J. Austral. Math. Soc. (*Series A*) **54** (1993), 304–320

ON MINIMAL FITTING CLASSES OF PERIODIC, INFINITE GROUPS

BRENDAN McCANN

(Received 15 November 1990; revised 31 May 1991)

Communicated by H. Lausch

Abstract

Examples of Fitting classes generated by certain periodic infinite groups are presented in this paper. The groups in question are finite extensions of direct products of quasi-cyclic (Prüfer) groups and are, in particular, Černikov groups. The methods applied follow closely those used for constructing Fitting classes of finite groups, especially in the case of nilpotent length three.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 F 16, 20 F 17, 20 D 35.

Introduction

A Fitting class of finite groups is a set, $\mathfrak{X}$, of finite groups with the following properties:

(a) If $G \in \mathfrak{X}$ and $H \cong G$ then $H \in \mathfrak{X}$;
(b) If $N \trianglelefteq G \in \mathfrak{X}$ then $N \in \mathfrak{X}$;
(c) If $G = G_1 G_2$ with $G_i \trianglelefteq G$ and $G_i \in \mathfrak{X}$ then $G \in \mathfrak{X}$.

Minimal Fitting classes of finite groups of nilpotent length three have been studied in, among other papers, Bryce [3], Bryce, Cossey and Ormerod [4], Dark [5] and the present author in [7, 8, 9]. The construction of Dark in [5] has proved to be a basic source of inspiration for most other minimal Fitting class constructions.

In this paper we examine Fitting classes based on certain infinite groups. The groups in question are elements of the class $\mathfrak{C}$ of soluble Černikov groups. In particular, they are periodic (that is, every element has finite

https://doi.org/10.1017/S1446788700031803 Published online by Cambridge University Press

order) and it is to be noted emphatically that their structure is such that the methods of [3, 5, 8] may be applied in an analogous, albeit at times suitably modified, fashion.

The class $\mathfrak{C}$ is a subclass of the class of $\mathfrak{S}_1$-groups, for which a theory of injectors has been developed in, for instance, Beidleman, Karbe and Tomkinson [1], Beidleman and Tomkinson [2] and Menegazzo and Newell [10]. Following Menegazzo and Newell, we define a Fitting class of $\mathfrak{C}$-groups to be a subclass $\mathfrak{X}$ of $\mathfrak{C}$ such that:

(i) If $N \trianglelefteq G \in \mathfrak{X}$ then $N \in \mathfrak{X}$;
(ii) If G is an element of $\mathfrak{C}$ such that $G = \langle N_k | k \in K \rangle$, where $N_k \trianglelefteq G$ and $N_k \in \mathfrak{X}$ for all k in K, then $G \in \mathfrak{X}$.

Note that, since subgroups and factor groups of Černikov groups are also Černikov groups, the class $\mathfrak{C}$ itself is a Fitting class of $\mathfrak{C}$-groups (which is closed with respect to factor groups). We note also that Beidleman, Karbe and Tomkinson use a different definition of a Fitting class of $\mathfrak{S}_1$-groups, in that the normality conditions in (i) and (ii) are replaced by analogous ascendancy conditions. The above definition is used in this paper since it mirrors more precisely the definition of a finite Fitting class given above.

For a given $\mathfrak{C}$-group, G, we define $\mathfrak{Fit}(G)$ by:

$$\mathfrak{Fit}(G) = \bigcap_{G \in \mathfrak{X}\text{, a Fitting class of } \mathfrak{C}\text{-groups}} \mathfrak{X} .$$

Thus $\mathfrak{Fit}(G)$ is a Fitting class of $\mathfrak{C}$-groups which contains G and is clearly the minimal such class. In what follows the term Fitting class will be used, for the sake of brevity, to denote a Fitting class of $\mathfrak{C}$-groups. We begin with the Fitting class generated by C_p. Section 2 then provides us with some useful definitions and results. The example of Section 3 shows that certain periodic infinite, metanilpotent Fitting classes are not as complicated as their finite counterparts, while Section 4 provides us with a Fitting class construction similar to that of [7].

I would like to acknowledge the support and encouragement of the mathematics seminar at Waterford Regional College while writing this paper. I would also like to thank the referee for his/her comments which have helped improve this paper.

1. $\mathfrak{Fit}(C_p)$

The following well-known result provides information about Černikov soluble groups which will be used later on. For the prime p, we recall the definition of the Prüfer, or quasi-cyclic, p-group:

$$C_{p^\infty} = \langle x_1, x_2, \ldots | x_1^p = 1, \; x_{i+1}^p = x_i, \; i = 1, \ldots \rangle.$$

Proposition 1.1. *If $G \in \mathfrak{C}$ then there exists a normal subgroup $H \trianglelefteq G$ such that G/H is finite and H is divisible of finite rank, that is $H = H_1 \times \cdots \times H_t$, for a suitable t, where $H_i \cong C_{p_i^\infty}$, for suitable primes, $p_1, \ldots, p_t$. In particular, H is characteristic in G since it is the unique maximal normal divisible subgroup of G.*

Using the description of C_{p^∞} given above, we see that $C_{p^\infty} = \langle N_i | i = 1, 2, \ldots \rangle$, where $N_i = \langle x_i \rangle \cong C_{p^i}$, $i = 1, \ldots$. This is clearly a normal product and, since $C_{p^i} \in \mathfrak{Fit}(C_p)$, for $i = 1, \ldots$, we conclude that $C_{p^\infty} \in \mathfrak{Fit}(C_p)$. In addition, since $C_p \cong \langle x_1 \rangle \trianglelefteq C_{p^\infty}$, we see that $C_p \in \mathfrak{Fit}(C_{p^\infty})$. We thus conclude that $\mathfrak{Fit}(C_p) = \mathfrak{Fit}(C_{p^\infty})$. Proposition 1.2 now gives us a more detailed description of this minimal Fitting class.

Proposition 1.2. *$\mathfrak{Fit}(C_p)$ is the class of groups which satisfy:*

(i) $G = AB$;
(ii) $A \cong C_{p^\infty} \times \cdots \times C_{p^\infty}$ *with t factors for a suitable t (with possibly $t = 0$, that is $G = B$)*;
(iii) *B is a finite p-group*;
(iv) $[A, B] = 1$.

(Thus G is the central product of a finite p-group with the direct product of finitely many copies of C_{p^∞}.)

Proof. Let $\mathfrak{X}$ be the class of groups described above. Since $\mathfrak{Fit}(C_p) = \mathfrak{Fit}(C_{p^\infty})$, and since it is well known that all finite p-groups are in $\mathfrak{Fit}(C_p)$, we see that each element of $\mathfrak{X}$ is a central, whence normal, product of elements of $\mathfrak{Fit}(C_p)$. Thus we see $C_p \in \mathfrak{X} \subseteq \mathfrak{Fit}(C_p)$, and it remains to show that $\mathfrak{X}$ is a Fitting class.

First we demonstrate closure with respect to normal products. Let G be an $\mathfrak{C}$-group with $G = \langle N_k | k \in K \rangle$, where $N_k \in \mathfrak{X}$ and $N_k \trianglelefteq G$ for each $k \in K$. Then G is clearly a p-group. Assuming that G is not finite, we see by Proposition 1.1 that there exists $H \trianglelefteq G$ such that $H = H_1 \times \cdots \times H_t$ (t finite), with $H_i \cong C_{p^\infty}$, $i = 1, \ldots, t$, and G/H is finite. Thus there exists a finite subset $S \subseteq K$ such that:

$$G = H \langle N_s | s \in S \rangle.$$

We let $N_s = A_s B_s$, where A_s and B_s are as in the statement of the proposition. Now, $A_s/(A_s \cap H) \cong A_s H/H$ which is isomorphic to a subgroup

https://doi.org/10.1017/S1446788700031803 Published online by Cambridge University Press

of G/H, which is finite. Since A_s is divisible, it has no non-trivial finite factor groups, so we conclude that $A_s \leq H$. If we let p^m be the exponent of the finite group B_s, then $D_s = \langle n \in N_s | o(n)|p^m \rangle$ is a finite characteristic subgroup of N_s, so H normalises D_s. Since $H/C_H(D_s)$ is isomorphic to a subgroup of $\mathrm{Aut}(D_s)$, which is finite, and since H has no non-trivial finite factor groups, we conclude that H centralises D_s. We let $B = \langle D_s | s \in S \rangle$. Then B is a finite normal subgroup of G such that $[B, H] = 1$ and $G = BH$. We conclude that G is an element of $\mathfrak{X}$.

To demonstrate closure with respect to normal subgroups, we let $G = AB$ be as in the statement of the proposition and let $N \trianglelefteq G$. Applying Proposition 1.1, we let $A_1 \trianglelefteq N$ be such that N/A_1 is finite and A_1 is divisible of finite rank. As above, $A_1 \leq A \leq Z(G)$. Since N/A_1 is finite, there exists a finite set $\{n_1, \dots, n_f\} \subseteq N$ with $N = A_1\langle n_1, \dots, n_f \rangle$. We let $B_1 = \langle n_1, \dots, n_f \rangle$. By Robinson [11, 5.4.11], B_1 is finite. Thus $N = A_1 B_1$, where A_1 and B_1 satisfy (ii), (iii) and (iv).

The following facts about $\mathfrak{Fit}(C_p)$ may be of interest.

Remark 1.3. $\mathfrak{Fit}(C_p)$ is closed with respect to factor groups and subgroups.

Proof. Let $G = AB$ be as in Proposition 1.2. If $N \trianglelefteq G$, then $G/N = (AN/N)(BN/N)$. Now $AN/N \cong A/(N \cap A)$ is divisible, since it is isomorphic to a factor group of the divisible group A. Also BN/N is clearly finite, so we see that $G/N \in \mathfrak{Fit}(C_p)$. In addition if U is any subgroup of G then U is subnormal in G, since G is nilpotent. Thus, by (sub)normal subgroups, U is also an element of $\mathfrak{Fit}(C_p)$.

2. Preliminary results

Proposition 2.1. *Let p, q and r be distinct prime numbers and let $\mathfrak{T}$ be the class of $\mathfrak{C}$-groups G which satisfy:*

(i) *G has a normal subgroup N with $N \in \mathfrak{Fit}(C_r)$;*
(ii) *G has as normal subgroup M with $M/N \in \mathfrak{Fit}(C_q)$;*
(iii) *$G/M \in \mathfrak{Fit}(C_p)$.*

Then $\mathfrak{T}$ is a Fitting class of $\mathfrak{C}$-groups which is closed with respect to factor groups.

The proof of Proposition 2.1 follows more or less by definition, along with Remark 1.3. The groups we will be dealing with will mainly be in the universe $\mathfrak{T}$, for some suitable primes p, q and r.

https://doi.org/10.1017/S1446788700031803 Published online by Cambridge University Press

DEFINITION 2.2. If G is a periodic group and π is a set of primes then

$$0^{\pi}(G) = \langle g \in G | g \text{ is a } \pi'\text{-element} \rangle .$$

(If π contains just one prime p, then it is customary to use the notation $0^{p}(G)$).

PROPOSITION 2.3. *Let G be a periodic group. Then*

(i) *If $M \trianglelefteq G$ is such that G/M is a π-group, then $0^{\pi}(G) \leq M$;*
(ii) *$0^{\pi}(G)$ is a characteristic subgroup of G;*
(iii) *If $N \trianglelefteq G$ then $0^{\pi}(N) \trianglelefteq 0^{\pi}(G)$;*
(iv) *If $G = \langle G_k | k \in K \rangle$ where, for all $k \in K$, $G_k \trianglelefteq G$, then $0^{\pi}(G) = \langle 0^{\pi}(G_k) | k \in K \rangle$.*

These straightforward properties of $0^{\pi}(G)$ will be used without comment in the rest of the paper.

DEFINITION 2.4. For the group G we define $\overline{G}$ by:

$$\overline{G} = \langle N \trianglelefteq G | [G, N] = N \rangle .$$

$\overline{G}$ is a characteristic subgroup of G and, if G is finite, $\overline{G}$ coincides with the nilpotent residual.

PROPOSITION 2.5. (i) *If $M \trianglelefteq G$ is such that G/M is nilpotent then $\overline{G} \leq M$;*

(ii) *$\overline{(G/\overline{G})} = 1_{G/\overline{G}}$;*
(iii) *If $H \leq G$ then $\overline{H} \leq \overline{G}$.*

LEMMA 2.6. *If G is a soluble periodic group and π is a set of primes, then $G/\overline{G} = P/\overline{G} \times Q/\overline{G}$, where $P/\overline{G}$ is a π-group and $Q/\overline{G}$ is a π'-group.*

PROOF. For notational convenience we assume, by Proposition 2.5(ii), that $\overline{G} = 1$. Let x_1 and x_2 be any π-elements of G. Then $\langle x_1, x_2 \rangle$ is a finitely generated, periodic, soluble group. By Robinson [11, 5.4.11], $\langle x_1, x_2 \rangle$ is finite. Now, $\overline{\langle x_1, x_2 \rangle} \leq \overline{G} = 1$, so $\langle x_1, x_2 \rangle$ is nilpotent. In particular $\langle x_1, x_2 \rangle$ is a π-subgroup of G, so $x_1 x_2$ is a π-element. We let $P = \{g \in G | g$ is a π-element$\}$ and $Q = \{g \in G | g$ is a π'-element$\}$. From above we see that P and Q are, respectively, a characteristic π-subgroup and a characteristic π'-subgroup of G and, since G is periodic, we conclude that $G = P \times Q$.

The next result is an application of Proposition 2.6.

LEMMA 2.7. *Let* $\mathfrak{T}$ *be the class described in Proposition* 2.1. *Then if* $G \in \mathfrak{T}$, $0^p[0^{p'}(G)] = 0^{p'}(G)$.

The proposition which follows is, like Lemma 2.7, a restatement of a result for finite groups. The proof is the same as that of [7, II.6].

PROPOSITION 2.8. *If* $\overline{G} = AB$, *where* A *and* B *are normal subgroups of* G *and both* A *and* B *satisfy the minimal condition for normal subgroups, then, for* $N \trianglelefteq G$, $\overline{N} = (\overline{N} \cap A)(\overline{N} \cap B)$.

The final results of this section are standard results from finite group theory which are adapted to cover some particular infinite cases.

PROPOSITION 2.9 (cf. Corollary to the theorem of Krull-Remak-Schmidt, Huppert [6, I.12.6]). *If* $G = G_1 \times \cdots \times G_n$, *where* $1 \neq G_i$ *is directly indecomposable and* $Z(G_i) = 1$ $(i = 1, \ldots, n)$, *then, apart from reordering the factors, this is the only direct decomposition of* G *into finitely many non-trivial, indecomposable factors. In particular, if* $\sigma \in \mathrm{Aut}(G)$, *then* $G_i^\sigma \in \{G_1, \ldots, G_n\}$ $(i = 1, \ldots, n)$.

PROOF. Let $G = A_1 \times \cdots \times A_m$ be a decomposition of G into finitely many non-trivial, indecomposable factors. Since $Z(G) = 1$, we have $Z(A_j) = 1$, $j = 1, \ldots, m$. We let $\pi_j \colon G \to A_j$ be the projection of G onto A_j and let $W = G_2 \times \cdots \times G_n$. Then $A_j = G_1^{\pi_j} W^{\pi_j}$. Now, $G_1^{\pi_j}$ and W^{π_j} commute elementwise so, since $Z(A_j) = 1$, we have $G_1^{\pi_j} \cap W^{\pi_j} = 1$. Thus $A_j = G_1^{\pi_j} \times W^{\pi_j}$ and, since A_j is directly indecomposable, we have either $G_1^{\pi_j} = 1$ and $W^{\pi_j} = A_j$ or $G_1^{\pi_j} = A_j$ and $W^{\pi_j} = 1$.

We assume the indices to be such that $G_1^{\pi_j} = A_j$, $j = 1, \ldots, s$ and $G_1^{\pi_j} = 1$, $j = s + 1, \ldots, m$. Then we have:

$$G_1 \leq A_1 \times \cdots \times A_s \quad \text{and} \quad W \leq A_{s+1} \times \cdots \times A_m,$$

whence, in fact, $G_1 = A_1 \times \cdots \times A_s$ and $W = A_{s+1} \times \cdots \times A_m$. Since G_1 is directly indecomposable we now have $G_1 = A_1$ and sufficient repetition of this argument will complete the proof in a finite number of steps.

PROPOSITION 2.10. (cf. Theorems of Zassenhaus, Huppert [6, I.18.1, 2]). *Let* G *be an* $\mathfrak{C}$*-group which has a normal* π*-subgroup* N *such that* G/N *is a* π'*-group. Then* N *has a complement in* G *and any two complements for* N *in* G *are conjugate in* G.

PROOF. Applying Proposition 1.1, we let $H \trianglelefteq G$ be such that G/H is finite and H is divisible of finite rank. We see that $H = P \times Q$, where P is a π-group and Q is a π'-group. We now work modulo Q, that is we let $G^* = G/Q$ and $N^* = NQ/N(\cong N)$. Then G^*/N^* is a finite π'-group. We let $G^*/N^* = \{g_1N^*, \dots, g_kN^*\}$, where k is a suitable natural number. Then, by Robinson [11, 5.4.11], $W = \langle g_1, \dots, g_k\rangle$ is a finite group. In addition, $G^* = WN^*$. $W \cap N^*$ is a π-group and $W/(W \cap N^*)$ is isomorphic to G^*/N^*, which is a π'-group so by Huppert [6, I.18.1], $W \cap N^*$ has a complement K^* in W. Clearly K^* is a complement for N^* in G^*. We let $K^* = K/Q$ and see that K is a complement for N in G.

Now let K_1 and K_2 be any two complements for N in G. Since the K_i are maximal π'-subgroups of G, $Q \leq K_i$ $(i = 1, 2)$. We let $K_i^* = K_i/Q$ and see that K_1^* and K_2^* are complements for N^* in G^* and, in particular, are finite. As above, we see that $D = \langle K_1^*, K_2^*\rangle$ is also finite. Thus K_1^* and K_2^* are both complements for $D \cap N^*$ in the finite group D, so by Huppert [6, I.18.12], K_1^* and K_2^* are conjugate in D and hence in G^*. It follows that K_1 and K_2 are conjugate in G.

3. A metanilpotent Fitting class

PROPOSITION 3.1. *Let q and r be distinct primes and let $\mathfrak{X}$ be the class of groups which satisfy the following conditions:*

(i) *G has a normal subgroup, M, such that $M \in \mathfrak{Fit}(C_r)$;*
(ii) *$G/M \in \mathfrak{Fit}(C_q)$;*
(iii) *$0^q[0^{q'}(G)] = A_1 \times \cdots \times A_s$, ($s = 0$ means that $0^q[0^{q'}(G)] = 1$);*
(iv) *$A_i \cong C_{r^\infty}$, $i = 1, \dots, s$.*

Then $\mathfrak{X}$ is a Fitting class of $\mathfrak{C}$-groups.

(For an example, see the group $U\langle y\rangle$ of the next section).

PROOF. We note that the groups which satisfy (i) and (ii) form a Fitting class, so we shall concentrate on (iii) and (iv). We first demonstrate closure with respect to normal subgroups. Let $N \trianglelefteq G \in \mathfrak{X}$ and let $W = 0^q[0^{q'}(N)]$. Then $W \leq 0^q[0^{q'}(G)]$, so by Robinson [11, 4.2.11], $W = B_1 \times \cdots \times B_t \times D$, say, where $B_i \cong C_{r^\infty}$ and D is an finite abelian r-group. Let Q be a complement to W in $0^{q'}(N)$. Note that $0^{q'}(N)/W$ is a q-group, so such a complement exists by Proposition 2.10. From Proposition 1.2 and the proof of Proposition 2.10, we see that Q is of the form $Q = Q_1Q_2$, where Q_1 is a finite Q-group and Q_2 is a divisible q-group with $Q_2 \leq Z[0^{q'}(N)]$. If we

https://doi.org/10.1017/S1446788700031803 Published online by Cambridge University Press

assume $W \neq 1$, then we also have $Q_1 \neq 1$, since $0^{q'}(N) = QW$. We may assume, in fact, $D \neq 1$, since otherwise we are finished.

If D is not elementary abelian we let r^k be the exponent of D and let $R_k = \langle w \in W | w^{r^k} = 1\rangle$. Then R_k is a finite, characteristic subgroup of W and so is normal in G. We let $\Phi = \Phi(R_k)$ (the Frattini subgroup of R_k). Then Φ is normal in G and, since Φ is finite, $W/\Phi = B_1\Phi/\Phi \times \cdots \times B_t\Phi/\Phi \times D\Phi/\Phi$, where $B_i\Phi/\Phi \cong C_{r^\infty}$ and $D\Phi/\Phi$ is a non-trivial, elementary abelian r-group. Since $0^q[0^{q'}(N/\Phi)] = 0^q[0^{q'}(N)]/\Phi$, we can work modulo Φ so, for ease of notation, we assume $\Phi = 1$.

We let $B = B_1 \times \cdots \times B_t$. Then $B = \langle w^{r^k} | w \in W\rangle$ is a characteristic subgroup of W. Now let $R = \langle w \in W | o(w) = r\rangle$. Then R is a finite, normal elementary abelian r-subgroup of N. By Maschke's Theorem, Huppert [6, I.17.9], $R = (B \cap R) \times K$, where K is a complement for $B \cap R$ in R which is normalised by Q_1, and hence by Q. We see $D \cong K$ $(\neq 1)$ and we have: $W = B \times K$. Now we must have $[Q, K] = K$. Otherwise, again by Maschke's Theorem, $K = K_1 \times K_2$, where Q_1 normalises both K_1 and K_2, $K_2 \neq 1$ and $[Q_1, K_1] = 1$. Then $0^{q'}(N)/(B \cap K_2)$ is nilpotent and has a Sylow r-subgroup isomorphic to K_1. But, by definition, $0^{q'}(N)$ and all its factor groups are generated by q-elements, and a contradiction ensues.

We let $H = \langle g \in 0^q[0^{q'}(G)] | g^r \in K\rangle$. Then, by the description of $0^q[0^{q'}(G)]$ given by (iii) and (iv), H is a finite subgroup of $0^q[0^{q'}(G)]$. In addition H is normalised by Q and $H^r = K$, where $H^r = \langle h^r | h \in H\rangle$, or, equivalently, $K = \Phi(H)$. We also have $B \cap H = B \cap R$, so, again by Maschke's Theorem, we have a decomposition: $H/K = (B \cap R)K/K \times H_1K$, where H_1/K is normalised by the induced action of Q and $H_1^r = K$ (since $((B \cap R)K)^r = R^r = 1$).

Thus H_1 is normalised by Q, $\Phi(H_1) = K$, and:

$$H_1 \cap W = H_1 \cap BK = K(H_1 \cap B) = K.$$

Moreover H_1 is an r-group which is not centralised by Q so, by a result of Philip Hall (Huppert [6, III.3.18]), Q_1, and hence Q, induces a non-trivial group of automorphisms on $H_1/\Phi(H_1) = H_1/K$. It follows that, for $S = [Q, H_1]$, we have $K \lneqq S$. In addition:

$$S \leq [0^{q'}(N), G] \leq 0^{q'}(N),$$

and, by comparison of orders, $S \leq W$. This is a contradiction to the fact that $K = W \cap H_1$. Thus we conclude $D = 1$, whence $N \in \mathfrak{X}$.

To demonstrate closure with regard to normal products, we let G be an element of $\mathfrak{C}$ such that $G = \langle G_k | k \in K\rangle$, where $G_k \trianglelefteq G$ and $G_k \in \mathfrak{X}$, for all $k \in K$. We let

$$0^q[0^{q'}(G_k)] = A_{k1} \times \cdots \times A_{ks_k},$$

where the A_{kj} are isomorphic to C_{r^∞}. Thus, for $W = 0^q[0^{q'}(G)]$, we have $W = \langle 0^q[0^{q'}(G_k)] | k \in K\rangle = \langle A_{kj} | k \in K,\ j = 1, \dots, s_k\rangle$. Since, by (i), (ii) and Remark 1.3, W is an element of $\mathfrak{Fit}(C_r)$, we have, as in the proof of Proposition 1.3, $W = A_1 \times \cdots \times A_s$, for a suitable s, where $A_i \cong C_{r^\infty}$, $i = 1, \dots, s$. Thus $0^q[0^{q'}(G)]$ is of the desired form, so $G \in \mathfrak{X}$.

The following corollary is a straightforward deduction from Proposition 3.1 and Lemma 2.7.

COROLLARY 3.2. *Let q and r be distinct primes and let G be an element of the class $\mathfrak{X}$, as described in Proposition* 3.1. *Then all finite groups in $\mathfrak{Fit}(G)$ are nilpotent.* (*Equivalently, if $\mathfrak{F}$ is the class of finite groups and $\mathfrak{N}$ is the class of nilpotent groups, then $\mathfrak{Fit}(G) \cap \mathfrak{F} \subseteq \mathfrak{N}$*).

We note here that this paper deals only with finite extensions of divisible groups of finite rank. It is probable that the construction of Fitting classes similar to those dealt with in this paper, but based on, say, torsion-free extensions of divisible groups will prove to be a somewhat more difficult task.

4. Some minimal Fitting classes of nilpotent length three

For the remainder of this paper we introduce the following notation: p, q and r will be distinct primes with:

(i) $q \top (r^p - 1)$;
(ii) $q \nmid (r^n - 1)$, $0 \leq n < p$;
(iii) $p \nmid (r - 1)$.

(For example: $p = 3$, $q = 7$, $r = 107$). We let

$$U \cong \underbrace{C_{r^\infty} \times \cdots \times C_{r^\infty}}_{p}$$

and $U_i = \{u \in U | u^{r^i} = 1\}$, $i = 1, \dots$. Then U_i is a characteristic subgroup of U, with

$$U_i \cong \underbrace{C_{r^i} \times \cdots \times C_{r^i}}_{p}.$$

In addition $U_i \leq U_{i+1}$, $i = 1, \dots$ and $U = \bigcup_{i=1}^{\infty} U_i$.

https://doi.org/10.1017/S1446788700031803 Published online by Cambridge University Press

We note that each element a of $\mathrm{Aut}(U)$ can be considered as a sequence $a = (a_1, a_2, \ldots)$, where $a_i \in \mathrm{Aut}(U_i)$ and $a_{i+1}|_{U_i} = a_i$ (and where the composition of automorphisms is componentwise). Equivalently, the elements of $\mathrm{Aut}(U)$ may be considered as invertible $p \times p$ matrices over the ring of p-adic integers.

Lemma 4.1. *If A and B are finite subgroups of $\mathrm{Aut}(U)$ such that $A|_{U_i} = B|_{U_i}$, for all i, then $A = B$.*

Proof. Let $b = (b_1, b_2, \ldots)$ be an element of B such that $b \notin A$. Note that, for $a = (a_1, a_2, \ldots)$, if $a_i = b_i$, then

$$a_i|_{U_{i-1}} = a_{i-1} = b_i|_{U_{i-1}} = b_{i-1},$$

so $a_{i-j} = b_{i-j}$, $j = 1, \ldots, i-1$. By contrast, if $a_i \neq b_i$, then $a_{i+k} \neq b_{i+k}$ for all k. Now, since b is assumed not to be in A, there exists, for each $a \in A$, a subscript i_a with $a_{i_a} \neq b_{i_a}$. We let $i_m = \max\{i_a | a \in A\}$. Since A is finite, i_m is a well-defined natural number. But then

$$b_{i_m} \in B|_{U_{i_m}} = A|_{U_{i_m}},$$

so $b_{i_m} = a_{i_m}$, for some $a \in A$, which is a contradiction. This demonstrates $B \leq A$, and $A \leq B$ is similarly shown.

Lemma 4.2. *If $a = (a_1, a_2, \ldots)$ is an r'-automorphism of U (that is $o(a)$ is finite and coprime to r), then $a_1 \neq 1$.*

Proof. Assume $a_1 = 1$. Then we show inductively that $a_i = 1$, for all i. So assume $a_i = 1$ (for some $i \geq 1$). Then $a_{i+1}|_{U_i} = a_i$, so a_{i+1} centralises U_i. We let x be any element of U_{i+1}. Since $x^r \in U_i$ we have $(x^r)^{a_{i+1}} = x^r = (x^{a_{i+1}})^r$, so $\langle x, x^{a_{i+1}} \rangle$ has order at most $ro(x)$ and so is isomorphic either to $\langle x \rangle$ or to $\langle x \rangle \times C_r$. Since $i \geq 1$ and $U_i = \{u \in U_{i+1} | u^{r^i} = 1\}$, we see, in particular, that $x^{-1}x^{a_{i+1}} \in U_i$, whence $x^{a_{i+1}} = xu$, for some $u \in U_i$. But, since a_{i+1} centralises U_i, we see that x is centralised by $a_{i+1}^{o(u)}$. But $o(u)$ is a power of r and a_{i+1} is an r'-automorphism of U_{i+1}, so we conclude $x^{a_{i+1}} = x$, whence $a_{i+1} = 1$.

Lemma 4.3. *For $Z_i = C_{\mathrm{Aut}(U_{i+1})}(U_i)$, we have $\mathrm{Aut}(U_{i+1})/Z_i \cong \mathrm{Aut}(U_i)$. (Equivalently, every automorphism of U_i is induced by an automorphism of U_{i+1}.)*

Proof. We have:

$$\begin{aligned}|\mathrm{Aut}(U_i)| &= |\mathrm{Aut}(\underbrace{C_{r^i} \times \cdots \times C_{r^i}}_{p})| \\ &= (r^{pi} - r^{p(i-1)})(r^{pi} - r^{p(i-1)+1}) \cdots (r^{pi} - r^{p(i-1)+p-1}) \\ &= r^{p^2(i-1)}(r^p - 1)(r^p - r) \cdots (r^p - r^{p-1}) = r^{p^2(i-1)}|\mathrm{Aut}(U_1)|,\end{aligned}$$

since U_1 is elementary of abelian of order r^p (whence $\mathrm{Aut}(U_1)$ is isomorphic to the general linear group $\mathrm{GL}(p, r)$). In particular we see $|\mathrm{Aut}(U_{i+1})| = r^{p^2}|\mathrm{Aut}(U_i)|$.

If we let $\{u_1, \dots, u_p\}$ be a minimal generating set for U_{i+1}, then Z_i consists precisely of those automorphisms b which satisfy $u_1^b = u_1 z_1, \dots, u_p^b = u_p z_p$, where $z_1, \dots, z_p$ are any elements of U_1. Thus there are r^{p^2} such automorphisms and, since $\mathrm{Aut}(U_{i+1})/Z_i$ is isomorphic to a subgroup of $\mathrm{Aut}(U_i)$, we conclude, by comparison of orders that $\mathrm{Aut}(U_{i+1})/Z_i \cong \mathrm{Aut}(U_i)$.

We now let $B = \langle x, y | x^p = y^q = 1, x^{-1}yx = y^r\rangle$ and note that if s is such that $s^p \equiv 1 \bmod(q)$ and $s \not\equiv 1 \bmod(q)$, then $B \cong \langle x_1, y_1 | x_1^p = y_1^q = 1, x_1^{-1}y_1x_1 = y_1^s\rangle$ (that is, all non-abelian groups of order pq are isomorphic to B).

To construct a group on which to base a Fitting class of nilpotent length three, we will use the information about $\mathrm{Aut}(U)$ contained in the next result.

Proposition 4.4. (i) $\mathrm{Aut}(U)$ *has a unique, non-empty conjugacy class of Sylow q-subgroups, all isomorphic to* C_q;

(ii) $\mathrm{Aut}(U)$ *has a unique, non-empty conjugacy class of subgroups isomorphic to* B (*as above*);

(iii) $\mathrm{Aut}(U)$ *does not have any subgroups isomorphic to* $B \times C_p$.

Proof. By our conditions on p, q and r, we see that $q \top |\mathrm{Aut}(U_1)|$, so, by Lemma 4.2, to show that $\mathrm{Aut}(U)$ has a Sylow q-subgroup (that is, a maximal q-subgroup) is equivalent to showing that $\mathrm{Aut}(U)$ has an element of order q. We show inductively that there is a sequence $a = (a_1, a_2, \dots)$ with $a_i \in \mathrm{Aut}(U_1)$, $o(a_i) = q$, and $a_{i+1}|_{U_i} = a_i$, $i = 1, \dots$.

Now, $\mathrm{Aut}(U_1)$ does have an element a_1 of order q, so we assume that we have found a suitable $a_i \in \mathrm{Aut}(U_i)$. By Lemma 4.3 there exists an element $d \in \mathrm{Aut}(U_{i+1})$ with $d|_{U_i} = a_i$. Since $|Z_i| = r^{p^2}$ we see that there is a suitable power, k, of d such that $o(d^k) = q$ and $d^k|_{U_i} = a_i$, so we may let $a_{i+1} = d^k$. Then, for $a = (a_1, a_2, \ldots)$, $\langle a \rangle$ is a Sylow q-subgroup of $\mathrm{Aut}(U)$ which is isomorphic to C_q.

Now let $\langle b \rangle$ be another Sylow q-subgroup of $\mathrm{Aut}(U)$, where b is given by the sequence $b = (b_1, b_2, \ldots)$. Since, by Lemma 4.2, $\langle a_1 \rangle$ and $\langle b_1 \rangle$ are Sylow q-subgroups of the finite group $\mathrm{Aut}(U_1)$, there exists $w_1 \in \mathrm{Aut}(U_1)$ with $\langle a_1 \rangle^{w_1} = \langle b_1 \rangle$. Again, suppose we have found $w_i \in \mathrm{Aut}(U_i)$ such that $\langle a_i \rangle^{w_i} = \langle b_i \rangle$ and $w_i|_{U_{i-1}} = w_{i-1}$. Then, by Lemma 4.3, there exists an element $d \in \mathrm{Aut}(U_{i+1})$ such that $\langle a_{i+1} \rangle^d \leq \langle b_{i+1} \rangle Z_i$ and $d|_{U_i} = w_i$. Thus $\langle a_{i+1} \rangle^d$ and $\langle b_{i+1} \rangle$ are Sylow q-subgroups of the finite group $\langle b_{i+1} \rangle Z_i$, so there exists $z \in Z_i$ with $\langle a_{i+1} \rangle^{dz} = \langle b_{i+1} \rangle$. Since $dz|_{U_i} = d|_{U_i} = w_i$, we can set $w_{i+1} = dz$.

We let $w = (w_1, w_2, \ldots)$ be the element of $\mathrm{Aut}(U)$ thus constructed and see that $\langle a \rangle^w|_{U_i} = \langle b_i \rangle = \langle b \rangle|_{U_i}$, for all i, so, by Lemma 4.1, $\langle a \rangle^w = \langle b \rangle$, that is $\langle a \rangle$ and $\langle b \rangle$ are conjugate in $\mathrm{Aut}(U)$.

Part (ii) is proved in a manner similar to part (i). We note first that by, say, [9, II.6], there exists a subgroup $B_1 \leq \mathrm{Aut}(U_1)$ with $B_1 \cong B$. Assuming a suitable B_i has been found, we apply Lemma 4.3 to show that there exists $D \leq \mathrm{Aut}(U_{i+1})$ with $Z_i \leq D$ and:

$$D|_{U_i} = B_i \quad \text{and} \quad D/Z_i \cong B.$$

Now $(|Z_i|, |D/Z_i|) = 1$, so by Zassenhaus' Theorems (Huppert [6, I.18.1, 2]), there exists a complement B_{i+1} to Z_i in D, and clearly $B_{i+1}|_{U_i} = B_i$. For the groups B_i, $i = 1, 2, \ldots$, found in this manner, we let $B^* = \{(b_1, b_2, \ldots) | b_i \in B_i, b_{i+1}|_{U_i} = b_i, \ i = 1, \ldots\}$. Since for $b_i \in B_i$, there exists a unique $b_{i+1} \in B_{i+1}$ with $b_{i+1}|_{U_i} = b_i$, we conclude that B^* is a subgroup of $\mathrm{Aut}(U)$ with $B^* \cong B$.

From now on we identify B with B^* as a subgroup of $\mathrm{Aut}(U)$ and let $H \leq \mathrm{Aut}(U)$ with $H \cong B$. We let $H_i = H|_{U_i}$ and, by Lemma 4.2, have that $H_i \cong B_i$, $i = 1, 2, \ldots$. From, say, [9, II.2 and II.6], we see that H_1 and B_1 are conjugate in $\mathrm{Aut}(U_1)$, so we let $w_1 \in \mathrm{Aut}(U_1)$ be such that $(H_1)^{w_1} = B_1$. Again assume we have found a suitable $w_i \in \mathrm{Aut}(U_i)$ with $(H_i)^{w_i} = B_i$ and $w_i|_{U_{i-1}} = w_{i-1}$. Applying Lemma 4.3 we see that there is a $d \in \mathrm{Aut}(U_{i+1})$ with $H_{i+1}^d \leq B_{i+1} Z_i$ and $d|_{U_i} = w_i$. Again by Zassenhaus' theorems, all complements for Z_i in $B_{i+1} Z_i$ are conjugate, so there exists $z \in Z_i$ with $H_{i+1}^{dz} = B_{i+1}$, so we may let $w_{i+1} = dz$. We again let $w = (w_1, w_2, \ldots)$ and apply Lemma 4.1 to see $H^w = B$.

Finally, our third claim is just a corollary to Lemma 4.2, since our conditions on p, q and r imply by, say, [9, II.3], that $\mathrm{Aut}(U_1)$ does not have a subgroup isomorphic to $B \times C_p$.

Keeping the primes p, q and r fixed, we define the group G by $G = U \rtimes B$, where, by Proposition 4.4, we identify B with any suitable subgroup of $\mathrm{Aut}(U)$. We note that U_1 is the unique minimal normal subgroup of $U\langle y\rangle$ (where $\langle y\rangle$ is the (normal) Sylow q-subgroup of B), and that U_{i+1}/U_i is the unique minimal normal subgroup of $U\langle y\rangle/U_i$. In particular, we have that $[U, y] = U$, whence $G' = U\langle y\rangle$. We also have $0^p[0^{p'}(G)] = U\langle y\rangle$.

Our next result shows that certain "natural" candidates are indeed elements of $\mathfrak{Fit}(G)$. The proof is similar to that of [7, III.3], so we just include a sketch here.

LEMMA 4.5. *Let H be such that:*

(i) $H \in \mathfrak{E}$;
(ii) *There exists a normal subgroup* $N \trianglelefteq H$ *with* $N = D_1 \times \cdots \times D_t$;
(iii) $H/N \in \mathfrak{Fit}(C_p)$;
(iv) $D_i \trianglelefteq H$, $i = 1, \dots, t$;
(v) $D_i \cong U\langle y\rangle$;
(vi) *Let* $W_i = D_i'$ $(\cong U)$, *then* $H/C_H(W_i) \cong B$.

Then $H \in \mathfrak{Fit}(G)$.

PROOF. We note first that it can be shown that $C_p \in \mathfrak{Fit}(G)$, whence $\mathfrak{Fit}(C_p) \subseteq \mathfrak{Fit}(G)$. Thus if H is a p-group (that is, $t = 0$), then $H \in \mathfrak{Fit}(C_p)$ whence $H \in \mathfrak{Fit}(G)$. Now suppose that $t = 1$. Ignoring the subscripts, we let $C = C_H(W)$. Then $C/W \in \mathfrak{Fit}(C_p)$ and W is an r-group, so, by Proposition 2.10, there exists a complement K for W in C, whence $C = W \times K$. By Propositions 4.4(ii) and 2.10, we see that $H/K \cong G$. In addition H/D is an element of $\mathfrak{Fit}(C_p)$, so H $(= H/(K \cap D))$ can be seen to be subnormally embedded in $H/K \times H/D$, where both direct factors are in $\mathfrak{Fit}(G)$. Thus $H \in \mathfrak{Fit}(G)$.

For the general case, we let $M_i = D_1 \times \cdots \times D_{i-1} \times D_{i+1} \times \cdots \times D_t$ and from above we have that $H/M_i \in \mathfrak{Fit}(G)$, $i = 1, \dots, t$. Finally we have $H = H/\bigcap_{i=1}^t M_i$, so H can be subnormally embedded in $H/M_1 \times \cdots \times H/M_t$. We conclude $H \in \mathfrak{Fit}(G)$.

LEMMA 4.6. *Let H be the normal product $H = D_1 \cdots D_s$, where $D_i \cong U\langle y\rangle$ and $D_i \trianglelefteq H$, $i = 1, \dots, s$. Then, for a suitable enumeration, $H = D_1 \times \cdots \times D_t \times Q$, where $t \leq s$ and Q is a (finite) q-group.*

PROOF. We proceed by induction as in, say, the proof of [7, III.2]. Suppose $\langle D_1, \dots, D_j\rangle = D_1 \times \cdots \times D_{t_1} \times Q_j$, where Q_j is a q-group and $t_1 \leq j$. Then if $D_{j+1} \bigcap \langle D_1, \dots, D_j \rangle = 1$, we have

$$\langle D_1, \dots, D_{j+1} \rangle = \langle D_1, \dots, D_j \rangle \times D_{j+1} = D_1 \times \dots \times D_{t_1} \times D_{j+1} \times Q_j,$$

and, apart from reordering the indices, we are finished. Now, if y_1 is an element of order q in D_{j+1} and $W = D'_{j+1}$, then $[W, y_1] = W$ and, for $W_i = \{w \in W | w^{r^i} = 1\}$, $i = 1, \dots$, we have that W_1 is the unique minimal normal subgroup of D_{j+1}. Assuming $1 \neq D_{j+1} \bigcap \langle D_1, \dots, D_j \rangle$, we have $W_1 \leq \langle D_1, \dots, D_j \rangle$, and, since W_1 is an r-group, we have $W_1 \leq D'_1 \times \dots \times D'_{t_1}$. Since $[W_1, y_1] = W_1$, we see:

$$\begin{aligned} 1 \neq [y_1, D'_1 \times \dots \times D'_{t_1}] &= [y_1, D'_1] \times \dots \times [y_1, D'_{t_1}] \\ &\leq (W \cap D'_1) \times \dots \times (W \cap D'_{t_1}) \quad \text{(by normality)}. \end{aligned}$$

Hence we may conclude $1 \neq W \cap D'_1$. Thus the unique minimal normal subgroups of D_{j+1} and D_1 must be identical and, in particular, $1 = D_{j+1} \cap (D_2 \times \dots \times D_{t_1} \times Q_j)$. But now we must have $W_i \leq D'_1$, $i = 2, \dots$, since otherwise we may apply a result of Philip Hall (Huppert [6, III.3.18]) and the minimal normality of W_{i+1}/W_i to gain the contradiction $[y_1, W_1] = 1$. Thus $W \leq D'_1$ and since, by comparison of orders, each finite subgroup of D'_1 is contained in W, we conclude $W = D'_1$.

By Proposition 2.10 there exists a complement K to W in $D_1 D_{j+1}$ and we see that K is isomorphic to either C_q or $C_q \times C_q$. In the first case we conclude $D_1 = D_{j+1}$. In the second we note, by our conditions on p, q and r, that $C_K(W) \cong C_q$. We let $\langle z \rangle = C_k(W)$. Then $\langle z \rangle = Z(D_1 D_{j+1})$ and $D_1 D_{j+1} = D_1 \times \langle z \rangle$. In particular, $\langle z \rangle \trianglelefteq H$. If we let $Q_{j+1} = Q_j \langle z \rangle$, we may conclude that

$$\langle D_1, \dots, D_{j+1} \rangle = D_1 \times \dots \times D_{t_1} \times Q_{j+1}.$$

Construction 4.7. *Let G be as described above. Then $\mathfrak{Fit}(G)$ is the class of groups, H, which satisfy*:

(i) $H \in \mathfrak{T}$ *(as in Proposition* 2.1);
(ii) $0^p(H) \in \mathfrak{Fit}(U\langle y \rangle)$;
(iii) $0^p[0^{p'}(H)] = D_1 \times \dots \times D_t$, *for a suitable t, where*
  (a) $D_i \cong U\langle y \rangle \quad (i = 1, \dots, t)$;
  (b) $D_i \trianglelefteq 0^{p'}(H)$;
  (c) *Let* $W_i = D'_i$, *then* $0^{p'}(H)/C_{0^{p'}(H)}(W_i) \cong B$.

Proof. Our proof is similar to that of [7, IV.1]. If we let $\mathfrak{X}$ be the class described above, then certainly $G \in \mathfrak{X}$ and, for $H \in \mathfrak{X}$, we have $H =$

$0^p(H)0^{p'}(H)$. By Lemma 4.5, $0^{p'}(H) \in \mathfrak{Fit}(G)$, so H is the normal product of two elements of $\mathfrak{Fit}(G)$, whence $H \in \mathfrak{Fit}(G)$. Thus $G \in \mathfrak{X} \subseteq \mathfrak{Fit}(G)$, and it remains to show that $\mathfrak{X}$ is a Fitting class (of $\mathfrak{C}$-groups). We note that (i) and (ii) are Fitting class properties, so we shall concentrate on property (iii).

If $N \trianglelefteq H \in \mathfrak{X}$ then, since $U\langle y\rangle$ satisfies the minimal condition for normal subgroups, we may apply Lemma 2.7 and Proposition 2.8 to see that

$$0^p[0^{p'}(N)] = (0^p[0^{p'}(N)] \cap D_1) \times \cdots \times (0^p[0^{p'}(N)] \cap D_t),$$

(where it is assumed that H is the group given in the statement of the construction). Let $I = 0^p[0^{p'}(N)] \cap D_1$, say. Suppose $I \neq 1$. If $0^{p'}(N)$ centralises I, then $0^{p'}(N)/(0^p[0^{p'}(N)] \cap (D_2 \times \cdots \times D_t))$ will have a p'-factor group isomorphic to I, which is a contradiction. In particular, $0^{p'}(N)$ does not centralise $W_1 \cap I$. We let $C_1 = C_{0^{p'}(H)}(W_1)$ and see that $0^{p'}(N)C_1/C_1$ is a non-trivial normal subgroup of $0^{p'}(H)C_1/C_1$, which is generated by p-elements. Thus $B \cong 0^{p'}(N)C_1/C_1 \cong 0^{p'}(N)/C_{0^{p'}(N)}(W_1)$. By normality, we see $W_1 = [W_1, 0^{p'}(N)] \leq 0^p[0^{p'}(N)]$ and noting that $0^{p'}(N) \cap (D_2 \times \cdots \times D_t)$ centralises W_1, we conclude $D_1 \leq 0^p[0^{p'}(N)]$. In this manner we see $N \in \mathfrak{X}$, so $\mathfrak{X}$ is closed with respect to normal subgroups.

For closure with respect to normal products, we let $H \in \mathfrak{C}$ be such that $H = \langle H_k | k \in K\rangle$, where $H_k \trianglelefteq H$ and $H_k \in \mathfrak{X}$ for all $k \in K$. We let

$$0^p[0^{p'}(H_k)] = D_{k1} \times \cdots \times D_{kt_k},$$

where the D_{kj} are isomorphic to $U\langle y\rangle$ and (a), (b) and (c) hold in the respective residuals. We let $W_{kj} = D'_{kj}$. By Proposition 3.1, the W_{kj} centralise each other elementwise, so by Proposition 2.9, each D_{kj} is normalised by $\langle W_{kj} \mid k \in K, j = 1, \ldots, t_k\rangle$. Now let us fix two arbitrary indices in K and denote them by 1 and 2. We show that D_{11}, say, is normalised by $0^{p'}(H_2)$.

Let y_1 be a q-element of, say, D_{21}. If y_1 does not normalise D_{11} then, by Proposition 2.9, we can assume $D_{11}^{y_1} = D_{12}$ and $D_{12}^{y_1} = D_{13}$. If x is an element of order q in D_{11}, then $[x, y_1, y_1] = x(x^{-2})x^{y_1^2}$ is an element of D_{21}. By normality we then have:

$$\begin{aligned} W_{11} \times W_{12} \times W_{13} &= [W_{11} \times W_{12} \times W_{13}, [x, y_1, y_1]] \\ &\leq [H_1, D_{21}, D_{21}] \leq D_{21}, \end{aligned}$$

and the contradiction $W_{11} \times W_{12} \times W_{13} \leq W_{21}$ ensues. From this we conclude that the D_{ij} are normal in $0^p[0^{p'}(H)] = \langle 0^p[0^{p'}(H_k)] | k \in K\rangle$. Finally, if s

is any p-element of $0^{p'}(H_2)$ with, say, $D_{11}^s = D_{12}$, then $[x, s] = x^{-1}x^s$ is a q-element of $0^{p'}(H_2)$, (where x, as above, is an element of order q in D_{11}). Thus $W_{11} \times W_{12} = [W_{11} \times W_{12}, [x, s]] \leq 0^p[0^{p'}(H_2)]$. As in the proof of Lemma 4.6, we may now assume $W_{11} = W_{21}$, but since s normalises D_{21}, W_{21} is also normalised by s and the contradiction $W_{11}^s = W_{11}$ follows. We infer that D_{11} is normalised by $0^{p'}(H_2)$. Since 1 and 2 were arbitrary indices from K, we conclude that the D_{kj} are normal in $0^{p'}(H) = \langle 0^{p'}(H_k) | k \in K\rangle$.

Now, $0^p[0^{p'}(H)]$ $(= \langle D_{kj} | k \in K,\ j = 1, \ldots, t_k\rangle)$ is an element of $\mathfrak{T}$, so there exists a normal subgroup $N \trianglelefteq 0^p[0^{p'}(H)]$ with $N \in \mathfrak{Fit}(C_r)$ and $0^p[0^{p'}(H)]/N \in \mathfrak{Fit}(C_q)$. In addition $0^p[0^{p'}(H)]/N$ is generated by $\langle D_{kj}N/N | k \in K,\ j = 1, \ldots, t_k\rangle$ and, for each choice of k and j, $D_{kj}N/N \cong D_{kj}/(D_{kj} \cap N) = D_{kj}/W_{kj} \cong \langle y \rangle \cong C_q$. Thus $0^p[0^{p'}(H)]/N$ is generated by normal subgroups isomorphic to C_q and hence has exponent q. Since $0^p[0^{p'}(H)]$ is Černikov, $0^p[0^{p'}(H)]/N$ must be finite. Thus $0^p[0^{p'}(H)]$ is generated by a finite subset of the D_{kj}, modulo N, say: $0^p[0^{p'}(H)] = \langle D_1, \ldots, D_s, N\rangle$ where $D_1, \ldots, D_s$ are among the D_{kj}. But then we see:

$$\begin{aligned} 0^p[0^{p'}(H)] &= 0^{q'}(0^p[0^{p'}(H)]) \\ &= \langle 0^{q'}(D_1), \ldots, 0^{q'}(D_s), 0^{q'}(N)\rangle = \langle D_1, \ldots, D_s\rangle, \end{aligned}$$

so we apply Lemma 4.6 to see that:

$$0^p[0^{p'}(H)] = D_1 \times \cdots \times D_t \times Q,$$

where $t \leq s$ and Q is finite q-group. Clearly Q is characteristic in $0^{p'}(H)$ and, since $Q \cap 0^{p'}(H_k) = Q \cap 0^p[0^{p'}(H_k)] = 1$, we have $[Q, 0^{p'}(H_k)] \leq Q \cap 0^{p'}(H_k) = 1$ for each choice of k, whence $Q \leq Z(0^{p'}(H))$. If we now work modulo $D_1 \times \cdots \times D_t$, we see that $0^{p'}(H)/Q$ is an element of $\mathfrak{Fit}(C_p)$, so, in particular, $0^{p'}(H)/Q$ is nilpotent. But since Q is central in $0^{p'}(H)$, we have that $0^{p'}(H)$ is itself nilpotent and has Q as a direct factor. However, $0^{p'}(H)$ has no non-trivial p'-factor groups, so we conclude $Q = 1$.

Finally we let $C_i = C_{0^{p'}(H)}(W_i)$, where $W_i = D_i'$. Then, since $Q = 1$, we see, as in the proof of Lemma 4.6, that either $D_i \leq 0^p[0^{p'}(H_k)]$ or $D_i \cap 0^p[0^{p'}(H_k)] = 1$. Since the H_k are in $\mathfrak{X}$, we conclude that $0^{p'}(H)/C_i = \langle 0^{p'}(H_i)C_i/C_i | k \in K\rangle$ is either isomorphic to B or is the normal product of subgroups, all isomorphic to B. In the latter case we see that $0^{p'}(H)/C_i$ has a subgroup isomorphic to $B \times B$ or $B \times C_p$. However, the second and third possibilities are ruled out by Proposition 4.4, so we conclude $O^{p'}(H)/C_i \cong B$.

Thus properties (iii)(a), (b) and (c) have been demonstrated and we conclude that $\mathfrak{X}$ is a Fitting class.

It should be emphasised that we have been mainly concerned with applying well-known methods of finite Fitting class construction in an infinite context. The periodicity and structure of the groups in question, in particular the fact that we have been dealing with finite extensions of divisible groups of finite rank, has played a major role in allowing the constructions to go through. By contrast, the construction of Fitting classes based on non-periodic groups seems likely to pose more challenging and novel problems.

References

[1] J. C. Beidleman, M. J. Karbe and M. J. Tomkinson, 'Fitting classes of $\mathfrak{S}_1$-groups, I', *Illinois J. Math.* **33** (1989), 333–359.

[2] J. C. Beidleman and M. J. Tomkinson, 'Hypercentral and nilpotent injectors of $\mathfrak{S}_1$-groups', Comm. Algebra **18** (1990) 4307–4321.

[3] R. A. Bryce, 'The fitting class generated by a finite soluble group' (A. N. U. Math. Research Report Series, No. 34, 1988).

[4] R. A. Bryce, J. Cossey and E. A. Ormerod, 'Fitting classes after Dark', in: *Group Theory–Proceedings of the Singapore group theory conference*, 1987 (eds. N. K. Cheng and Y. K. Leong) (de Gruyter, Berlin, 1989) pp. 293–321.

[5] R. S. Dark, 'Some examples in the theory of injectors of finite soluble groups', *Math. Z.* **127** (1972), 145–156.

[6] B. Huppert, *Endliche Gruppen* I (Springer, Berlin 1967).

[7] B. McCann, 'Examples of minimal Fitting classes of finite groups', *Arch. Math.* **49** (1987), 179–186.

[8] B. McCann, 'A Fitting class construction', Ann. Mat. Pura Appl. **157** (1990), 27–61.

[9] B. McCann, 'Examples of normal products of finite groups and an application to Fitting classes', in: *Proc. Int. Group Theory Conf., Bressanone/Brixen*, 1989 (Supp. ai Rend. del Circolo Mat. di Palermo, Serie II, No. 23, 1990) pp. 237–252.

[10] F. Menegazzo and M. L. Newell, 'Injectors in Fitting Classes of $\mathfrak{S}_1$-groups', to appear.

[11] D. J. S. Robinson, *A course in the theory of groups* (Springer, Berlin, 1982).

Department of Physical and Quantitative Sciences
Regional Technical College
Cork Road
Waterford
Ireland