Can. J. Math., Vol. XXVII, No. 5, 1975, pp. 1075–1082

A POINTWISE ERGODIC THEOREM IN L_p-SPACES

M. A. AKCOGLU

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be a measure space and $L_p = L_p(X, \mathscr{F}, \mu)$, $1 \leqq p \leqq \infty$, the usual Banach spaces. A linear operator $T : L_p \to L_p$ is called a positive contraction if it transforms non-negative functions into non-negative functions and if its norm is not more than one. The purpose of this note is to show that if $1 < p < \infty$ and if $T : L_p \to L_p$ is a positive contraction then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^k f$$

exists a.e. for each $f \in L_p$. Related results in this direction were obtained by E. M. Stein [**6**], A. Ionescu-Tulcea [**5**], R. V. Chacon and J. Olsen [**3**] and by R. V. Chacon and S. A. McGrath [**4**]. It has been shown by Burkholder [**2**] that this theorem is false if T is not positive (see also [**1**]).

It is well known (see, e.g. [**5**]) that to prove this result it is enough to prove the following theorem.

(1.1) THEOREM. *Let $1 < p < \infty$ and let $T : L_p \to L_p$ be a positive contraction. For each $f \in L_p$, let*

$$\bar{f}(x) = \sup_{n \geqslant 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \right| .$$

Then

$$\|\bar{f}\| \leqq \frac{p}{p-1} \|f\| .$$

If the statement of this theorem is true for an operator then we will say that the Dominated (Ergodic) Estimate holds for this operator. In Section 2 it will be shown that the Dominated Estimate holds for a positive contraction on a finite dimensional L_p-space, by reducing this case to the case considered by A. Ionescu-Tulcea in [**5**]. In Section 3 the theorem will be proved for the general case.

Theorem 1.1 answers a question raised by Chacon and McGrath in [**4**]. This question was the starting point of the present work. It may be noted that the elementary results (2.4) and (2.5) also appeared in [**4**], but in a different context.

Received March 26, 1974 and in revised form, July 10, 1974. This research was supported by NRC Grant A-3974.

The author is grateful to Professor Louis Sucheston for very stimulating discussions on this problem.

2. Finite dimensional L_p-spaces. In this section the indices i and j will range through the integers 1, 2, ..., n where $n \geqq 1$ is an arbitrary but fixed integer. Let m_i's be strictly positive fixed numbers and introduce a norm on $\mathbf{R}^n$ as $||r||_p = [\sum_i |r_i|^p m_i]^{1/p}$, $r = (r_i) \in \mathbf{R}^n$. We denote the resulting L_p-space by l_p. Let $T : l_p \to l_p$ be a positive contraction determined by a matrix (T_{ij}) as $(Tr)_j = \sum_i T_{ij} r_i$. First we will assume the following two conditions:

(2.1) $T_{ij} > 0$

(2.2) $||T||_p = 1.$

(2.3) LEMMA. *There exists a vector $u \in l_p^+$ so that $||u||_p = ||Tu||_p = 1$.*

Proof. Let $K = \max ||Tu||_p$ where the maximum is taken over the vectors $u \in l_p^+$ with $||u||_p = 1$. It is clear that $K \leqq 1$ and if $K = 1$ then the lemma would follow. If $K < 1$, then for any $r \in l_p$, $||Tr||_p = ||Tr^+ - Tr^-||_p \leqq ||Tr^+ + Tr^-||_p < ||r^+ + r^-||_p = ||r||_n$. This contradicts (2.2).

We now also consider the adjoint space l_p^* and identify this, as usual, with l_q, $q = p(p-1)^{-1}$, so that a vector $s \in l_q$ acts as functional on l_p defined as $(r, s) = \sum_i r_i s_i m_i$, $r \in l_p$. The adjoint of T will be a positive contraction $T^* : l_q \to l_q$ so that $(Tr, s) = (r, T^*s)$ for all $r \in l_p$, $s \in l_q$. Since $(Tr, s) = \sum_j m_j s_j \sum_i T_{ij} r_i = \sum_i m_i r_i \sum_j (mj/mi) T_{ij} s_j$, we obtain that

$$(T^*s)_i = \sum_j \frac{mj}{mi} T_{ij} s_j.$$

For each $r \in l_p^+$ there is a unique $r^* \in l_q^+$ so that $(r, r^*) = ||r||_p^p = ||r^*||_q^q$. This vector is given as $r^*_i = r_i^{p-1}$.

(2.4) LEMMA. *Let u be the vector obtained in Lemma (2.3) and let $v = Tu$. Then $u^* = T^*v^*$ and both u and v have strictly positive coordinates.*

Proof. Since $1 = (v, v^*) = (Tu, v^*) = (u, T^*v^*)$ and since $||T^*v^*||_q \leqq 1$ we see that $u^* = T^*v^*$. Because of (2.1), $v = Tu$ and $u^* = T^*v^*$ have strictly positive coordinates. Hence u also has strictly positive coordinates.

(2.5) COROLLARY. *There exist two vectors $u = (u_i)$ and $v = (v_j)$ with strictly positive coordinates so that*

(2.6) $$v_j = \sum_i T_{ij} u_i,$$

(2.7) $$m_i u_i^{p-1} = \sum_j m_j T_{ij} v_j^{p-1}.$$

Given a positive contraction $T : l_p \to l_p$ satisfying (2.1) and (2.2) we are now going to construct a measure space $(Z, \mathscr{F}, \mu)$ and a transformation $\tau : Z \to Z$.

The space Z will be a subset of the two dimensional cartesian space Oxy. The σ-algebra $\mathscr{F}$ will be the class of two dimensional Borel subsets of Z and the measure μ will be the restriction of the two dimensional Lebesgue measure to $\mathscr{F}$. One and two dimensional Lebesgue measures will be denoted with l and l^2, respectively and no distinction will be made between l^2 and μ. The differentials of l and l^2 will also be denoted as dx and $dxdy$, respectively.

Let I_i's be n disjoint intervals on the x-axis and J_i's n disjoint intervals on the y-axis so that $l(I_i) = m_i$ and $l(J_i) = 1$. Let $I = \bigcup_i I_i$ and $E_i = I_i \times J_i$ and finally $Z = \bigcup_i E_i$.

To define the transformation $\tau : Z \to Z$, let

$$\xi_{ij} = T_{ij}\frac{u_i}{v_j}, \tag{2.8}$$

$$\eta_{ij} = T_{ij}\frac{v_j^{p-1}}{u_i^{p-1}}\frac{m_j}{m_i}, \tag{2.9}$$

where (u_i) and (v_j) are as given by Corollary (2.5), and note that $\sum_i \xi_{ij} = 1$ and $\sum_j \eta_{ij} = 1$. Now divide each I_j into n disjoint subintervals $I_{1j}, I_{2j}, \ldots, I_{nj}$ and also divide each J_i into n disjoint subintervals $J_{i1}, J_{i2}, \ldots, J_{in}$ so that $l(I_{ij}) = \xi_{ij} m_j$ and $l(J_{ij}) = \eta_{ij}$. Let $R_{ij} = I_i \times J_{ij}$ and $S_{ij} = I_{ij} \times J_j$. Note that $E_i = \bigcup_j R_{ij}$ and $E_j = \bigcup_i S_{ij}$.

For each (i, j) we can now find four constants $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ so that the affine transformation

$$\tau_{ij}(x, y) = (a_{ij}x + b_{ij}, c_{ij}y + d_{ij})$$

transforms R_{ij} onto S_{ij}, up to an l^2-null set. Let $\tau : Z \to Z$ be defined as τ_{ij} on each R_{ij}.

The transformation $\tau : Z \to Z$ is invertible and measurable in both directions. Also, $\mu(B) = 0$ if and only if $\mu(\tau^{-1}B) = \mu(\tau B) = 0$. Let $\nu = \mu\tau^{-1}$ be the measure obtained by transporting μ by τ. It is clear that ν is absolutely continuous with respect to μ and its Radon Nikodym derivative is given as

$$\rho = \frac{d\nu}{d\mu} = \sum_{i,j} \frac{\mu(R_{ij})}{\mu(S_{ij})}\chi_{S_{ij}} = \sum_{i,j}\left(\frac{v_j}{u_i}\right)^p \chi_{S_{ij}}$$

where χ denotes the characteristic function of a set.

This transformation $\tau : Z \to Z$ is an automorphism, in the terminology of [**5**]. Hence $(Qf)(x,y) = [\rho(x,y)]^{1/p} f(\tau^{-1}(x,y))$, $(x,y) \in Z, f \in L_p = L_p(Z, \mathscr{F}, \mu)$ defines a positive invertible isometry $Q : L_p \to L_p$, for which the Dominated Ergodic Estimate holds [**5**]. From this fact we will now obtain the same theorem for T as follows.

Let $\{E_i\}$ be the partition of Z as defined above and let $E : L_p \to L_p$ be the conditional expectation operator with respect to $\{E_i\}$. More explicitly, let

$$Ef = \sum_i \chi_{E_i}\frac{1}{m_i}\int_{E_i} f\,d\mu, \quad f \in L_p.$$

Note that this operator is a positive contraction of L_1 and L_∞ simultaneously, and hence a positive contraction $E : L_p \to L_p$ for each p, $1 < p < \infty$.

(2.10) LEMMA. *Let $f, g \in L_p$ be two functions on Z depending only on the x-coordinate of a point $(x, y) \in Z$. If $Ef = Eg$ then $EQf = EQg$.*

Proof. Let $F : I \to \mathbf{R}$ be a function so that $f(x, y) = F(x)$, $(x, y) \in Z$. We will compute EQf as follows:

$$(Qf)(x, y) = \sum_{i,j} \frac{v_j}{u_i} f(\tau_{ij}^{-1}(x, y))\chi_{S_{ij}}(x, y)$$

$$\begin{aligned}\int_{E_j} Qf\, d\mu &= \sum_i \frac{v_j}{u_i} \int_{S_{ij}} f(\tau_{ij}^{-1}(x, y))dxdy \\ &= \sum_i \frac{v_j}{u_i} \frac{\mu(S_{ij})}{\mu(R_{ij})} \int_{R_{ij}} f(x, y)dxdy \\ &= \sum_i \left(\frac{v_j}{u_i}\right)^{1-p} \eta_{ij} \int_{I_i} F(x)dx \\ &= \sum_i T_{ij} \frac{m_j}{m_i} \int_{E_i} fd\mu.\end{aligned}$$

This means that

$$(2.11) \quad EQf = \sum_j \chi_{E_j} \sum_i T_{ij} \frac{1}{m_i} \int_{E_i} fd\mu.$$

If $Ef = Eg$ then

$$\int_{E_i} fd\mu = \int_{E_i} gd\mu,$$

which shows that $EQf = EQg$.

(2.12) LEMMA. *If $r \in l_p$ then*

$$EQ\Big(\sum_i r_i\chi_{E_i}\Big) = \sum_j (Tr)_j\chi_{E_j}.$$

Proof. This follows directly from the formula (2.11) obtained above.

(2.13) THEOREM. *For any $r \in l_p$ and for any integer $k \geqq 0$,*

$$EQ^k \sum_i r_i\chi_{E_i} = \sum_j (T^k r)_j\chi_{E_j}.$$

Proof. We apply induction on k. The theorem is trivial for $k = 0$. Assume that it is true for an integer k. First note that if $f \in L_p$ depends only on the x-coordinate then the same is also true for Qf, as follows from the definition of Q. Hence $Q^k\sum_i r_i\chi_{E_i}$ depends only on the x-coordinate. Hence, by Lemmas

https://doi.org/10.4153/CJM-1975-112-7 Published online by Cambridge University Press

(2.10) and (2.12) and by the induction hypothesis,

$$EQ^{k+1} \sum_i r_i \chi_{Ei} = EQEQ^k \sum_i r_i \chi_{Ei}$$
$$= EQ \sum_i (T^k r)_i \chi_{Ei}$$
$$= \sum_j (T^{k+1} r)_j \chi_{Ej}.$$

(2.14) THEOREM. *The Dominated Ergodic Estimate holds for a positive contraction* $T : l_p \to l_p$ *satisfying* (2.1) and (2.2).

Proof. It is enough to prove the theorem for a positive vector. Let $r \in l_p^+$ and let

$$\bar{r}_i = \sup_{K \geqq 1} \frac{1}{K} \sum_{k=0}^{K-1} (T^k r)_i.$$

We have to show that

$$||\bar{r}||_p \leqq \frac{p}{p-1} ||r||_p.$$

Define f $\in L_p^+(Z)$ as $f = \sum_i r_i \chi_{Ei}$, and let

$$\bar{f} = \sup_{K \geqq 1} \frac{1}{K} \sum_{k=0}^{K-1} Q^k f.$$

Then we have that $||f||_p = ||r||_p$ and hence that

$$||\bar{f}||_p \leqq \frac{p}{p-1} ||f||_p = \frac{p}{p-1} ||r||_p,$$

by the result of [**5**].

Now, since

$$\frac{1}{K} \sum_{k=0}^{K-1} EQ^k f \leqq E\bar{f}$$

for all $K \geqq 1$, we also have that

$$\sup_{K \geqq 1} \frac{1}{K} \sum_{k=0}^{K-1} EQ^k f \leqq E\bar{f}.$$

But

$$\sup_{K \geqq 1} \frac{1}{K} \sum_{k=0}^{K-1} EQ^k f = \sup_{K \geqq 1} \frac{1}{K} \sum_{k=0}^{K-1} \sum_i (T^k r)_i \chi_{Ei}$$
$$= \sum_i \bar{r}_i \chi_{Ei}.$$

Therefore

$$||\bar{r}||_p = \left|\left| \sum_i \bar{r}_i \chi_{Ei} \right|\right|_p \leqq ||E\bar{f}||_p \leqq \frac{p}{p-1} ||r||_p.$$

(2.15) THEOREM. *The Dominated Ergodic Estimate holds for any positive contraction $T : l_p \to l_p$ without any additional hypotheses.*

Proof. First assume that $T_{ij} > 0$ but $||T||_p = 1/\lambda$, $\lambda > 1$. Then the operator λT satisfies the hypotheses of the previous theorem and we obtain

$$\left|\left| \sup_{K \geqslant 1} \frac{1}{K} \sum_{k=0}^{K-1} \lambda^k T^k r \right|\right|_p \leqq \frac{p}{p-1} ||r||_p,$$

for any $r \in l_p^+$. But this clearly implies the same estimate for T.

Next suppose that $T_{ij} = 0$ for some (i, j) and that the Dominated Estimate does not hold for T. This means that there is an integer $N \geqq 1$ and an $r \in l_p^+$ so that

$$\left|\left| \sup_{1 \leqslant K \leqslant N} \frac{1}{K} \sum_{k=0}^{K-1} T^k r \right|\right|_p > \frac{p}{p-1} ||r||_p.$$

Then there is a number $c < 1$ so that the same inequality holds if T is replaced by cT. But then there is an $e > 0$ so that $T'_{ij} = cT_{ij} + e$ still defines a positive contraction $T' : l_p \to l_p$ for which the Dominated Estimate does not hold. This is a contradiction, since T' has strictly positive entries.

3. General L_p-spaces. Let $(X, \mathscr{F}, \mu)$ be a measure space and let $T : L_p \to L_p$ be a positive contraction of $L_p = L_p(X, \mathscr{F}, \mu)$. If $\{E_1, \ldots, E_n\}$ is a finite partition of X then the conditional expectation operator will be defined as

$$Ef = \sum_i \alpha_i \chi_{E i}, \quad f \in L_p,$$

where

$$\alpha_i = \begin{cases} 0, & \text{if } \mu(E_i) = 0 \text{ or } \mu(E_i) = \infty, \\ \dfrac{1}{\mu(E_i)} \displaystyle\int_{E i} f d\mu, & \text{if } 0 < \mu(E) < \infty. \end{cases}$$

As before, $E : L_p \to L_p$ is a positive contraction. Furthermore, if $f_1, \ldots, f_K$ are finitely many members of L_p and if $e > 0$ then there is a conditional expectation E so that

$$|| f_k - Ef_k|| < e, \quad k = 1, \ldots, K.$$

(3.1) LEMMA. *Given an $e > 0$, an integer $K \geqq 1$ and an $f \in L_p$ then there is a conditional expectation E so that*

$$||T^k f - (ET)^k Ef|| < e$$

for all k, $0 \leqq k \leqq K - 1$.

Proof. Choose E so that $||T^k f - ET^k f|| < e/K$ for all $k = 0, 1, \ldots, K - 1$. We will show that $||T^k f - (ET)^k Ef|| < e(k + 1)/K$ for all $k = 0, \ldots, K - 1$. The proof is by induction. The result is true for $k = 0$. If it is true for k, then

$$||T^{k+1}f - T(ET)^k Ef|| < e(k+1)/K$$

and hence

$$||ET^{k+1}f - (ET)^{k+1}Ef|| < e(k+1)/K.$$

But also,

$$||T^{k+1}f - ET^{k+1}f|| < e/k,$$

which gives that

$$||T^{k+1}f - (ET)^{k+1}Ef||_p < e(k+2)/K.$$

(3.2) LEMMA. *Given an $e > 0$ and finitely many functions $f_1, \ldots, f_k$ in L_p^+, there is an $S > 0$ so that if $g_1, \ldots, g_k$ are K functions in L_p^+ satisfying $|| f_k - g_k||_p < S$, $k = 1, \ldots, K$, then*

$$\left|\left| \max_{1 \leqslant k \leqslant K} f_k - \max_{1 \leqslant k \leqslant K} g_k \right|\right|_p < e.$$

Proof. First choose a set $A \in \mathscr{F}$ so that $\mu(A) < \infty$ and so that

$$\sum_{k=1}^{K} ||\chi_{A^c} f_k|| < \frac{e}{10}.$$

Next choose a $\lambda > 0$ so that $\mu(B) < \lambda$ implies that $\sum_{k=1}^{K}||\chi_B f_k|| < e/10$. Then let

$$0 < S < \min\left(\frac{e}{10K}, \left[\frac{\lambda}{K\mu(A)}\right]^{1/p} \frac{e}{10}\right).$$

Assume that $||g_k - f_k|| < S$ for each $k = 1, \ldots, K$.
Let

$$B_k = \left\{x \mid |f_k(x) - g_k(x)|^p > \frac{e^p}{10^p \mu(A)}\right\}.$$

Then

$$\frac{e^p}{10^p \mu(A)} \mu(B_k) < \frac{\lambda e^p}{K 10^p \mu(A)}.$$

This means that if $B = \bigcup_{k=1}^{K} B_k$ then $\mu(B) < \lambda$.
Now let

$$\bar{f} = \max_{1 \leqslant k \leqslant K} f_k \quad \text{and} \quad \bar{g} = \max_{1 \leqslant k \leqslant K} g_k.$$

Note that

$$|\bar{f}(x) - \bar{g}(x)|^p < \frac{e^p}{10^p \mu(A)} \quad \text{if } x \in R = A \cap B^c$$

and

$$|\bar{f}(x) - \bar{g}(x)| < \sum_{k=1}^{K} (f_k(x) + g_k(x)) \quad \text{if } x \in S = R^c = A^c \cap B.$$

(This last inequality is true everywhere, but we need it only on S.) Hence $||\bar{f} - \bar{g}|| \leqq ||\chi_R(\bar{f} - \bar{g})|| + ||\chi_S(\bar{f} - \bar{g})||$. But $||\chi_R(\bar{f} - \bar{g})|| < e/10$ and

$$\begin{aligned} ||\chi_S(\bar{f} - \bar{g})|| &\leqq \sum_{k=1}^{K} (||\chi_S f_k|| + ||\chi_S g_k||) \\ &\leqq \sum_{k=1}^{K} (2||\chi_S f_k|| + ||\chi_S(g_k - f_k)||) \\ &\leqq \sum_{k=1}^{K} 2||\chi_S f_k|| + \frac{e}{10}. \end{aligned}$$

Since $||\chi_S f_k|| \leqq ||\chi_A c f_k|| + ||\chi_B f_k||$, this shows that $||\bar{f} - \bar{g}|| < 2e/10 + 2e/10 + 2e/10 < e$.

(3.3) THEOREM. *The Dominated Estimate holds for any positive contraction* $T : L_p \to L_p$.

Proof. If the Dominated Estimate does not hold for T then there is an integer N and a function $f \in L_p^+$ so that

$$\left|\left| \max_{1 \leqslant K \leqslant N} \frac{1}{K} \sum_{k=0}^{K-1} T^k f \right|\right| > \frac{p}{p-1} ||f||.$$

Then, by the previous two lemmas, there is a conditional expectation E so that

$$\left|\left| \max_{1 \leqslant K \leqslant N} \frac{1}{K} \sum_{k=0}^{K-1} (ET)^k Ef \right|\right| > \frac{p}{p-1} ||Ef||.$$

Let $\{E_1, \ldots, E_{n'}\}$ be the partition corresponding to E, and let $\{E_1, \ldots, E_n\}$ be the atoms of this partition with finite non-zero measures. The class of L_p functions which are constant on these atoms can be identified with l_p and ET defines a positive contraction on this l_p. Hence the last inequality contradicts Theorem (2.15).

REFERENCES

1. M. A. Akcoglu and L. Sucheston, *On the dominated ergodic theorem in L_2 space* (to appear in Proc. Amer. Math. Soc.).
2. D. L. Burkholder, *Semi-Gaussian subspaces*, Trans. Amer. Math. Soc. *104* (1962), 123–131.
3. R. V. Chacon and J. Olsen, *Dominated estimates of positive contractions*, Proc. Amer. Math. Soc. *20* (1969), 266–271.
4. R. V. Chacon and S. A. McGrath, *Estimates of positive contractions*, Pacific J. Math. *30* (1969), 609–620.
5. A. Ionescu-Tulcea, *Ergodic properties of isometries in L_p-spaces*, Bull. Amer. Math. Soc. *70* (1964), 366–371.
6. E. M. Stein, *On the maximal ergodic theorem*, Proc. Nat. Acad. Sci. U.S.A. *47* (1961), 1894–1897.

University of Toronto,
Toronto, Ontario