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SOME GEOMETRIC PROPRTIES OF A SEQUENCE SPACE RELATED TO ℓ^p

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The sequence space $m(\phi)$, introduced and studied by W.L.C. Sargent in 1960, is closely related to the space ℓ^p. In this paper we obtain an explicit formula for the Hausdorff measure of noncompactness of any bounded subset in $m(\phi)$. We also show that $m(\phi)$ enjoys the weak Banach-Saks property, while $C(m(\phi)) = 2$. This shows that the condition $C(X) < 2$, known to be sufficient for the space X to have the weak Banach-Saks property, is not a necessary one.

1. PRELIMINARIES AND INTRODUCTION

Let X be a Banach space and $S(X)$ and $B(X)$ be the unit sphere and the unit ball of X, respectively. Let ℓ^0 be the set of all real sequences.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence $\{x_n\}$ in X admits a subsequence $z = \{z_n\}$ such that the sequence $\{t_k(z)\}$ is convergent in norm in X (see [**4**]), where

$$t_k(z) = \frac{1}{k}(z_1 + z_2 + \cdots + z_k).$$

A Banach space X is said to have the *weak Banach-Saks property* whenever given any weakly null sequence $\{x_n\} \subset X$ there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that the sequence $\{t_k(z)\}$ converges to zero strongly.

A new geometric constant $C(X)$ concerning the Banach-Saks property was introduced in [**3**]

$$C(X) = \sup\Big\{A\big(\{x_n\}\big) : \{x_n\} \text{ is a weakly null sequence in } S(X)\Big\},$$

where for a sequence $\{x_n\} \subset X$, we define [**1**]

$$A\big(\{x_n\}\big) = \lim_{n\to\infty} \inf\big\{\|x_i + x_j\| : i, j \geqslant n,\ i \neq j\big\}$$

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For any Köthe sequence space $X, C(X) \leqslant D(X) \leqslant 2$ (see [**3**, Theorem 2]), where

$$D(X) = \sup\Big\{\operatorname{sep}\big(\{x_n\}\big) : \{x_n\} \subset S(X)\Big\},$$

and

$$\operatorname{sep}\big(\{x_n\}\big) = \inf\big\{\|x_n - x_m\| : n \neq m\big\}.$$

Let Q be a bounded subset of X. Then the *Hausdorff measure of noncompactness* (see [**2, 5**]) of the set Q denoted by $\chi(Q)$ is defined as

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon- \text{ net in } X\}.$$

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ for which $c_n(\sigma) = 1$ if $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further, let

$$\mathcal{C}_s = \left\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leqslant s\right\},$$

the set of those σ whose support has cardinality at most s, and let

$$\Phi = \left\{\phi = \{\phi_n\} \in \ell^0 : \phi_1 > 0, \Delta\phi_k \geqslant 0 \text{ and } \Delta\Big(\frac{\phi_k}{k}\Big) \leqslant 0 \quad (k = 1, 2, \ldots)\right\},$$

where $\Delta\phi_n = \phi_n - \phi_{n-1}$.

For $\phi \in \Phi$, we define the following sequence space, introduce in [**7**],

$$m(\phi) = \left\{x = \{x_n\} \in \ell^0 : \sup_{s \geqslant 1} \sup_{\sigma \in \mathcal{C}_s}\Big(\frac{1}{\phi_s}\sum_{n \in \sigma}|x_n|\Big) < \infty\right\}.$$

It is easy to see that the space $m(\phi)$ is a Köthe sequence space, indeed a BK-space with respect to its natural norm. Sargent [**7**] established the relationship of this space to the space ℓ^p $(1 \leqslant p \leqslant \infty)$ and characterised some matrix transformations. In [**6**] matrix classes $(X, m(\phi))$ have been characterised, where X is any FK-space.

In this paper we shall compute the Hausdorff measure of noncompactness in the space $m(\phi)$ and also study some geometric properties of $m(\phi)$.

2. Main Results

Theorem 1. *Let Q be a bounded subset of $m(\phi)$. Then*

$$\chi(Q) = \lim_{k\to\infty} \sup_{x \in Q}\Big(\sup_{s > k} \sup_{\tau \in \mathcal{C}_s} \frac{1}{\phi_s}\sum_{n \in \tau}|x_n|\Big), \tag{1.1}$$

Proof: Let us define the operator $P_k : m(\phi) \to m(\phi)$ by $P_k(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$ for $(x_1, x_2, \ldots) \in m(\phi)$. Then clearly

$$Q \subset P_kQ + (I - P_k)Q. \tag{1.2}$$

It follows from (1.2) and the basic properties of χ that

$$\begin{aligned}\chi(Q) &\leqslant \chi(P_kQ)+\chi\big((I-P_k)Q\big)=\chi\big((I-P_k)Q\big)\\ &\leqslant \operatorname{diam}\big((I-P_k)Q\big)=\sup_{x\in Q}\big\|(I-P_k)x\big\|,\end{aligned}\tag{1.3}$$

where

$$\big\|(I-P_k)x\big\|=\sup_{s>k}\sup_{\tau\in C_s}\frac{1}{\phi_s}\sum_{n\in\tau}|x_n|.$$

So we have

$$\chi(Q)\leqslant \lim_{k\to\infty}\sup_{x\in Q}\big\|(I-P_k)x\big\|.\tag{1.4}$$

Conversely, let $\varepsilon>0$ and $\{z_1,z_2,\dots,z_j\}$ be a $[\chi(Q)+\varepsilon]$-net of Q. Then

$$Q\subset\{z_1,z_2,\dots,z_j\}+\big[\chi(Q)+\varepsilon\big]B\big(m(\phi)\big).\tag{1.5}$$

Hence

$$\sup_{x\in Q}\big\|(I-P_k)x\big\|\leqslant\sup_{1\leqslant i\leqslant j}\big\|(I-P_k)z_i\big\|+\big[\chi(Q)+\varepsilon\big].\tag{1.6}$$

Finally, (1.6) implies that

$$\lim_{k\to\infty}\sup_{x\in Q}\big\|(I-P_k)x\big\|\leqslant\chi(Q)+\varepsilon.\tag{1.7}$$

Since ε is arbitrary, (1.4) and (1.7) yield (1.1). ∎

THEOREM 2. *The space $m(\phi)$ has the weak Banach-Saks property.*

PROOF. Let $\{\varepsilon_n\}$ be a sequence of positive numbers for which $\sum_{n=1}^{\infty}\varepsilon_n\leqslant 1/2$. Let $\{x_n\}$ be a weakly null sequence in $B\big(m(\phi)\big)$. Set $x_0=0$ and $z_1=x_1$. Then there exists $s_1\in\mathbb{N}$ such that

$$\Big\|\sum_{i\in\tau_1}z_1(i)e_i\Big\|_{m(\phi)}<\varepsilon_1,$$

where τ_1 consist of the elements of σ which exceed s_1.

Since $x_n\overset{\omega}{\to}0$ implies $x\to0$ coordinatewise, there is an $n_2\in\mathbb{N}$ such that

$$\Big\|\sum_{i=1}^{s_1}x_n(i)e_i\Big\|_{m(\phi)}<\varepsilon_1,$$

when $n\geqslant n_2$. Set $z_2=x_{n_2}$. Then there exists a $s_2>s_1$ such that

$$\Big\|\sum_{i\in\tau_2}z_2(i)e_i\Big\|_{m(\phi)}<\varepsilon_2,$$

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where τ_2 consist of all elements of σ which exceed s_2. Again using the fact $x_n \to 0$ coordinatewise, there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=1}^{s_2} x_n(i) e_i \right\|_{m(\phi)} < \varepsilon_2,$$

when $n \geqslant n_3$.

Continuing this process, we can find two increasing sequences $\{s_i\}$ and $\{n_i\}$ such that

$$\left\| \sum_{i=1}^{s_j} x_n(i) e_i \right\|_{m(\phi)} < \varepsilon_j, \text{ for each } n \geqslant n_{j+1},$$

and

$$\left\| \sum_{i \in \tau_j} z_i(i) e_i \right\|_{m(\phi)} < \varepsilon_j,$$

where $z_i = x_{n_j}$ and τ_j consist of elements of σ which exceed s_j.

Since $\varepsilon_{j-1} + \varepsilon_j < 1$, we have

$$\frac{1}{\phi_s} \sum_{n \in \sigma} |z_j(n)| \leqslant \varepsilon_{j-1} + \varepsilon_j < 1,$$

for all $j \in \mathbb{N}$. Hence

$$\begin{aligned}
\left\| \sum_{j=1}^{n} z_j \right\|_{m(\phi)} &= \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{s_{j-1}} z_j(i) e_i + \sum_{i=s_{j-1}+1}^{s_j} z_j(i) e_i + \sum_{i \in \tau_j} z_j(i) e_i \right) \right\|_{m(\phi)} \\
&\leqslant \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{s_{j-1}} z_j(i) e_i \right) \right\|_{m(\phi)} + \left\| \sum_{j=1}^{n} \left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i) e_i \right) \right\|_{m(\phi)} \\
&\qquad + \left\| \sum_{j=1}^{n} \left(\sum_{i \in \tau_j} z_j(i) e_i \right) \right\|_{m(\phi)}, \\
&\leqslant \left\| \sum_{j=1}^{n} \left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i) e_i \right) \right\|_{m(\phi)} + 2 \sum_{j=1}^{n} \varepsilon_j,
\end{aligned}$$

and

$$\left\| \sum_{j=1}^{n} \left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i) e_i \right) \right\|_{m(\phi)} \leqslant \sum_{j=1}^{n} \left\| \sum_{i \in \tau_j} z_j(i) e_i \right\|_{m(\phi)} < \sum_{j=1}^{n} \varepsilon_j.$$

Therefore

$$\left\| \sum_{j=1}^{n} z_j \right\|_{m(\phi)} \leqslant 3 \sum_{j=1}^{n} \varepsilon_j,$$

and

$$\left\| \frac{1}{n} \sum_{j=1}^{n} z_j \right\|_{m(\phi)} \leqslant \frac{3}{n} \sum_{j=1}^{n} \varepsilon_j \to 0 \quad (n \to \infty).$$

This completes the proof of the theorem. ∎

THEOREM 3. *For the Banach space $X = m(\phi)$,*

$$C(X) = 2.$$

PROOF: Let us consider a sequence $u = (u_1, u_2, \ldots, u_s, 0, 0, \ldots)$ such that $|u_1| + |u_2| + \cdots + |u_s| = \phi_s$. Then $\Delta\phi_s = \phi_s - \phi_{s-1} = |u_s|$. Therefore $u \in m(\phi)$, since $\Delta\phi \in m(\phi)$, (see [**7**, Lemma 6]). Further $\|u\|_{m(\phi)} = 1$. Define

$$x_n = (\underbrace{0, 0, \ldots, 0}_{sn}, u_1, u_2, \ldots, u_s, 0, \ldots)$$

$(n \in \sigma)$. Then $x_n \overset{\omega}{\to} 0$ and

$$\|x_k + x_l\|_{m(\phi)} = 2\|u\|_{m(\phi)} = 2,$$

that is, $\|x_k + x_l\|_{m(\phi)} = 2 (k \neq l)$ implies $A(\{x_n\}) = 2$. Hence $C(m(\phi)) \geqslant 2$. Further, $\|x_k - x_l\|_{m(\phi)} = 2$, and hence $D(m(\phi)) \leqslant 2$. Since $C(m(\phi)) \leqslant D(m(\phi)) \leqslant 2$, we conclude that $C(m(\phi)) = 2$. ∎

REMARK. In [**3**], it was shown that any Banach space X with $C(X) < 2$ has the weak Banach-Saks property. Our Theorems 2 and 3 show that the converse of this statement need not be true; that is, if a Banach space X has the weak Banach-Saks property then $C(X) < 2$ is not necessarily true.

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