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INTERSECTIONS OF α-SPACES

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Abstract

Let $\bar{\beta}$ be an infinite r.e. repère, $\bar{W}$ an infinite dimensional r.e. space such that $\bar{W} \leqq L(\bar{\beta})$. A condition is derived that is both necessary and sufficient for the existence of an infinite subset $\beta \subset \bar{\beta}$ such that $L(\beta) \cap \bar{W}$ is not an α-space. Examples which satisfy this condition are exhibited, proving that the class of α-spaces is not closed under intersections.

Introduction

Dekker (1969) and (1971), introduced and studied an $\aleph_0$-dimensional recursive vector space $\bar{U}_F$ over a countable field F. Briefly, it consists of an infinite recursive set ε_F of numbers (that is, non-negative integers), an operation $+$ from $\varepsilon_F \oplus \varepsilon_F$ into ε_F and an operation $\cdot$ from $F \times \varepsilon_F$ into ε_F. If the field F is identified with a recursive set, both $+$ and $\cdot$ are partial recursive functions. Let β be a subset of ε_F. We call β a *repère*, if it is linearly independent; β is a *r.e. repère* if β is a r.e. set, and β is an *α-repère* if it is included in some r.e. repère. A subspace V of $\bar{U}_F$ is an *α-space*, if it has at least one *α-basis*, that is, at least one basis which is also an α-repère. A subspace V is *isolic* if it includes no infinite r.e. repère; it is r.e. if it is r.e. as a set. The word "space" is used in the sense of "subspace of $\bar{U}_F$," and we denote "W is a subspace of V" by "$W \leqq V$." We usually write (0) for $\{0\}$, and $\bar{U}$ for $\bar{U}_F$. We identify $a(n)$ and a_n, for every function $a(n)$; and a bar over a set (or space) is generally intended to indicate recursive enumerability. We write "L.C." for "linear combination" and "L.C.N.Z.C." for "linear combination with non-zero coefficients." Let $\alpha \subset \varepsilon_F$. If $\alpha = \emptyset$, $L(\alpha) = (0)$. If $\alpha \neq \emptyset$, $L(\alpha)$ denotes the span of α, that is, the set of all L.C. (with coefficients in F) of finitely many elements of α. If $\alpha = \{a_0, \cdots\}$, we usually write $L(a_0, \cdots)$ instead of $L(\{a_0, \cdots\})$.

The results presented in this paper were taken from the author's doctoral dissertation written at Rutgers University under the direction of Professor J.C.E. Dekker.

The repères β and γ are *independent* if they are disjoint and their union is a repère. The spaces V and W are *independent* if $V \cap W = (0)$. The sets β and γ are *separable* [written: $\beta | \gamma$], if they can be separated by r.e. sets. The α-repères β and γ are *α-independent* [written: $\beta \parallel \gamma$], if they can be separated by independent r.e. repères. The spaces V and W are *α-independent* [written: $V \parallel W$], if there are independent r.e. spaces $\bar{V}$ and $\bar{W}$ such that $V \leqq \bar{V}$ and $W \leqq \bar{W}$.

Let S, C, V, W be spaces and consider the following three statements:

(a) V, W α-spaces $\Rightarrow V \cap W$ α-space,

(b) V α-space, W r.e. space $\Rightarrow V \cap W$ α-space,

(c) $S \oplus C = V$ and $S \parallel C$ and V an α-space $\Rightarrow$ both S and C are α-spaces.

Clearly, (a) implies (b); (c) is a conjecture that appears in Dekker (1971; page 493), and is established in Fowler (to appear) in the case S (or C) is isolic or r.e. Assume the hypothesis of (c), and suppose $\bar{W}$, $\bar{Z}$ are two independent r.e. spaces such that $S \leqq \bar{W}$, $C \leqq \bar{Z}$. It can be easily shown that $S = V \cap \bar{W}$, and $C = V \cap \bar{Z}$ hence (b) implies (c).

In this paper, we provide several counterexamples to (b); hence α-spaces are not closed under intersections, and the above approach to (c) is fruitless. More specifically, if $\bar{\beta}$ is an infinite r.e. repère and $\bar{W}$ is an infinite dimensional r.e. space such that $\bar{W} \leqq L(\bar{\beta})$, we derive a condition that is both necessary and sufficient for the existence of an infinite subset $\beta \subset \bar{\beta}$ such that $L(\beta) \cap \bar{W}$ is not an α-space. We exhibit examples in which this condition is satisfied, regardless of the cardinality of F. We take our notation from Dekker (1969) and (1971) and the reader is assumed to be familiar with their contents.

2. The condition

NOTATIONS. Let $p_0 = 2$, $p_n =$ the n-th odd prime for $n \geqq 1$. Then $\eta = \rho e_n$ is the recursive canonical basis for $\bar{U}_F$, where $e_n = p_n - 1$ (see the specific Gödel numbering used in Dekker (1969)). If β is a repère, $x \in L(\beta)$ and $\sigma \subset L(\beta)$, then

$\beta_x = \{b \in \beta |$ *x has a non-zero coordinate with respect to b if expressed as a* L.C.N.Z.C. of elements in $\beta\}$,

$\beta_\sigma = \cup \{\beta_x | x \in \sigma\}$.

DEFINITION. Let $\bar{W}$ be an $\aleph_0$-dimensional r.e. space and $\bar{\beta}$ a r.e. repère such that $\bar{W} \leqq L(\bar{\beta})$. Then $\bar{\beta}$ *has property* Δ with respect to $\bar{W}$ if there is no $1-1$ recursive function $d(n)$ enumerating a basis of $\bar{W}$ for which $\cup_{i \neq j}(\bar{\beta}_{d(i)} \cap \bar{\beta}_{d(j)})$ is finite.

REMARKS. (a) Let $\bar{W} \leqq L(\bar{\beta})$ where $\bar{W}$ is a r.e. space and $\bar{\beta}$ is a r.e. repère. Then $\bar{W} \leqq L(\bar{\beta}_{\bar{W}})$, $\bar{\beta}_{\bar{W}} \subset \bar{\beta}$, where $\bar{\beta}_{\bar{W}}$ is also a r.e. repère; moreover, $\bar{\beta}_x \subset \bar{\beta}_{\bar{W}}$ for every $x \in \bar{W}$. Hence $\bar{\beta}_{\bar{W}}$ has property Δ with respect to $\bar{W}$ if and only if $\bar{\beta}$ has property Δ with respect to $\bar{W}$.

(b) If $\bar{\beta}$ has property Δ with respect to $\bar{W}$ and $d(n)$ is a $1-1$ recursive function enumerating a basis of $\bar{W}$, then the sequence $\langle \bar{\beta}_{d(i)} \rangle$ of (finite, non-empty) sets does not have a tail of mutually disjoint sets.

DEFINITIONS.

(a) The r.e. space $\bar{W}$ is *decidable relative to* the r.e. space $\bar{V}$, if

(i) $\bar{W} \leqq \bar{V}$,

(ii) the set $\bar{V} \backslash \bar{W}$ is r.e.

(b) The r.e. space $\bar{W}$ is *recursive relative to* the r.e. space $\bar{V}$, if

(i) $\bar{W} \leqq \bar{V}$,

(ii) there is some r.e. space Z such that $Z \cap \bar{W} = (0)$ and $\bar{W} \oplus Z = \bar{V}$.

(c) If the r.e. space $\bar{W}$ is decidable (or recursive) relative to $\bar{U}_F$, we say that $\bar{W}$ is *decidable* (respectively *recursive*).

REMARKS.

(a) If $\bar{V}$ is an $\aleph_0$-dimensional r.e. space, there are many recursive isomorphisms from $\bar{V}$ onto $\bar{U}_F$; pick one, say h. Then $\bar{W}$ is decidable (or recursive) relative to $\bar{V}$ if and only if $h(\bar{W})$ is decidable (respectively recursive).

(b) Well-known results concerning decidable and recursive spaces carry over to the relative case by (a); in particular, the following two results due to Guhl (to appear):

(i) If F is finite, $\bar{W}$ recursive $\Leftrightarrow$ $\bar{W}$ decidable,

(ii) if F is infinite, $\bar{W}$ recursive $\Rightarrow$ $\bar{W}$ decidable, but not conversely.

PROPOSITION P1. *Let $\bar{W}$ be an $\aleph_0$-dimensional r.e. space and $\bar{\beta}$ a r.e. repère such that $\bar{W} \leqq L(\bar{\beta})$. Then $\bar{W}$ not recursive relative to $L(\bar{\beta}_W) \Rightarrow \bar{\beta}_W$ has property Δ with respect to $\bar{W}$.*

PROOF. We may assume without loss of generality that $\bar{\beta}_W = \bar{\beta}$. We shall prove the contrapositive, that is,

$\bar{\beta}$ does not have property Δ with respect to $\bar{W} \Rightarrow$

$\bar{W}$ recursive relative to $L(\bar{\beta})$.

Assume the hypothesis. Then there is a $1-1$ recursive function d_n ranging over some r.e. basis $\bar{\gamma}$ of $\bar{W}$ and a finite subset $\{b_0, \cdots, b_m\}$ of $\bar{\beta}$ such that

$$(\forall i)(\forall j)[i \neq j \Rightarrow \bar{\beta}_{d(i)} \bigcap \bar{\beta}_{d(j)} \subset \{b_0, \cdots, b_m\}].$$

Denote $\{b_0, \cdots, b_m\}$ by ρ.

Note that for each number j we can

(i) effectively test whether $\bar{\beta}_{d(j)} \subset \rho$,

(ii) if not $[\bar{\beta}_{d(j)} \subset \rho]$, effectively list both the elements of $\bar{\beta}_{d(j)} \cap \rho$ and those of $\bar{\beta}_{d(j)} \backslash \rho$. Define

$$\bar{\delta} = \{d_n \in \bar{\gamma} \mid \bar{\beta}_{d(n)} \subset \rho\}.$$

Then by (i) both $\bar{\delta}$ and $\bar{\gamma}\backslash\bar{\delta}$ are r.e. Only finitely many elements $d_n \in \bar{\gamma}$ have the property $\bar{\beta}_{d(n)} \subset \rho$; this follows from the fact that the span of all these elements $d_n \in \bar{\gamma}$ is a subspace of the finite dimensional space $L(\rho)$, while $\bar{\gamma}$ is an infinite repère. Thus $\bar{\delta}$ is a finite repère. Clearly if $d_n \in \bar{\delta}$, then $d_n \in L(\rho)$. Then we have $L(\bar{\delta}) \leqq L(\rho)$. Combining this with the fact that $\bar{\delta}$ and ρ are finite repères, we see that there is a finite repère $\bar{\alpha}_1$ such that $\bar{\delta} \subset \bar{\alpha}_1$ and $L(\bar{\alpha}_1) = L(\rho)$. The sets $\bar{\alpha}_1$ and $\bar{\alpha}_1\backslash\bar{\delta}$ are finite, hence r.e. We note that $\bar{\gamma}\backslash\bar{\delta}$ is infinite and r.e. For every $d_j \in \bar{\gamma}\backslash\bar{\delta}$, we have

$$(1) \qquad \begin{cases} \text{not } [\bar{\beta}_{d(j)} \subset \rho],\ \bar{\beta}_{d(j)}\backslash\rho \neq \varnothing \\ \bar{\beta}_{d(j)} = (\bar{\beta}_{d(j)}\backslash\rho) \cup (\bar{\beta}_{d(j)} \cap \rho), \\ d_j \in L(\bar{\beta}_{d(j)}),\ d_j \notin L(\bar{\beta}_{d(j)} \cap \rho). \end{cases}$$

For $d_j \in \bar{\gamma}\backslash\bar{\delta}$, put

$$c_j = \min\{\bar{\beta}_{d(j)}\backslash\rho\},\ \tau_j = [(\bar{\beta}_{d(j)}\backslash\rho)\backslash\{c_j\}] \cup \{d_j\}.$$

It follows that

$$(2) \qquad d_j \in \tau_j \text{ and } L(\bar{\beta}_{d(j)}) = L(\tau_j) \oplus L(\bar{\beta}_{d(j)} \cap \rho).$$

We now define

$$\bar{\alpha}_2 = \cup \{\tau_j \,|\, d_j \in \bar{\gamma}\backslash\bar{\delta}\}$$

and we claim that

(a) $L(\bar{\alpha}_1) + L(\bar{\alpha}_2) = L(\bar{\beta})$,
(b) $\bar{\alpha}_2$ is a r.e. repère,
(c) $L(\bar{\alpha}_1) \cap L(\bar{\alpha}_2) = (0)$,
(d) $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are disjoint and $\bar{\alpha}_1 \cup \bar{\alpha}_2$ is a r.e. basis for $L(\bar{\beta})$,
(e) $\bar{W}$ is recursive relative to $L(\bar{\beta})$.

Re (a).
$$\begin{aligned} L(\bar{\alpha}_1) + L(\bar{\alpha}_2) &= L(\rho) + L(\cup\{\tau_j \,|\, d_j \in \bar{\gamma}\backslash\bar{\delta}\}) \\ &= L(\rho) + \Sigma\{L(\tau_j) \,|\, d_j \in \bar{\gamma}\backslash\bar{\delta}\} \\ &= \Sigma\{L(\bar{\beta}_{d(j)}) \,|\, d_j \in \bar{\gamma}\}, \text{ since } \bar{\beta}_W = \bar{\beta},\ d_j \in \bar{\delta} \\ &\qquad \text{implies } L(\bar{\beta}_{d(j)}) \leqq L(\rho), \text{ and (2).} \end{aligned}$$

Hence $L(\bar{\alpha}_1) + L(\bar{\alpha}_2) = L(\bar{\beta})$, again since $\bar{\beta}_W = \bar{\beta}$.

Re (b). Let $\Gamma = \{\tau_j \,|\, d_j \in \bar{\gamma}\backslash\bar{\delta}\}$. Then Γ is a r.e. class of non-empty finite sets, hence $\bar{\alpha}_2$ is a r.e. set. It follows from the definition of τ_j that Γ consists of finite repères. To prove that $\bar{\alpha}_2$ is also a repère, it therefore suffices to show that

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$$(3)\quad \begin{cases} \text{if } d_{i(0)}, \cdots, d_{i(n)} \text{ are distinct elements of } \bar{\gamma}\backslash\delta, \text{ then} \\ L(\tau_{i(n)}) \cap [L(\tau_{i(0)} + \cdots + L(\tau_{i(n+1)})] = (\bar{0}). \end{cases}$$

Assume the hypothesis of (3) and suppose that

$$x \in L(\tau_{i(n)}) \cap [L(\tau_{i(0)} + \cdots + L(\tau_{i(n-1)})],$$

say

$$x = r_n d_{i(n)} + y_n = r_0 d_{i(0)} + \cdots + r_{n-1} d_{i(n-1)} + y_0 + \cdots + y_{n-1},$$

where $r_0, \cdots, r_n \in F$, and for every $k \leqq n$,

$$y_k \in L(\sigma_k), \text{ where } \sigma_k = (\bar{\beta}_{di(k)} \backslash \rho) \backslash \{c_{i(k)}\}.$$

Then

$$(4)\quad 0 = r_n d_{i(n)} - [r_0 d_{i(0)} + \cdots + r_{n-1} d_{i(n-1)}] + y_n - (y_0 + \cdots + y_{n-1}).$$

The family $\{(\bar{\beta}_{d(j)} \backslash \rho) \mid d_j \in \bar{\gamma} \backslash \delta\}$ consists of mutually disjoint finite subsets of $\bar{\beta}$, hence its union is a repère. This fact and the definition of c_j, for $d_j \in \bar{\gamma} \backslash \delta$ imply the two relations

(5) $\{\sigma_k \mid k \leqq n\}$ is a family of mutually disjoint finite subsets of $\bar{\beta}$, hence its union is a repère.

$$(6)\quad c_{i(k)} \notin (\cup \{\sigma_k \mid k \leqq n\}) \cup \rho, \text{ for } k \leqq n.$$

Let us now look at (4). By the definition of $\bar{\beta}_{di(n)}$, the element $d_{i(n)} \in \bar{\gamma} \backslash \delta$ has a non-zero coordinate with respect to each element of $\bar{\beta}_{di(n)}$ when expressed as a L.C. of elements in $\bar{\beta}$, in particular, with respect to $c_{i(n)}$. Suppose $d_{i(k)}$, for some $0 \leqq k \leqq n-1$, also had a non-zero coordinate with respect to $c_{i(n)}$ when expressed as a L.C. of elements in $\bar{\beta}$. Then $c_{i(n)} \in \bar{\beta}_{di(n)} \cap \bar{\beta}_{di(k)}$ implies that $c_{i(n)} \in \rho$, contrary to $c_{i(n)} \in \bar{\beta}_{di(n)} \backslash \rho$. Thus $d_{i(k)}$ has no non-zero coordinate w.r.t. $c_{i(n)}$ when expressed as a L.C. of elements in $\bar{\beta}$. We note that (6) implies that none of $y_0, \cdots, y_n$ has a non-zero coordinate with respect to $c_{i(n)}$ when expressed as a L.C. of elements in $\bar{\beta}$. Thus (4) implies that $r_n = 0$. Similarly we can prove that (4) implies that $r_0 = 0, \cdots, r_{n-1} = 0$. Using (4) once more we see that

$$y_n - (y_0 + \cdots + y_{n-1}) = 0.$$

This implies that $y_0 = 0, \cdots, y_n = 0$ by (5). Since $r_n = 0$ and $y_n = 0$, we conclude that $x = r_n d_{i(n)} + y_n = 0$. This completes the proof of (3) and thereby of (b).

Re (c). Recall that $L(\bar{\alpha}_1) = L(\rho)$. We wish to prove that

$$L(\rho) \cap L(\bar{\alpha}_2) = (0),$$

that is, that

$$x \in L(\rho) \cap L(\bar{\alpha}_2) \Rightarrow x = 0. \tag{7}$$

Assume the hypothesis, say

$$x = r_0 b_0 + \cdots + r_m b_m = s_0 d_{i(0)} + \cdots + s_n d_{i(n)} + y_0 + \cdots + y_n, \tag{8}$$

where $r_0, \cdots, r_m, s_0, \cdots, s_n \in F$, $i(0), \cdots, i(n)$ are distinct, and

$$\begin{cases} \text{for } k \leqq n,\ y_k \in L(\sigma_k), \text{ where} \\ \sigma_k = (\bar{\beta}_{di(k)} \backslash \rho) \backslash \{c_{i(k)}\}. \end{cases} \tag{9}$$

As observed in the proof of (b), $d_{i(n)}$ has a non-zero coordinate with respect to the element $c_{i(n)} \in \bar{\beta}_{di(n)} \backslash \rho$, while none of $d_{i(0)}, \cdots, d_{i(n-1)}, y_0, \cdots, y_n$ has a non-zero coordinate w.r.t. $c_{i(k)}$ when expressed as a L.C. of elements in $\bar{\beta}$. Since $c_{i(n)} \notin \rho = \{b_0, \cdots, b_m\}$, it follows from (8) that $s_n = 0$. Similarly we prove that $s_0 = 0, \cdots, s_{n-1} = 0$. Then (8) yields

$$x = r_0 b_0 + \cdots + r_m b_m = y_0 + \cdots + y_n. \tag{10}$$

However, $(\cup \{\sigma_k | k \leqq n\}) \cap \rho = \emptyset$, while $(\cup \{\sigma_k | k \leqq n) \cup \rho \subset \bar{\beta}$ imply $x = 0$ since $\bar{\beta}$ is a repère. This completes the proof of (c).

Re (d). Since $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are repères, neither contains 0. Thus (c) implies that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are disjoint and that $\bar{\alpha}_1 \cup \bar{\alpha}_2$ is a repère. The set $\bar{\alpha}_1 \cup \bar{\alpha}_2$ is r.e., since both $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are r.e. Finally, $\bar{\alpha}_1 \cup \bar{\alpha}_2$ is a basis of $L(\bar{\beta})$ by (a).

Re (e). We have $\bar{\gamma} = \bar{\delta} \cap (\bar{\gamma} \backslash \bar{\delta})$. Also, $\bar{\delta} \subset \bar{\alpha}_1$ by the definition of $\bar{\alpha}_1$, and $\bar{\gamma} \backslash \bar{\delta} \subset \bar{\alpha}_2$ by the definition of $\bar{\alpha}_2$. Then $\bar{\gamma} \subset \bar{\alpha}_1 \cup \bar{\alpha}_2$, where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are disjoint, hence

$$(\bar{\alpha}_1 \cup \bar{\alpha}_2) \backslash \bar{\gamma} = (\bar{\alpha}_1 \backslash \bar{\delta}) \cup [\bar{\alpha}_2 \backslash (\bar{\gamma} \backslash \bar{\delta})].$$

However, $\bar{\alpha}_1 \backslash \bar{\delta}$ is r.e., and the definitions of $\bar{\alpha}_2$ and τ_j imply that $\bar{\alpha}_2 \backslash (\bar{\gamma} \backslash \bar{\delta})$ is r.e. We conclude that the set $(\bar{\alpha}_1 \cup \bar{\alpha}_2) \backslash \bar{\gamma}$ is also r.e. Thus,

$$L(\bar{\gamma}) \oplus L((\bar{\alpha}_1 \cup \bar{\alpha}_2) \backslash \bar{\gamma}) = L(\bar{\alpha}_1 \cup \bar{\alpha}_2),$$

where both spaces on the left are r.e. Hence

$$\bar{W} \oplus L((\bar{\alpha}_1 \cup \bar{\alpha}_2) \backslash \bar{\gamma}) = L(\bar{\beta}),$$

and $\bar{W}$ is recursive relative to $L(\bar{\beta})$.

Consider the following example: let $f(n)$ be a $1-1$ recursive function ranging over an infinite r.e. but not a recursive subset of $\varepsilon \backslash \{0\}$. Let $d(n) = e_0 + e_1$

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$+ \cdots + e_{f(n)}$. Then d_n is a $1-1$ recursive function. Let $\bar{W} = L\{d_n \mid n \in \varepsilon\}$. Then $\bar{W}$ is clearly r.e., but not decidable, and hence not recursive. Note that $\eta_{\bar{W}} = \eta$. By $P1$ then, η has property Δ w.r.t. $\bar{W}$.

The next proposition shows that the converse of P1 is false.

PROPOSITION P2. *There exists an $\aleph_0$-dimensional r.e. space $\bar{W}$, and a r.e. repère $\bar{\beta}$ such that $\bar{W} \leqq L(\bar{\beta})$ is recursive relative to $L(\bar{\beta})$ and $\bar{\beta}_{\bar{W}}$ has property Δ with respect to $\bar{W}$.*

PROOF. We define a function c_n by

$$\begin{array}{llll} c_0 = e_0 + e_2, & c_3 = e_1 + e_8, & c_6 = e_5 + e_{14}, & c_9 = e_7 + e_{20}, \\ c_1 = e_0 + e_4, & c_4 = e_3 + e_{10}, & c_7 = e_5 + e_{16}, & c_{10} = e_9 + e_{22}, \\ c_2 = e_1 + e_6, & c_5 = e_3 + e_{12}, & c_8 = e_7 + e_{18}, & c_{11} = e_9 + e_{24}, \qquad \text{etc.} \end{array}$$

Put

$$\bar{\delta}_1 = \rho c_n, \qquad \bar{W} = L(\bar{\delta}_1),$$
$$\bar{\delta}_2 = \{e_0, e_1, e_3, e_5, \cdots\}, \qquad \bar{Z} = L(\bar{\delta}_2).$$

Note that if $p_0, p_1, p_2, \cdots$ is the enumeration according to size of the set of all positive primes, then $e_n + e_m = p_n p_m - 1$ (see the specific Gödel numbering used in Dekker (1969)). Thus c_n is a strictly increasing recursive function, $\bar{\delta}_1$ an infinite and $\bar{W}$ an $\aleph_0$-dimensional r.e. space. It is readily seen that $\bar{\delta}_1$ is a repère; thus $\bar{\delta}_1$ is a r.e. basis of $\bar{W}$. Note that $\bar{W} + \bar{Z} = \bar{U}$. For every number n,

$$L(c_0, \cdots, c_n) \cap L(e_0, e_1, \cdots, e_{2n+1}) = (0),$$

hence $\bar{W} \cap \bar{Z} = (0)$. Thus $\bar{W} \oplus \bar{Z} = \bar{U}$ and since $\bar{Z}$ is clearly r.e., we conclude that $\bar{W}$ is recursive relative to $\bar{U}$. Furthermore, $\bar{W} \leqq L(\eta)$, where $\eta_{\bar{W}} = \eta$, hence $\bar{W}$ is recursive relative to $L(\eta_{\bar{W}})$. It remains to show that η has property Δ w.r.t. $\bar{W}$. Let $\bar{\gamma}$ be any r.e. basis of $\bar{W}$ and d_n any $1-1$ recursive function ranging over $\bar{\gamma}$.

Put

$$\sigma = \bigcup_{i \neq j} (\eta_{d(i)} \cap \eta_{d(j)}).$$

We now show that σ is infinite by proving for every $n \geqq 1$,

$$\{e_{2n-1}, e_{4n+2}, e_{4n+4}\} \cap \sigma \neq \emptyset.$$

Since the reasoning is similar for every $n \geqq 1$, we restrict our attention to the case $n = 1$, and prove

$$(11) \qquad \{e_1, e_6, e_8\} \cap \sigma \neq \emptyset.$$

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Consider the elements $c_2 = e_1 + e_6$ and $c_3 = e_1 + e_8$ of $\bar{W}$, say

$$c_2 = e_1 + e_6 = r_0 d_{i(0)} + \cdots + r_m d_{i(m)},$$

$$c_3 = e_1 + e_8 = s_0 d_{j(0)} + \cdots + s_l d_{j(l)} \quad ,$$

where $r_0, \cdots, r_m, s_0, \cdots, s_l \in F$, $i(0), \cdots, i(m)$ are distinct, and $j(0), \cdots, j(l)$ are distinct. Each of the elements $d_{i(0)}, \cdots, d_{i(m)}, d_{j(0)}, \cdots, d_{j(l)}$ belongs to $\bar{W}$, where $\bar{W} = L(\bar{\delta}_1)$. The only element of $\bar{\delta}_1$ which has e_6 as a term is c_2, hence at least one of $d_{i(0)}, \cdots, d_{i(m)}$ has a non-zero coordinate with respect to c_2 when expressed as a L.C.N.Z.C. of elements in $\bar{\delta}_1$. Choose one, say $d_{i(p)}$, where $0 \leqq p \leqq m$. Similarly, at least one of $d_{j(0)}, \cdots, d_{j(l)}$ must have a non-zero coordinate with respect to c_3, when expressed as a L.C.N.Z.C. of elements in $\bar{\delta}_1$. Choose one, say $d_{j(q)}$, where $0 \leqq q \leqq l$. Now assume that both $d_{i(p)}$ and $d_{j(q)}$ are expressed as L.C.N.Z.C. of elements in $\bar{\delta}_1$.

Case 1. $d_{i(p)}$ has coordinate 0 w.r.t. c_3 and $d_{j(q)}$ has coordinate 0 with respect to c_2. Then clearly $e_1 \in \eta_{di(p)} \cap \eta_{dj(q)} \subset \sigma$.

Case 2. Either $d_{i(p)}$ has a non-zero cordinate with respect to c_3 or $d_{j(q)}$ has a non-zero coordinate with respect to c_2. We may assume without loss of generality that the former holds. Since $c_2 = e_1 + e_6$ does not have e_8 as a term, at least one of $d_{i(s)}$, for $0 \leqq s \leqq m$ and $s \neq p$, must also have a non-zero coordinate w.r.t. c_3. Then $e_8 \in \eta_{di(p)} \cap \eta_{di(s)} \subset \sigma$.

3. The equivalence

We have proved the existence of $\aleph_0$-dimensional r.e. spaces $\bar{W}$ and r.e. repères $\bar{\beta}$ such that

$$\bar{W} \leqq L(\bar{\beta}) \text{ and } \bar{\beta} \text{ has property } \Delta \text{ with respect to } \bar{W}.$$

In both of our examples, $\bar{\beta} = \bar{\beta}_W = \eta$; in one case $\bar{W}$ was recursive relative to $L(\bar{\beta}_W)$ and in the other case it was not. Now suppose that $\bar{W}$ is an $\aleph_0$-dimensional r.e. space and $\bar{\beta}$ a r.e. repère such that $\bar{W} \leqq L(\bar{\beta})$. Consider the statement

(∗) there is an infinite subset β of $\bar{\beta}$ such that $L(\beta) \cap \bar{W}$ is not an α-space.

This section is devoted to showing that (∗) holds if and only if $\bar{\beta}$ has property Δ with respect to $\bar{W}$. We first demonstrate the sufficiency. The technique was developed with the help of insight gained by reading Soare's proof of Osofsky's result concerning the existence of non-α-spaces (see Soare (1974; Section 1)).

Proposition P3. *The intersection of a r.e. space and an α-space need not be an α-space.*

Proof. Let $\bar{W}$ be an $\aleph_0$-dimensional r.e. space and $\bar{\beta}$ a r.e. repère such that $\bar{W} \leqq L(\bar{\beta})$ and $\bar{\beta}$ has property Δ with respect to $\bar{W}$. We may assume w.l.g. that $\bar{\beta} = \bar{\beta}_W$. Let all infinite r.e. repères in $\bar{W}$ be enumerated without repetitions in the sequence $\langle \bar{\alpha}_0, \bar{\alpha}_1, \cdots \rangle$. Let a_{nm} be a function of two variables such that for every n, a_{nm} is a $1-1$ recursive function of m with $\bar{\alpha}_n$ as range. Let b_n be a $1-1$ recursive function ranging over $\bar{\beta}$. We shall write

$$\bar{\beta}_{nm} = \bar{\beta}_{a(n,m)},$$

that is,

$$\bar{\beta}_{nm} = \{b \in \bar{\beta} \mid a_{nm} \text{ has a non-zero coordinate w.r.t. } b \text{ when expressed as a L.C.N.Z.C. of elements in } \bar{\beta}\}.$$

We shall define by induction an infinite sequence $\langle x_0, x_1, \cdots \rangle$ of elements in $\bar{W}$. For every number k, we define

$$A_k = \begin{cases} \varnothing, \text{ if } x_k \notin L(\bar{\alpha}_k), \\ \{a \in \bar{\alpha}_k \mid x_k \text{ has a non-zero coordinate with respect to } a, \\ \quad \text{if expressed as a L.C.N.Z.C. of elements in } \bar{\alpha}_k\}, \\ \quad \text{otherwise.} \end{cases}$$

$$B_k = \{b \in \bar{\beta} \mid x_k \text{ has a non-zero coordinate with respect to } b, \text{ when expressed as a L.C.N.Z.C. of elements in } \bar{\beta}\}.$$

The goal of the following construction is to choose for every number n, an element x_n in $\bar{W}$ in such a manner that if

$$\beta = \bigcup_{k=0}^{\infty} B_k$$

and $S = L(\beta) \cap \bar{W}$, then

$$(\forall n)[x_n \in S \text{ and } x_n \notin L(\bar{\alpha}_n \cap S)].$$

Since every α-basis of S is of the form $\bar{\alpha}_n \cap S$, for some n, this would imply that S is not an α-space. The sequence $\langle x_0, x_1, \cdots, \rangle$ of elements in $\bar{W}$ we wish to define is such that for every number n,

$(1, n)$ $x_0, \cdots, x_n$ are distinct and linearly independent,

$(2, n)$ $(\forall i \leqq n)[A_i \neq \varnothing \Rightarrow A_i \backslash L(\bigcup_{j \leqq n} B_j) \neq \varnothing]$,

$(3, n)$ $(\forall i \leqq n)[x_i \in L(\bar{\alpha}_i) \Leftrightarrow \operatorname{codim} \bar{W} L(\bar{\alpha}_i) < \aleph_0]$.

Basis: $n = 0$. If $\operatorname{codim}_{\bar{W}} L(\bar{\alpha}_0) = \aleph_0$, we define

$$x_0 = \min[\bar{W} \backslash L(\bar{\alpha}_0)].$$

Then x_0 exists, since $L(\bar{\alpha}_0) < \bar{W}$ and (1,0) holds, because $x_0 \neq 0$. The fact that $x_0 \notin L(\bar{\alpha}_0)$ implies that $A_0 = \varnothing$, hence (2,0) is true. Finally, (3,0) holds, since $x_0 \notin L(\bar{\alpha}_0)$. Now consider the case that $\operatorname{codim} \bar{W} L(\bar{\alpha}_0) < \aleph_0$. Then the r.e. repère $\bar{\alpha}_0$ can be extended to a r.e. basis α_0' of $\bar{W}$ such that $\alpha_0' \backslash \bar{\alpha}_0$ is finite. By remark (b) following the definition of property Δ, there are two distinct elements a_{0i} and a_{0j} in $\alpha_0' \cap \bar{\alpha}_0$ such that $\bar{\beta}_{0i} \cap \bar{\beta}_{0j} \neq \varnothing$. Let

$$a_{0i} = rb_p; + r_0 b_{i(0)} + \cdots + r_k b_{i(k)},$$

$$a_{0j} = sb_p + s_0 b_{j(0)} + \cdots + s_l b_{j(l)},$$

where $r, r_0, \cdots, r_k, s, s_0, \cdots, s_l \in F \backslash (0)$, $p \notin \{i_0, \cdots, i_k\}$ and $p \notin \{j_0, \cdots, j_l\}$. Define

$$x_0 = r^{-1} a_{0i} - s^{-1} a_{0j}.$$

Note that a_{0i}, a_{0j} are distinct elements of a repère, namely $\bar{\alpha}_0$; this implies (1,0). The element a_{0i} has a non-zero coordinate with respect to b_p, but $b_p \notin B_0$ by definition of x_0. Hence $a_{0i} \notin L(B_0)$ and since $A_0 = \{a_{0i}, a_{0j}\}$ we conclude that $a_{0i} \in A_0 \backslash L(B_0)$; thus (2,0) holds. Finally, (3,0) is true, for $x \in L(\bar{\alpha}_0)$ by the definition of x_0.

Inductive Step. As inductive hypothesis, assume that $n \geqq 1$ and elements $x_0, \cdots, x_{n-1}$ have been defined such that

(1, $n-1$) $x_0, \cdots, x_{n-1}$ are distinct and linearly independent,

(2, $n-1$) $(\forall i \leqq n-1)[A_i \neq \varnothing \Rightarrow A_i \backslash L\left(\bigcup_{j \leqq n-1} B_j\right) \neq \varnothing]$,

(3, $n-1$) $(\forall i \leqq n-1)[x_i \in L(\bar{\alpha}_i) \Leftrightarrow \operatorname{codim}_{\bar{W}} L(\bar{\alpha}_i) < \aleph_0]$.

Case 1. $\operatorname{Codim}_{\bar{W}} L(\bar{\alpha}_n) = \aleph_0$.

Suppose x_n is any element such that

$$\text{(i)} \qquad x_n \in \bar{W} \backslash L(\bar{\alpha}_n)$$

and

$$\text{(ii)} \quad x_n \notin L\left(\bigcup_{j \leqq n-1} B_j\right).$$

Such an element x_n exists, since $L(\bar{\alpha}_n)$ has infinite co-dimension with respect to $\bar{W}$ and $B_0 \cup \cdots \cup B_{n-1}$ is a finite set. Then $(3, n)$ holds by (i). By the definition of B_j for $j \leqq n-1$,

$$\{x_0, \cdots, x_{n-1}\} \subset L\left(\bigcup_{j \leqq n-1} B_j\right),$$

hence $x_n \notin L(x_0, \cdots, x_{n-1})$ by (ii); thus $(1, n)$ holds. Since A_n will be empty for each such element x_n, condition $(2, n)$ is equivalent to

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(iii) $$(\forall_i \leqq n-1)\left[A_i \neq \varnothing \Rightarrow A_i \backslash L\Big(\bigcup_{j \leqq n} B_j\Big) \neq \varnothing\right].$$

We now show that an element x_n satisfying (i) and (ii) can be chosen so that (iii) holds as well. Put for $i \leqq n-1$,

$$C_i = \begin{cases} \varnothing, \text{ if } A_i = \varnothing \\ \{b \in \bar{\beta} \mid \text{some } a \in A_i \backslash L\Big(\bigcup_{j \leqq n-1} B_j\Big) \text{ has a non-zero coordinate} \\ \quad \text{w.r.t. } b \text{ if expressed as a L.C.N.Z.C. of elements in } \bar{\beta}\}, \\ \quad \text{otherwise.} \end{cases}$$

Then $C_0 \cup \cdots \cup C_{n-1}$ is a finite set, since $A_0, \cdots, A_{n-1}$ are finite. Define

$$x_n = (\mu y)\left[y \in \bar{W} \backslash L(\bar{\alpha}_n) \text{ and } \bar{\beta}_y \cap \bigcup_{j \leqq n-1} (B_j \cup C_j) = \varnothing\right].$$

Note that x_n exists, since $L(\bar{\alpha}_n)$ has infinite codimension with respect to $\bar{W}$, while $\cup_{j \leqq n-1}(B_j \cup C_j)$ is a finite set, say of cardinality p. Then by linear algebra, we can find $p+1$ elements $\langle y_0, \cdots, y_p \rangle$ distinct and linearly independent such that $L(y_0, \cdots, y_p) \cap L(\bar{\alpha}_n) = (0)$ and such that at least one non-zero $z \in L(y_0, \cdots, y_p)$ satisfies

$$\bar{\beta}_z \cap \bigcup_{j \leqq n-1} (B_j \cup C_j) = \varnothing.$$

It follows from the definitions of x_n that $x_n \neq 0$ and

$$B_n \cap \Big(\bigcup_{j \leqq n-1} (B_j \cup C_j)\Big) = \varnothing.$$

Thus, from $x_n \in L(B_n)$ we conclude that (ii) holds. It remains to be shown that (iii) is true. We claim that

(12) $$A_i \backslash L\Big(\bigcup_{j \leqq n-1} B_j\Big) \subset A_i \backslash L\Big(\bigcup_{j \leqq n} B_j\Big), \text{ for } i \leqq n-1.$$

For let us assume that

$$a \in A_i \backslash L\Big(\bigcup_{j \leqq n-1} B_j\Big), \text{ where } i \leqq n-1.$$

Then a only has non-zero coordinates w.r.t. elements of $\bar{\beta}$ which belong to C_i, hence to $\cup_{j \leqq n-1} C_j$. All elements in

$$L\Big(\bigcup_{j \leqq n} B_j\Big) \backslash L\Big(\bigcup_{j \leqq n-1} B_j\Big)$$

have at least one non-zero coordinate w.r.t. some element in B_n, where

$$B_n \cap \bigcup_{j \leqq n-1} (B_j \cup C_j) = \varnothing.$$

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Hence

$$a \in A_i \backslash L\left(\bigcup_{j \leq n} B_j\right).$$

This proves (12). Then (iii) follows from $(2, n-1)$ and (12). Summarizing, we see that x_n has been defined so that the conditions $(1, n)$, $(2, n)$ and $(3, n)$ are satisfied.

Case 2. $\operatorname{Codim}_{\bar{W}} L(\bar{\alpha}_n) < \aleph_0$.
Let d be the finite codimension of $L(\bar{\alpha}_n)$ relative to $\bar{W}$. Then the r.e. repère $\bar{\alpha}_n$ can be extended to a r.e. basis α_n' of $\bar{W}$ by adjoining d distinct elements, say $h_0, \cdots, h_{d-1}$. We shall use the following enumeration of α_n' without repetitions:

(III) $$h_0, \cdots, h_{d-1}, a_{n0}, a_{n1}, \cdots.$$

We define

$$m = (\mu x)(\forall y)\left[y > x \Rightarrow b_y \notin \bigcup_{j \leq n-1} (B_j \cup C_j)\right].$$

Since $\bar{\beta}$ has property Δ with respect to $\bar{W}$, we can, no matter how far out we go in (III), find two distinct elements c and e in α_n' such that

$$\bar{\beta}_c \cap \bar{\beta}_e \neq \varnothing \text{ and not } [\bar{\beta}_c \cap \bar{\beta}_e \subset \{b_0, \cdots, b_m\}].$$

In particular, we want to go out a finite distance $t+1$ in (III), that is, to $a_{n,t-d}$ such that

(i) all the remaining elements of (III) are in $\bar{\alpha}_n$,
(ii) all the remaining elements of (III) are *not* in

$$L\left[\bigcup_{j \leq n-1} (B_j \cup C_j)\right],$$

let $t - d = h$. We distinguish two cases.

Subcase 2.1. There exist distinct elements $i, j > h$ such that

$$\bar{\beta}_{ni} \cap \bar{\beta}_{nj} \neq \varnothing \text{ and } (\bar{\beta}_{ni} \cup \bar{\beta}_{nj}) \cap \bigcup_{k \leq n-1} (B_k \cup C_k) = \varnothing.$$

We select such an ordered pair $\langle i, j \rangle$ of elements. Let

$$b_p \in \bar{\beta}_{ni} \cap \bar{\beta}_{nj},$$

and

$$a_{ni} = rb_p + r_0 b_{i(0)} + \cdots + r_k b_{i(k)},$$

$$a_{nj} = sb_p + s_0 b_{j(0)} + \cdots + s_l b_{j(l)},$$

where $r, r_0, \cdots, r_k, s, s_0, \cdots, s_l \in F\backslash(0)$, $p \notin \{i_0, \cdots, i_k\}$ and $p \notin \{j_0, \cdots, j_l\}$. Define

$$x_n = r^{-1} a_{ni} - s^{-1} a_{nj}.$$

We proceed to show that $(1, n)$, $(2, n)$ and $(3, n)$ hold.

Re $(1, n)$. $x_n \neq 0$, since a_{ni} and a_{nj} are distinct elements of a repère, namely $\bar{\alpha}_n$. By the definition of a_{ni} and a_{nj},

$$(\bar{\beta}_{ni} \cap \bar{\beta}_{nj}) \cup \bigcup_{k \leqq n-1} (B_k \cup C_k) = \varnothing.$$

Using the two relations

$$x_n \in L(\bar{\beta}_{ni} \cup \bar{\beta}_{nj}) \Rightarrow x_n \notin L\left(\bigcup_{k \leqq n-1} B_k\right),$$

$$\{x_0, \cdots, x_{n-1}\} \subset L\left(\bigcup_{k \leqq n-1} B_k\right),$$

we conclude that $x_n \notin L(x_0, \cdots, x_{n-1})$. This implies $(1, n)$.

Re $(2, n)$. We wish to prove

$$(\forall_i \leqq n)\left[A_i \neq \varnothing \Rightarrow A_i \backslash L\left(\bigcup_{k \leqq n} B_k\right) \neq \varnothing\right],$$

and we split this up into two parts, namely

(a) $$A_n \neq \varnothing \Rightarrow A_n \backslash L\left(\bigcup_{k \leqq n} B_k\right) \neq \varnothing,$$

(b) $$(\forall_i \leqq n-1)\left[A_i \neq \varnothing \Rightarrow A_i \backslash L\left(\bigcup_{k \leqq n} B_k\right) \neq \varnothing\right].$$

Re (a). $A_n = \{a_{ni}, a_{nj}\}$, hence $A_n \neq \varnothing$. We have to prove

$$A_n \backslash L\left(\bigcup_{k \leqq n} B_k\right) \neq \varnothing.$$

Since $a_{ni} \in A_n$, it suffices to show that

(13) $$a_{ni} \notin L\left(\bigcup_{k \leqq n} B_k\right).$$

The definition of x_n implies $b_p \notin B_n$. Moreover,

$$b_p \in \bar{\beta}_{ni} \text{ and } \bar{\beta}_{ni} \cap \left(\bigcup_{k \leqq n-1} B_k\right) = \varnothing \Rightarrow b_p \notin \bigcup_{k \leqq n-1} B_k.$$

It follows that $b_p \notin \cup_{k \leqq n} B_k$. However, $b_p \in \bar{\beta}_{ni}$, that is , a_{ni} has a non-zero coordinate with respect to b_p when expressed as a L.C.N.Z.C. of elements in $\bar{\beta}$. Thus (13) and (a) are true.

Re (b). Recall that we know by the inductive hypothesis

$$(\forall i \leqq n-1)\left[A_i \neq \varnothing \Rightarrow A_i \backslash L\left(\bigcup_{k \leqq n-1} B_k\right) \neq \varnothing\right].$$

It therefore suffices to prove

(14) $$A_i \backslash L\left(\bigcup_{k \leqq n-1} B_k\right) \subset A_i \backslash L\left(\bigcup_{k \leqq n} B_k\right), \text{ for } i \leqq n-1.$$

Assume that a belongs to the left side of (14), where $i \leqq n-1$. Then a only has non-zero coordinates with respect to elements in C_i. If on the other hand,

$$a \in L\left(\bigcup_{k \leqq n} B_k\right) \backslash L\left(\bigcup_{k \leqq n-1} B_k\right),$$

then a has at least one non-zero coordinate with respects to some element in B_n, where

$$B_n \subset \bar{\beta}_{ni} \cup \bar{\beta}_{nj} \text{ and } (\bar{\beta}_{ni} \cup \bar{\beta}_{nj}) \cap C_i = \emptyset.$$

We conclude that if a belongs to the left side of (14), for some $i \leqq n-1$, then

$$a \notin L\left(\bigcup_{k \leqq n} B_k\right) \backslash L\left(\bigcup_{k \leqq n-1} B_k\right),$$

so that a also belongs to the right side of (14).

Re $(3, n)$. $x_n \in L(\bar{\alpha}_n)$, since $\{a_{ni}, a_{nj}\} \subset \bar{\alpha}_n$. Thus $(3, n)$ holds.

Subcase 2.2. We have

$$(\forall i, j > h)\left[\bar{\beta}_{ni} \cap \bar{\beta}_{nj} \neq \emptyset \Rightarrow (\bar{\beta}_{ni} \cup \bar{\beta}_{nj}) \cap \bigcup_{k \leqq n-1} (B_k \cup C_k) \neq \emptyset\right].$$

Let $p = \operatorname{card} \cup_{k \leqq n-1} (B_k \cup C_k)$. We now choose $p+1$ ordered pairs $\langle a_{n,i(s)}, a_{n,j(s)}\rangle$, for $s \leqq p$, of elements in $\bar{\alpha}_n$ such that $a_{n,i(0)}, a_{n,j(0)}, \cdots, a_{n,i(p)}, a_{n,j(p)}$ are distinct and

(c) $i(s), j(s) > h$ for $s \leqq p$,

(d) $\bar{\beta}_{n,i(s)} \cap \bar{\beta}_{n,j(s)} \neq \emptyset$, for $s \leqq p$,

(e) $(\forall s \leqq p)(\exists x)[b_x \in \bar{\beta}_{n,i(s)} \cap \bar{\beta}_{n,j(s)} \text{ and } b_x \notin \{b_0, \cdots, b_m\}]$.

Note that by the definition of m,

$$b_x \notin \{b_0, \cdots, b_m\} \Rightarrow b_x \notin \bigcup_{j \leqq n-1} (B_j \cup C_j).$$

Define

$$m(s) = (\mu x)[b_x \in \bar{\beta}_{n,i(s)} \cap \bar{\beta}_{n,j(s)} \text{ and } b_x \notin \{b_0, \cdots, b_m\}], \text{ for } s \leqq p,$$

$$\Gamma = \{b_{m(0)}, \cdots, b_{m(p)}\} \cup \bigcup_{j \leqq n-1} (B_j \cup C_j),\ q = \operatorname{card} \Gamma,$$

$$D = \{a_{n,i(0)}, a_{n,j(0)}, \cdots, a_{n,i(p)}, a_{n,j(p)}\}.$$

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According to the definition of $b_{m(s)}$,

$$\text{(f)}\quad \text{for } s \leqq p, \begin{cases} b_{m(s)} \in \bar{\beta}_{n,i(s)} \cap \bar{\beta}_{n,j(s)} \text{ and} \\ b_{m(s)} \notin \{b_0, \cdots, b_m\}, \\ b_{m(s)} \notin \bigcap_{j \leqq n-1} (B_j \cup C_j). \end{cases}$$

The elements $b_{m(0)}, \cdots, b_{m(p)}$ are not necessarily distinct, but none of them belongs to $\bigcup_{j \leqq n-1} (B_j \cup C_j)$, hence

$$\text{(g)} \qquad p + 1 \leqq q \leqq 2p + 1.$$

We proceed to prove

(h) there is an element $y \in L(D) \setminus (0)$ such that when expressed as a L.C.N.Z.C. of elements in $\bar{\beta}$, y has coordinate 0 with respect to each element in Γ, that is, $\bar{\beta}_y \cap \Gamma = \varnothing$.

To prove (h), we put

$$\hat{\beta} = \bar{\beta}_D \cup \Gamma, \quad \hat{V} = L(\hat{\beta}).$$

Then $\hat{\beta}$ is a finite subset of $\bar{\beta}$; let $l = \operatorname{card}(\hat{\beta})$. Clearly

$$L(D) \leqq L(\hat{\beta}) = \hat{V} \text{ and } \dim \hat{V} = l.$$

Let $b_{c(1)}, \cdots, b_{c(l)}$ be an enumeration without repetitions of the basis $\hat{\beta}$ of $\hat{V}$ such that

$$\Gamma = \{b_{c(1)}, \cdots, b_{c(q)}\}, \; \hat{\beta} \setminus \Gamma = \{b_{c(q+1)}, \cdots, b_{c(l)}\}.$$

Every element v of $\hat{V}$ can be uniquely expressed in the form

$$v = r_1 b_{c(1)} + \cdots + r_l b_{c(l)}, \text{ where } r_1, \cdots, r_l \in F.$$

Let

$$\hat{W} = \{v \in \hat{V} \mid r_1 = 0, \cdots, r_q = 0\},$$

then $\dim \hat{W} = l - q$, and

$$\dim[\hat{W} \cap L(D)] + \dim[\hat{W} + L(D)]$$

$$= \dim \hat{W} + \dim L(D) = l - q + 2p + 2,$$

$$q \leqq 2p + 1 \Rightarrow \dim[\hat{W} \cap L(D)] + \dim[\hat{W} + L(D)] \geqq l + 1,$$

$$\hat{W} + L(D) \leqq \hat{V} \Rightarrow \dim[\hat{W} + L(D)] \leqq l.$$

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Hence $\dim[\hat{W} \cap L(D)] \geqq 1$, that is, $(0) < \hat{W} \cap L(D)$. Then every non-zero element $y \in \hat{W} \cap L(D)$ satisfies the requirements. This completes the proof of (h). Define

$$x_n = (\mu y)[y \in (\hat{W} \cap L(D)) \setminus (0)].$$

Then we shall show that $(1, n)$, $(2, n)$ and $3, n)$ hold.

Re $(1, n)$. B_n is disjoint from Γ, hence also from $\cup_{j \leqq n-1} B_j$. Since $x_n \neq 0$, we obtain

$$x_n \notin L\Big(\bigcup_{j \leqq n-1} B_j\Big), \; \{x_0, \cdots, x_{n-1}\} \subset L\Big(\bigcup_{j \leqq n-1} B_j\Big),$$

and $(1, n)$ follows in the usual way.

Re $(2, n)$. We wish to prove

$$(\forall i \leqq n)\Big[A_i \neq \varnothing \Rightarrow A_i \setminus L\Big(\bigcup_{j \leqq n} B_j\Big) \neq \varnothing\Big],$$

and we split this up into two parts, namely

(i) $$A_n \neq \varnothing \Rightarrow A_n \setminus L\Big(\bigcup_{k \leqq n} B_k\Big) \neq \varnothing,$$

(j) $$(\forall i \leqq n-1)\Big[A_i \neq \varnothing \Rightarrow A_i \setminus \Big(\bigcup_{k \leqq n} B_k\Big) \neq \varnothing\Big].$$

Re (i). Since $x_n \in L(D) \setminus (0)$, we know that $A_n \neq \varnothing$, hence all we have to show is

$$A_n \setminus L\Big(\bigcup_{j \leqq n} B_j\Big) \neq \varnothing.$$

Since $x_n \neq 0$, there is a number $t \leqq p$ such that $a_{n,i(t)} \in A_n$ or $a_{n,j(t)} \in A_n$; we may assume without loss of generality that $a_{n,i(t)} \in A_n$. It now suffices to prove that

(15) $$a_{n,i(t)} \notin L\Big(\bigcup_{j \leqq n} B_j\Big).$$

By the definition of $b_{m(t)}$, the element $a_{n,i(t)}$ has a non-zero coordinate with respect to $b_{m(t)}$. However, $b_{m(t)}$ does not belong to $\cup_{j \leqq n-1} B_j$ by (f). Moreover, x_n has coordinate 0 with respect to each element in Γ, in particular with respect to $b_{m(t)}$; this implies $b_{m(t)} \notin B_n$. Hence $b_{m(t)} \notin \cup_{j \leqq n} B_j$ and we conclude that (15) holds.

Re (j). Recall that by the inductive hypothesis

$$(\forall i \leqq n-1)\Big[A_i \neq \varnothing \Rightarrow A_i \setminus L\Big(\bigcup_{k \leqq n-1} B_k\Big) \neq \varnothing\Big].$$

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It therefore suffices to prove

$$A_i \backslash L\Big(\bigcup_{k \leqq n-1} B_k\Big) \subset A_i \backslash L\Big(\bigcup_{k \leqq n} B_k\Big) \text{ for } i \leqq n-1. \tag{16}$$

Assume that a belongs to the left side of (16), where $i \leqq n-1$. Then a only has non-zero coordinates with respect to elements in C_i, hence in Γ. If, on the other hand,

$$a \in L\Big(\bigcup_{k \leqq n} B_k\Big) \backslash L\Big(\bigcup_{k \leqq n-1} B_k\Big),$$

then a has at least one non-zero coordinate with respect to element of B_n, where B_n is disjoint from Γ. Hence

$$a \notin L\Big(\bigcup_{k \leqq n} B_k\Big) \backslash L\Big(\bigcup_{k \leqq n-1} B_k\Big),$$

and a belongs to the right side of (16).

Re $(3, n)$. $x_n \in L(D) \backslash (0)$, where $D \subset \bar{\alpha}_n$, hence $x \in L(\bar{\alpha}_n)$ and $(3, n)$ holds.

This completes the inductive step. We have defined an infinite sequence $\langle x_0, x_1, \cdots \rangle$ of elements in $\bar{W}$ such that for every n,

$(1, n)$ $x_0, \cdots, x_n$ are distinct and linearly independent,

$(2, n)$ $(\forall_i \leqq n)[A_i \neq \varnothing \Rightarrow A_i \backslash L(\cup_{j \leqq n} B_j) \neq \varnothing]$,

$(3, n)$ $(\forall i \leqq n)[x_i \in L(\bar{\alpha}_i) \Leftrightarrow \operatorname{codim}_{\bar{W}} L(\bar{\alpha}_i) < \aleph_0]$.

We claim that

$$x_0, x_1, \cdots \text{ are all distinct and linearly independent,} \tag{17}$$

$$(\forall n)\Big[A_n \neq \varnothing \Rightarrow A_n \backslash L\Big(\bigcup_{k=0}^{\infty} B_k\Big) \neq \varnothing\Big], \tag{18}$$

$$(\forall n)[x_n \in L(\bar{\alpha}_n) \Leftrightarrow \operatorname{codim}_{\bar{W}} L(\bar{\alpha}_n) < \aleph_0]. \tag{19}$$

Relations (17) and (19) follow immediately from the fact that $(1, n)$ and $(3, n)$ hold for every n. We now establish (18). Suppose $A_k \neq \varnothing$ and $A_k \backslash L(\cup_{j=0}^{\infty} B_j) = \varnothing$, that is $A_k \subset L(\cup_{j=\infty}^{\infty} B_j)$. Since $A_k, B_0, B_1, \cdots$ are finite sets, there is a number $m \geqq k$ such that

$$A_k \subset L\Big(\bigcup_{j \leqq m} B_j\Big), \text{ that is, } A_k \backslash L\Big(\bigcup_{j \leqq m} B_j\Big) = \varnothing,$$

contrary to $(2, m)$.

We define $\beta = \cup_{k=0}^{\infty} B_k$, $V = L(\beta)$, $S = V \cap \bar{W}$. Clearly, $x_n \in (B_n)$, for every n, hence $\{x_0, x_1, \cdots\} \subset V$. The elements $x_0, x_1, \cdots$ also belong to $\bar{W}$, hence

$\{x_0, x_1, \cdots\} \subset S$. Thus S is an $\aleph_0$-dimensional space by (17), hence so is V; then β is an infinite subset of $\bar{\beta}$ and $\beta = \bar{\beta}_S$. Relation (18) can be rewritten as

$$(\forall n)[A_n \neq \emptyset \Rightarrow A_n \backslash V \neq \emptyset].$$

Since $A_n \subset \bar{W}$, $A_n \backslash V = A_n \backslash (V \cap \bar{W})$ and we obtain

$$(\forall n)[A_n \neq \emptyset \Rightarrow A_n \backslash S \neq \emptyset]. \tag{20}$$

We claim that S is not an α-space. For suppose it were. Then S would have an α-basis of the form $\bar{\alpha} \cap S$, for some infinite r.e. repère $\bar{\alpha}$ in $\bar{W}$, say $\bar{\alpha} = \bar{\alpha}_n$. Hence $S = L(\bar{\alpha}_n \cap S)$. Since $x_n \in S$ we obtain $x_n \in L(\bar{\alpha}_n)$. However $x_n \neq 0$, hence $A_n \neq \emptyset$. We now have a contradiction, for

$$x_n \in L(\bar{\alpha}_n \cap S) \Rightarrow A_n \subset S,$$

$$A_n \neq \emptyset \Rightarrow A_n \backslash S \neq \emptyset, \text{ by (20).}$$

We conclude that S is not an α-space.

Let $\bar{W}$ be an $\aleph_0$-dimensional r.e. space and $\bar{\beta}$ a r.e. repère such that $\bar{W} \leqq L(\bar{\beta})$. According to the proof of P3

$$\begin{aligned}&\bar{\beta} \text{ has property } \Delta \text{ with respect to } \bar{W} \Rightarrow\\ &(\exists \beta)[\beta \subset \bar{\beta} \text{ and } \beta \text{ is infinite and}\\ &L(\beta) \cap \bar{W} \text{ is not an } \alpha\text{-space}].\end{aligned}$$

We conclude this section by proving the converse of this condition. We shall need the following two lemmas.

LEMMA L4. *Let A, B, W be spaces such that A is finite dimensional and $A \cap B = (0)$. Then $B \cap W$ has finite codimension in $(A \oplus B) \cap W$.*

PROOF. Assume the hypothesis and suppose that $B \cap W$ has infinite co-dimension with respect to $(A \oplus B) \cap W$. Then there is an $\aleph_0$-dimensional space C such that

$$(B \cap W) \cap C = (0) \text{ and } (B \cap W) \oplus C = (A \oplus B) \cap W.$$

Let $y_0, y_1, \cdots$ be distinct elements in C such that $\{y_0, y_1, \cdots\}$ is a basis of C. Define for $n \in \varepsilon$, the elements a_n and b_n by

$$y_n = a_n + b_n, \; a_n \in A, \; b_n \in B.$$

Let $m = \dim(A)$. Then $\langle a_0, \cdots, a_m \rangle$ is a linearly dependent sequence of elements in A, hence there exist elements $r_0, \cdots, r_m \in F$, at least one of which is non-zero, such that $r_0 a_0 + \cdots + r_m a_m = 0$. This implies

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$$r_0y_0 + \cdots + r_my_m = r_0b_0 + \cdots + r_mb_m,$$
$$r_0y_0 + \cdots + r_my_m \in B \cap C.$$

But $C \leqq W$ and $(B \cap W) \cap C = (0)$ imply that $B \cap C = (0)$. It follows that $y_0, \cdots, y_m$ are linearly dependent, contrary to the fact that they are distinct elements of a repère. Hence $B \cap W$ cannot have infinite codimension in $(A \oplus B) \cap W$.

LEMMA L5. *Let* $\Gamma = \{V_i \mid i \in I\}$ *be a non-empty family of distinct α-spaces, where* $I = \{0, \cdots, n-1\}$ *if card* $\Gamma = n > 0$ *and* $I = \varepsilon$ *otherwise. Let* $S = \cap \Gamma$. *Then for all finite dimensional spaces B,*

$$S \parallel B \Leftrightarrow S \cap B = (0).$$

PROOF. (a) $\Rightarrow$ This is clear from the definition of $S \parallel B$. (b) $\Leftarrow$ If $\dim(B) = 0$, we are done, so assume $\dim(B) = m \geqq 1$. We establish the result by induction on m.

Basis step. $m = 1$. Then $B = L(p)$ for some $p \notin S$. Then there must be at least one $V_i \in \Gamma$ such that $p \notin V_i$. Pick one, say V_j, and let α_j be an α-basis for V_j, and $\alpha_j \subset \bar{\alpha}_j$, where $\bar{\alpha}_j$ is a r.e. repère. If $p \notin L(\bar{\alpha}_j)$, we are done since $S \parallel B$ by $\langle L(\bar{\alpha}_j), B\rangle$. If $p \in L(\bar{\alpha}_j)$, let $p = r_0a_0 + \cdots + r_ka_k$, where $r_0, \cdots, r_k \in F\backslash(0)$ and $a_0, \cdots, a_k \in \bar{\alpha}_j$. Now $p \notin L(\alpha_j) = V_j$ implies that at least one of $a_0, \cdots, a_k$ is not an element of α_j, say a_0. Then $S \parallel B$ by $\langle L(\bar{\alpha}_j \backslash \{a_0\}), B\rangle$.

Inductive hypothesis. Assume $S \cap B = (0)$ implies $S \parallel B$ for all B such that $\dim(B) \leqq k$.

Inductive step. Suppose $\dim(B) = k + 1$. Let $b \in B\backslash(0)$. Then by the induction hypothesis applied to $L(b)$, there exists a r.e. space $\bar{W}$ such that $S \leqq \bar{W}$ and $L(b) \cap \bar{W} = (0)$. Thus

$$(0) \leqq \bar{W} \cap B < B,\ 0 \leqq \dim((\bar{W} \cap B) \leqq k.$$

If $\bar{W} \cap B = (0)$, we are done. So assume $(0) < \bar{W} \cap B < B$. By the induction hypothesis applied to $\bar{W} \cap B$, there exists a r.e. space $\bar{V}$ such that $S \leqq \bar{V}$ and $\bar{V} \cap (\bar{W} \cap B) = (0)$. Hence $S \parallel B$ by $\langle \bar{W} \cap \bar{V}, B\rangle$ since $(\bar{W} \cap \bar{V}) \cap B = (0)$ while $S \leqq \bar{W} \cap \bar{V}$.

PROPOSITION P6. *Let* $\bar{W}$ *be an* $\aleph_0$*-dimensional r.e. space and* $\bar{\beta}$ *a r.e. repère such that* $\bar{W} \leqq L(\bar{\beta})$. *If there is an infinite subset* β *of* $\bar{\beta}$ *such that* $L(\beta) \cap \bar{W}$ *is not an α-space, then* $\bar{\beta}$ *has property* Δ *with respect to* $\bar{W}$.

PROOF. We may assume without loss of generality that $\bar{\beta} = \bar{\beta}_W$. We shall prove the contrapositive. Suppose $\bar{\beta}$ does not have property Δ with respect to $\bar{W}$.

This means that there is a $1-1$ recursive function d_n enumerating a r.e. bases $\bar{\gamma}$ of $\bar{W}$, and a finite subset $\{b_0, \cdots, b_m\}$ of $\bar{\beta}$ such that

$$(21) \qquad (\forall i)(\forall_j)[i \neq j \Rightarrow \bar{\beta}_{d(i)} \cap \bar{\beta}_{d(j)} \subset \{b_0, \cdots, b_m\}].$$

Define $\rho = \{b_0, \cdots, b_m\} \cap \bar{\beta}_{\bar{\gamma}}$. Then ρ is a finite subset of $\bar{\beta}$. Let β be any infinite subset of $\bar{\beta}$, and $S = L(\beta) \cap \bar{W}$. We wish to prove that S is an α-space. The sets β and ρ are repères, and $\beta \cup \rho$ is a repère since it is included in $\bar{\beta}$. Let $\rho' = \rho \backslash \beta$. Then

$$(22) \qquad L(\beta) \cap L(\rho') = (0), L(\beta \cup \rho) = L(\beta) \oplus L(\rho').$$

We proceed to show that

$$(23) \qquad S \leqq L(\bar{\gamma}_S) \leqq [L(\beta) \oplus L(\rho')] \cap \bar{W}.$$

The first inclusion of (23) is obvious, since $S \leqq \bar{W}$ and $\bar{\gamma}$ is a basis of $\bar{W}$. To prove the second inclusion we shall show that

$$(24) \qquad d_k \in \bar{\gamma}_S \Rightarrow d_k \in [L(\beta) \oplus L(\rho')],$$

for trivially, $d_k \in \bar{W}$. Assume the hypothesis of (24). Then there is an element x in S which, when expressed as a L.C.N.Z.C. of elements in $\bar{\gamma}$, has a non-zero coordinate with respect to d_k; let

$$(25) \qquad x = rd_k + s_0 d_{i(0)} + \cdots + s_n d_{i(n)},$$

where $r, s_0, \cdots, s_n \in F \backslash (0)$ and $k, i_0, \cdots, i_n$ are distinct. Since $x \in S$, it can also be expressed in the form

$$(26) \qquad x = t_0 b_{j(0)} + \cdots + t_p b_{j(p)},$$

where $t_0, \cdots, t_p \in F \backslash (0)$ and $b_{j(0)}, \cdots, b_{j(p)}$ are distinct elements of β. If we can prove

$$(27) \qquad \bar{\beta}_{d(k)} \subset \beta \cup \rho,$$

we are done, for then $d_k \in L(\bar{\beta}_{d(k)})$ and (27) imply $d_k \in L(\beta \cup \rho)$, hence $d_k \in L(\beta) \oplus L(\rho')$ by (22). To prove (27), suppose $b \in \bar{\beta}_{d(k)}$. Either $b \in \beta$, hence $b \in \beta \cup \rho$, or $b \in \bar{\beta}_{d(k)} \backslash \beta$. In the latter case, $b \notin \{b_{j(0)}, \cdots, b_{j(p)}\}$, since $b_{j(0)}, \cdots, b_{j(p)}$ all belong to β. Hence in (25), at least one of the $d_{i(0)}, \cdots, d_{i(n)}$ must also have a non-zero coordinate with respect to b, when expressed as a L.C.N.Z.C. of element in $\bar{\beta}$, say $d_{i(q)}$, where $0 \leqq q \leqq p$. Then we have by (21)

$$b \in \bar{\beta}_{d(k)} \cap \bar{\beta}_{d(i(q))} \Rightarrow b \in \{b_0, \cdots, b_m\}.$$

Since trivially, $b \in \bar{\beta}_{\bar{\gamma}}$, we conclude that $b \in \rho$; again $b \in \beta \cup \rho$. This completes the proof of (27), and thereby of (24). We have now established (23). If we take $A = L(\rho')$, $B = L(\beta)$, $W = \bar{W}$ in $L4$, then $A \cap B = (0)$ since $L(\beta) \cap L(\rho') = (0)$.

Hence $B \cap W$ has finite codimension in $(A \oplus B) \cap W$, i.e., $S = L(\beta) \cap \bar{W}$ has finite codimension in $[L(\beta) \oplus L(\rho')] \cap \bar{W}$. Then (23) implies that S also has finite codimension in the α-space $L(\bar{\gamma}_S)$. Thus there is a finite dimensional space E such that $S \cap E = (0)$ and $S \oplus E = L(\bar{\gamma}_S)$. Note that $S = L(\beta) \cap \bar{W}$, where $L(\beta)$ and $\bar{W}$ are α-spaces. Hence $S \parallel E$ by *L*5. We know $S \oplus E = L(\bar{\gamma}_S)$, $S \parallel E$ and $L(\bar{\gamma}_S)$ is an α-space. Since E is r.e. (and isolic!) we know by the established cases of the conjecture (c) mentioned in the Introduction that S is an α-space.

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