^3x_4 & \rho X_{22} = x_1^2x_2^2x_3x_4^6 & \rho X_{33} = x_2^2x_3^7x_4^2
\end{array}
$$

By eliminating ρ, x_1, x_2, x_3, and x_4 from these thirty-three equations and $F_4(x_1x_2x_3x_4) = 0$, we get as the thirty equations defining the surface:

$$
\left\| \begin{matrix} X_1 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{11} & X_{12} & X_{16} \\ X_5 & X_{21} & X_{22} & X_{24} & X_{25} & X_{26} & X_{28} & X_{29} & X_{31} \end{matrix} \right\| = 0
$$

$$
\left\| \begin{matrix} X_2 & X_9 & X_{13} & X_{14} & X_{17} & X_{18} & X_{19} \\ X_{13} & X_5 & X_7 & X_8 & X_{10} & X_{11} & X_{12} \end{matrix} \right\| = 0
$$

$$\left\|\begin{matrix} X_3 & X_{15} & X_{27} & X_{30} & X_{31} & X_{32} & X_{33} \\ X_{30} & X_6 & X_{20} & X_{22} & X_{23} & X_{24} & X_{25} \end{matrix}\right\| = 0$$

$$\left\|\begin{matrix} X_4 & X_{20} & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{21} & X_7 & X_{11} & X_{12} & X_{15} & X_{16} \end{matrix}\right\| = 0$$

$$\left\|\begin{matrix} X_6 & X_7 & X_8 & X_{10} & X_{11} \\ X_{23} & X_{25} & X_{26} & X_{28} & X_{29} \end{matrix}\right\| = 0$$

and equation $aX_{31} + bX_{25} + cX_{28} + dX_{29} = 0$. Designate by P'_3 the branch point of φ corresponding to the point P_3 on F_4. The coordinates of P'_3 are all zero except X_3.

The curves A_1 on F_4 pass through P_3 if $a_3 = 0$. The tangent plane at P_3 to F_4 is $x_2 = 0$. Now, the system of eleventh-degree surfaces passing through P_3 cuts $x_2 = 0$ in the curves $x_2 = 0$, and

$$a_1x_1^{11} + a_4x_4^{11} + a_5x_1^7x_3x_4^3 + a_{10}x_1^6x_3^4x_4 + a_{21}x_1^3x_3^2x_4^6 + a_{27}x_1^2x_3^5x_4^4 + a_{32}x_1x_3^8x_4^2 = 0.$$

For general values of the constants this is an eleventh-degree curve with a triple point at P_3, two branches being tangent to the line $x_2 = x_4 = 0$ and one to the line $x_2 = x_1 = 0$. When $a_5 = a_{10} = a_{21} = a_{27} = a_{32} = 0$, the plane eleventh-degree curve breaks up into eleven lines through P_3. These are all distinct except when either $a_1 = 0$ or $a_4 = 0$, when they coincide with $x_2 = x_4 = 0$ or $x_2 = x_1 = 0$, respectively. Since P_3 is imperfect, the $|A_1|$ through P_3 must have eleven distinct branches unless each branch touches one of the two invariant directions. In the plane $x_2 = 0$, the involution I_{11} is generated by the homography T_1, which is $x'_1 \colon x'_3 \colon x'_4 = x_1 \colon E^2x_3 \colon E^3x_4$.

By use of the plane quadratic transformation X, $y_1 \colon y_3 \colon y_4 = w_1w_4 \colon w_3^2 \colon w_1w_3$ and X^{-1}, $w_1 : w_3 : w_4 = y_4^2 : y_3y_4 : y_1y_3$ one gets

$$(w_1,w_3,w_4) \sim_{X^{-1}} (y_4^2,y_3y_4,y_1y_3) \sim_{T_1} (E^6y_4,E^5y_3y_4,E^2y_1y_3) \sim_X (E^6w_1,E^5w_3,E^2w_4)$$

or

$$x'_1 : x'_3 : x'_4 = E^4x_1 : E^3x_3 : x_4 \qquad \text{for } T_2.$$

Again $(w_1,w_3,w_4) \sim_{X^{-1}} (y_4^2,y_3y_4,y_1y_3) \sim_{T_2} (y_4^2,E^3y_3y_4,E^7y_1y_3) \sim_X (w_1,E^3w_3,E^7w_4)$ or T_3 is $x'_1 : x'_3 : x'_4 = x_1 : E^3x_3 : E^7x_4$. By use of XT_3X^{-1} one gets

$$(w_1,\ w_3,\ w_4) \sim (E^{14}w_1,\ E^{10}w_3,\ E^3w_4)$$

or T_4 is $x'_1 : x'_3 : x'_4 = E^{11}x_1 : E^7x_3 : x_4 = x_1 : E^7x_3 : x_4$.

Thus, the following theorem has just been established.

THEOREM 4. *The imperfect point of coincidence P_3 has an imperfect point in the first order neighbourhood along the $x_1 = x_2 = 0$ direction. It also has an imperfect point in the second order neighbourhood. In the third order neighbourhood there is a perfect point.*

Now, investigate the characteristics of the point adjacent to P_3 along the invariant direction $x_4 = x_2 = 0$. By use of YT_1Y^{-1}, where the transforma-

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tion Y is $y_1 : y_3 : y_4 = w_3w_4 : w_3^2 : w_1w_4$ and the inverse is $w_1 : w_3 : w_4 = y_3y_4 : y_1y_3 : y_1^2$, we get $(w_1,w_3,w_4) \sim_{Y^{-1}} (y_3y_4,y_1y_3,y_1^2) \sim_{T_1} (E^5y_3y_4,E^2y_1y_3,y_1^2) \sim_Y (E^5w_1, E^2w_3,w_4)$. We have an imperfect point. Define T'_2 as YT_1Y^{-1}. Now apply $XT'_2X^{-1} \equiv T''_2$ to our next order point, remembering that T'_2 may be written $x'_1 : x'_3 : x'_4 = E^5x_1 : E^2x_3 : x_4$. We obtain

$$(w_1,w_3,w_4) \sim_{X^{-1}} (y_4^2,y_3y_4,y_1y_3) \sim_{T'_2} (y_4^2,E^2y_3y_4,E^7y_1y_3) \sim_X (w_1,E^2w_3,E^7w_4).$$

This transformation T''_2 or $x'_1 : x'_3 : x'_4 = x_1 : E^2x_3 : E^7x_4$ gives evidence of another imperfect point. For the third order neighbourhood, we use $YT''_2Y^{-1} \equiv T'''_2$. This becomes $(w_1,w_3,w_4) \sim (E^9w_1,E^2w_3,w_4)$, denoting an imperfect point in the third order neighbourhood of P_3 along the $x_2 = x_4 = 0$ direction.

Finally, by use of $XT'''_2X^{-1} \equiv T^{\text{iv}}_2$ we get $(w_1, w_3, w_4) \sim (w_1, E^2w_3, E^{11}w_4)$ or (w_1,E^2w_3,w_4) since $E^{11} = 1$. This indicates a perfect point. We shall state our result in the following theorem.

THEOREM 5. *Along the invariant direction $x_2 = x_4 = 0$, there are no perfect points in either the first or second or third order neighbourhood of P_3. There is, however, a perfect point in the fourth order neighbourhood.*

The following theorem is self-evident.

THEOREM 6. *The imperfect point P_3 on F_4 has no perfect points in the neighbourhood of the first or second order. It does have one in the third order neighbourhood and one in the fourth order neighbourhood, however.*

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