MINIMAL DEPENDENT SETS

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1. Introduction

The subject matter of this note is the notion of a dependence structure on an abstract set. There are a number of different approaches to this topic and it is known that many of these lead to precisely the same structure [1]. Axioms are given here to specify the minimal dependent sets for such a structure. They are closely related to conditions introduced by Hassler Whitney in [1] and to a certain "elimination axiom" given by A. P. Robertson and J. D. Weston in [2]. Theorem 1 shows that a dependence structure may equally well be defined by means of axioms for the independent sets. Axiom (I) is adapted from a condition due to R. Rado [3]. Theorem 2 links our minimal dependent sets with Whitney's "circuits". Theorem 3 is an "elimination" theorem which generalizes the statement of our axiom (C_2). Theorem 4 is due to Rado ([3], Theorem 3, p. 307), and A. W. Ingleton [4]. It is shown here to follow from Theorem 3.

2. Axioms and theorems

Let X be a set. Let $\mathscr{C}$ be a set of non-empty finite subsets of X. Furthermore, let $\mathscr{C}$ satisfy the following two conditions.

(C_1) *No proper subset of a member of* $\mathscr{C}$ *is a member of* $\mathscr{C}$.

(C_2) *If* E *and* F *are distinct members of* $\mathscr{C}$ *and* $x \in E \cap F$, *then* $E \cup F$ *has a subset belonging to* $\mathscr{C}$ *but not containing* x.

Axiom (C_2) is the "elimination axiom" of Robertson and Weston [2] who apply it to a set $\mathscr{R}$ of finite subsets of a set X and use no other condition. For the case where the empty set is not a member of $\mathscr{R}$, it can be seen that the members of $\mathscr{C}$ are precisely the minimal members of $\mathscr{R}$, for it is clear that the "elimination axiom" must hold with $\mathscr{R}$ replaced by the set of its minimal members. In [2] the authors define "pure sets" as those non-empty subsets of X which fail to contain members of $\mathscr{R}$. Here we define independent sets to be those subsets of X which fail to contain

[1] A study of the various axioms for a dependence structure formed the topic of a M. Sc. thesis by the author at Monash University.

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members of $\mathscr{C}$. The "pure sets" are then precisely the non-empty independent sets. The following theorem characterizes these independent sets.

THEOREM 1. *A set $\mathscr{U}$ of subsets of a set X is the set of independent sets defined by a set $\mathscr{C}$ of non-empty finite subsets of X satisfying* (C_1) *and* (C_2) *if, and only if $\mathscr{U}$ is non-empty, $\mathscr{U}$ has the inductive property and $\mathscr{U}$ satisfies the following condition.*

(I) *If A and B are subsets of X such that $A \notin \mathscr{U}$, $B \notin \mathscr{U}$, and $A \cap B \in \mathscr{U}$, then for all elements $x \in A \cup B$ it follows that $(A \cup B)\backslash\{x\} \notin \mathscr{U}$.*

PROOF. Firstly, let $\mathscr{U}$ be the set of independent sets. Then $\mathscr{U}$ is non-empty since $\square \in \mathscr{U}$, where $\square$ is the empty set. Now if $A \in \mathscr{U}$, then clearly every subset of A belongs to $\mathscr{U}$. If $A \notin \mathscr{U}$, then a member of $\mathscr{C}$ contained in A is a finite subset of A which fails to belong to $\mathscr{U}$. Thus $\mathscr{U}$ has the inductive property. To verify condition (I) let $A \notin \mathscr{U}$, $B \notin \mathscr{U}$, $A \cap B \in \mathscr{U}$, $x \in A \cup B$. Then there exist sets $C \in \mathscr{C}$, $D \in \mathscr{C}$ such that $C \subseteqq A$, $D \subseteqq B$. Also $C \neq D$, since $A \cap B \in \mathscr{U}$. If $x \notin C \cap D$, then either $C \subseteqq (A \cup B)\backslash\{x\}$ or $D \subseteqq (A \cup B)\backslash\{x\}$ and so $(A \cup B)\backslash\{x\} \notin \mathscr{U}$. If $x \in C \cap D$, then from (C_2) there exists $E \in \mathscr{C}$ with $E \subseteqq (C \cup D)\backslash\{x\} \subseteqq (A \cup B)\backslash\{x\}$ so again $(A \cup B)\backslash\{x\} \notin \mathscr{U}$. Thus condition (I) holds. One may observe that the set $\mathscr{C}$ is precisely the set of subsets of X minimal with respect to not belonging to $\mathscr{U}$.

Secondly, let $\mathscr{U}$ be a set of subsets of X having the properties stated in the theorem, and let $\mathscr{C}$ be the set of subsets of X minimal with respect to not belonging to $\mathscr{U}$. Since $\mathscr{U}$ is non-empty and possesses the inductive property, it follows that $\square \in \mathscr{U}$. Hence $\square \notin \mathscr{C}$. Also from the inductive property, any subset not in $\mathscr{U}$ contains a finite subset not in $\mathscr{U}$. Hence $\mathscr{C}$ consists of non-empty finite sets. That (C_1) holds is clear from the definition of $\mathscr{C}$. In order to verify (C_2), let $E \in \mathscr{C}$, $F \in \mathscr{C}$, $E \neq F$, and $x \in E \cap F$. Then $E \notin \mathscr{U}$, $F \notin \mathscr{U}$ and also $E \cap F \in \mathscr{U}$ because $E \cap F$ is a proper subset of E. It follows from condition (I) that $(E \cup F)\backslash\{x\} \notin \mathscr{U}$. Then since it is a finite set, we have that $(E \cup F)\backslash\{x\}$ must contain a member of $\mathscr{C}$. To complete the proof of the theorem one obseıves that the set $\mathscr{U}$ consists precisely of those subsets of X which fail to contain members of $\mathscr{C}$.

In the paper [1], Whitney uses the following axiom (C'_2) together with (C_1) and refers to the members of $\mathscr{C}$ by the name "circuits". Also he restricts attention to the case where the set X is finite.

(C'_2) *If E and F are distinct members of $\mathscr{C}$, if $x \in E \cap F$ and if $y \in E\backslash F$, then $E \cup F$ has a subset belonging to $\mathscr{C}$ which contains y but fails to contain x.*

Since (C'_2) seems to impose a stronger condition on the set $\mathscr{C}$ than (C_2), the following theorem may be of some interest. In any case it provides the link between the two systems.

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THEOREM 2. *If $\mathscr{C}$ is a set of non-empty finite subsets of a set X, then conditions* (C_1) *and* (C_2) *together are equivalent to conditions* (C_1) *and* (C_2') *together.*

PROOF. It is clear that (C_2) follows from (C_2'). Suppose now that (C_1) and (C_2) hold but that (C_2') fails to hold and let m be the least integer such that for some pair of sets E and F belonging to $\mathscr{C}$ and satisfying $|E \cup F| = m$ there exist elements $x \in E \cap F$ and $y \in E \backslash F$ such that $E \cup F$ contains no member G of $\mathscr{C}$ satisfying $y \in G$ and $x \notin G$. By supposition such an integer exists, and we may assume that E, F, x and y have the stated properties. Then by (C_2) there exists a subset G of $E \cup F$ belonging to $\mathscr{C}$ and failing to contain x. But then $y \notin G$. From (C_1) we may choose $z \in G \backslash E$, and then using the fact that $|G \cup F| < m$ and the minimality of m we may apply (C_2') to the sets G and F and the elements $z \in G \cap F$ and $x \in F \backslash G$. Thus there exists a subset H of $G \cup F$ belonging to $\mathscr{C}$ and containing x but not containing z. But then $|E \cup H| < m$ since $z \notin E \cup H$ and we may apply (C_2') to the sets E and H and the elements $x \in E \cap H$ and $y \in E \backslash H$ to show the existence of a subset J of $E \cup H$ belonging to $\mathscr{C}$, containing y and failing to contain x. This, however, is a contradiction.

THEOREM 3. *If $A_1, A_2, \cdots, A_n$ are members of $\mathscr{C}$, if $n \geqq 2$ and if*

$$A_i \nsubseteq \bigcup \{A_j : j < i\}, \qquad i = 2, \cdots, n$$

holds, then for each subset B of X with $|B| = r < n$, there exist members $C_1, C_2, \cdots, C_{n-r}$ of $\mathscr{C}$ such that

$$C_i \subseteq \bigcup \{A_k : k = 1, \cdots, n\} \backslash B \text{ and}$$
$$C_i \nsubseteq \bigcup \{C_j : j \neq i\} \text{ hold for } i = 1, 2, \cdots, n-r.$$

PROOF. Let (n, r) denote the case of the theorem for which n members of the set $\mathscr{C}$ are considered and the set B consists of r elements.

Case $(n, 0)$. By the hypothesis and since $\square \notin \mathscr{C}$ we may choose $x_1 \in A_1$ and elements $x_i \in A_i \backslash \bigcup \{A_j : j < i\}$ for $i = 2, \cdots, n$. The following sets $B_{i,j}$ are defined for the indices $j = i+1, \cdots, n$ and $i = 1, 2, \cdots, n$. For $j = 2, \cdots, n$, if $x_1 \notin A_j$ put $B_{1,j} = A_j$ and if $x_1 \in A_j$ using (C_2') choose $B_{1,j} \in \mathscr{C}$ such that $B_{1,j} \subseteq A_1 \cup A_j$, $x_1 \notin B_{1,j}$, and $x_j \in B_{1,j}$. Then by induction, if $x_i \notin B_{i-1,j}$ define $B_{i,j} = B_{i-1,j}$ and if $x_i \in B_{i-1,j}$ choose $B_{i,j} \in \mathscr{C}$ such that $B_{i,j} \subseteq B_{i-1,i} \cup B_{i-1,j}$, $x_i \notin B_{i,j}$ and $x_j \in B_{i,j}$ where the choice is made possible by (C_2') and by the fact that $x_i \in B_{i-1,j} \cap B_{i-1,i}$ and $x_j \in B_{i-1,j} \backslash B_{i-1,i}$. Then let $C_1 = A_1$, $C_2 = B_{1,2}, \cdots, C_n = B_{n-1,n}$. Then for $i = 1, \ldots, n$ we have $x_i \in C_i$ and $x_i \notin \bigcup \{C_j : j = i+1, \cdots, n\}$. Also one observes that for $i = 2, \cdots, n$ we have $x_i \notin \bigcup \{C_j : j < i\}$. Thus for $i = 1, \cdots, n$ the relation $C_i \nsubseteq \bigcup \{C_j : j \neq i\}$ holds. Finally it is clear that

$C_i \subseteq \bigcup \{A_k : k = 1, \cdots, n\}$ must also hold and hence the sets $C_1, \cdots, C_n$ satisfy the requirements of the theorem in this case.

Case (n, r). This case of the theorem is shown to depend on the case $(n-1, r-1)$ and so ultimately on the case $(n-r, 0)$ which is a case already proved. Let $B \subseteq X$ and $|B| = r$. By the case $(n, 0)$ it may be assumed that the sets $A_1, \cdots, A_n$ already satisfy the relation $A_i \nsubseteq \bigcup \{A_j : j \neq i\}$. Let $x \in B$. If $x \notin \bigcup \{A_i : i = 1, \cdots, n\}$, the case reduces immediately to the case $(n-1, r-1)$. Otherwise, by the symmetry it may be assumed that $x \in A_n$. Now choose elements $y_i \in A_i \backslash \bigcup \{A_j : j \neq i\}$ for $i = 1, 2, \cdots, n-1$. Using (C_2') where necessary, there exist members $B_1, B_2, \cdots, B_{n-1}$ of $\mathscr{C}$ such that for $j = 1, 2, \cdots, n-1$ we have $B_j \subseteq A_n \cup A_j$, $x \notin B_j$ and $y_j \in B_j$. One then notices that $y_j \notin \bigcup \{B_i : i \neq j\}$ and so $B_j \nsubseteq \bigcup \{B_i : i \neq j\}$ for $j = 1, \cdots, n-1$. Applying the theorem for the case $(n-1, r-1)$ to these members $B_1, \cdots, B_{n-1}$ of $\mathscr{C}$ and to the set $B \backslash \{x\}$ we obtain members $C_1, C_2, \cdots, C_{n-r}$ of $\mathscr{C}$ having the required properties.

THEOREM 4. *If $A_1, A_2, \cdots, A_n$ are subsets of X which do not belong to the set $\mathscr{U}$ of independent sets, and if for $i = 2, 3, \cdots, n$ the sets $A_i \cap \bigcup \{A_j : j < i\}$ belong to $\mathscr{U}$, then for any subset B of X with $|B| < n$ it follows that $\bigcup \{A_i : i = 1, \cdots, n\} \backslash B$ is not a member of $\mathscr{U}$.*

PROOF. Let $B_1, B_2, \cdots, B_n$ be minimal dependent sets (i.e. members of $\mathscr{C}$) contained in $A_1, A_2, \cdots, A_n$ respectively. Then for $i = 1, 2, \cdots, n$ we have

$$B_i \cap \bigcup \{B_j : j < i\} \subseteqq A_i \cap \bigcup \{A_j : j < i\}.$$

Then since $B_i \notin \mathscr{U}$ it follows that $B_i \nsubseteq \bigcup \{B_j : j < i\}$ for $i = 2, 3, \cdots, n$. By theorem 3, if $|B| = r$, there are $(n-r) > 0$ minimal dependent sets contained in $\bigcup \{A_i : i = 1, 2, \cdots, n\} \backslash B$ and hence this latter set is not a member of $\mathscr{U}$.

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