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ASCENDING HNN-EXTENSIONS AND PROPERLY 3-REALISABLE GROUPS

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In this paper, we show that any ascending HNN-extension of a finitely presented group is properly 3-realisable. We recall that a finitely presented group G is said to be properly 3-realisable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\widetilde{K}$ has the proper homotopy type of a (PL) 3-manifold (with boundary).

1. INTRODUCTION

The most useful constructions in Combinatorial Group Theory are amalgamated free products and HNN-extensions, and they are the two basic examples in the theory of graphs of groups due to Bass and Serre (see [**9**]). We recall that given a group G and a subgroup $H \leqslant G$ together with monomorphisms (respectively homomorphisms) $\psi, \varphi : H \longrightarrow G$, the group determined by the presentation

$$\langle G, t; t^{-1}\psi(h)t = \varphi(h), h \in H \rangle$$

is an HNN-extension (respectively pseudo HNN-extension) of G over H, with stable letter t (see [**12**]). In case $H = G$ and $\psi = id_G$, this HNN-extension is called an *ascending* HNN-extension, and it will be denoted by $G*_{\varphi}$.

We are concerned with the behaviour of the property of being *properly 3-realisable* (for finitely presented groups) with respect to these constructions. Recall that a finitely presented group G is said to be properly 3-realisable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\widetilde{K}$ has the proper homotopy type of a p.l. 3-manifold. It is worth mentioning that the property of being properly 3-realisable has implications in the theory of cohomology of groups, in the sense that if G is properly 3-realisable then for some (equivalently any) compact 2-polyhedron K with $\pi_1(K) \cong G$ we have $H^2_c(\widetilde{K}; \mathbf{Z})$ free Abelian (by manifold duality arguments), and hence so is $H^2(G; \mathbf{Z}G)$ (see [**10**]). It is a long standing conjecture that $H^2(G; \mathbf{Z}G)$ be free Abelian for every finitely presented group G. See [**1, 6**] to learn more about properly 3-realisable groups.

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In [1] it was shown that amalgamated free products and HNN-extensions of properly 3-realisable groups over a finite cyclic group are properly 3-realisable. The main results of this paper follow.

Theorem 1.1. *Any ascending HNN-extension $G*_\varphi$ of a finitely presented group G is properly 3-realisable.*

Observe that the only property required is that G be finitely presented. More generally, the techniques used in the proof of Theorem 1.1 yield the following.

Theorem 1.2. *Let G be a 1-ended finitely presented group. If the fundamental pro-group of G at infinity is pro-(finitely generated free) and semistable, then G is properly 3-realisable.*

Corollary 1.3. *Let G be an infinite finitely presented group, without torsion in case G has infinitely many ends. If G is simply connected at infinity, then it is properly 3-realisable.*

Recall that, given a compact 2-polyhedron K with $\pi_1(K) \cong G$ and having $\widetilde{K}$ as universal cover, the number of ends of G is the number of ends of $\widetilde{K}$ which equals $0, 1, 2$ or ∞ (see [9, 15]); and G is said to be simply connected at infinity if $\widetilde{K}$ is so, that is, for every compact subpolyhedron $L \subset \widetilde{K}$ there is a compact subpolyhedron $J \supset L$ so that any map $S^1 \longrightarrow \widetilde{K} - J$ extends to a map $B^2 \longrightarrow \widetilde{K} - L$.

Note that if G is 1-ended, then any ascending HNN-extension of G is simply connected at infinity [14]. A particularly interesting example is Thompson's group

$$\begin{aligned} F &\equiv \langle x_k, k \geqslant 0; x_i^{-1} x_j x_i = x_{j+1}, \text{ if } i < j \rangle \\ &\cong \langle x_0, x_1; [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle \end{aligned}$$

whose cohomology modules $H^*(F, \mathbf{Z}F)$ are trivial [3] (and hence F is 1-ended, see [9]); and it is an infinitely iterated HNN-extension. See [4] to learn more about F and similar groups defined by Thompson.

2. The main results

In this section, we prove Theorems 1.1 and 1.2 and Corollary 1.3.

Proof of Theorem 1.1: Let G be a finitely presented group and $\varphi : G \longrightarrow G$ be a monomorphism. It is known, by ([14, Theorem 4.1]), that the corresponding ascending HNN-extension $G*_\varphi$ is either 1-ended or 2-ended, depending on whether G is infinite or not. In the second case, we know that $G*_\varphi$ is properly 3-realisable by ([1, Corollary 1.2]). Thus, we may assume that G is infinite and hence $G*_\varphi$ is 1-ended. Note that if $\varphi = id_G$, then $G*_\varphi = G \times \mathbf{Z}$ and the conclusion follows from ([6, Theorem 1.1]).

Let X be a compact 2-polyhedron having $\pi_1(X) \cong G$, and $\widetilde{X}$ as universal cover. Let $f : X \longrightarrow X$ be a cellular map inducing the homomorphism φ on the fundamental group, and let $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{X}$ be a proper lifting of f. Observe that such a map $\widetilde{f}$ exists as $f_* = \varphi : \pi_1(X) \longrightarrow \pi_1(X)$ is a monomorphism (see [**9**]). On the other hand, let Y denote the mapping torus of f, that is, Y is obtained as a quotient space from the disjoint union $(X \times [0,1]) \sqcup X$ by identifying $(x,0)$ with x and $(x,1)$ with $f(x)$, for all $x \in X$. One can check that $\pi_1(Y) \cong G*_\varphi$, and it is not hard to see that the universal cover $\widetilde{Y}$ of Y can be seen as a collection of copies of the mapping cylinder $M_{\widetilde{f}}$ of $\widetilde{f}$ attached each other along the copies $q(\widetilde{X} \times \{0\}), q(\widetilde{X}) \subset M_{\widetilde{f}}$ of $\widetilde{X}$, where $q : (\widetilde{X} \times [0,1]) \sqcup \widetilde{X} \longrightarrow M_{\widetilde{f}}$ is the corresponding quotient map for a generic copy of $M_{\widetilde{f}}$ in $\widetilde{Y}$, that is, $(z,1)$ and $\widetilde{f}(z)$ $(z \in \widetilde{X})$ get identified via q (see [**14**]). Moreover, no two copies of $M_{\widetilde{f}}$ in $\widetilde{Y}$ intersect along the corresponding subcomplex $q(\widetilde{X} \times \{0\})$ of each of them.

We are going to compute the pro-isomorphism type of the fundamental pro-group $pro-\pi_1(\widetilde{Y})$, see the Appendix. For simplicity, we shall take care of neither base rays nor base points in what follows.

A collection $\widetilde{X}_a, \widetilde{X}_{a+1}, \dots, \widetilde{X}_{a+r}$ $(r \geqslant 1)$ of copies of $\widetilde{X}$ in $\widetilde{Y}$ form a *chain* within $\widetilde{Y}$ if for each $0 \leqslant j \leqslant r-1$, there are copies $M_{a,j}$ of $M_{\widetilde{f}}$ in $\widetilde{Y}$ having $\widetilde{X}_{a+j}, \widetilde{X}_{a+j+1}$ as the corresponding subcomplexes $q(\widetilde{X} \times \{0\}), q(\widetilde{X}) \subset M_{\widetilde{f}}$ respectively. Observe that such a chain is unique from $\widetilde{X}_a$ to $\widetilde{X}_{a+r}$, as $\widetilde{Y}$ is simply connected. On the other hand, we say that two different copies $\widetilde{X}_a, \widetilde{X}_b$ of $\widetilde{X}$ in $\widetilde{Y}$ are *at the same level* if there is $r \geqslant 1$ and two chains $\widetilde{X}_a, \widetilde{X}_{a+1}, \dots, \widetilde{X}_{a+r}$ and $\widetilde{X}_b, \widetilde{X}_{b+1}, \dots, \widetilde{X}_{b+r}$ within $\widetilde{Y}$ with $\widetilde{X}_{a+r} = \widetilde{X}_{b+r}$. From now on, we shall fix an infinite chain within $\widetilde{Y}$, that is, we fix copies $\widetilde{X}_n (n \in \mathbf{Z})$ of $\widetilde{X}$ in $\widetilde{Y}$ so that for each $n \in \mathbf{Z}$, there is a copy M_n of $M_{\widetilde{f}}$ in $\widetilde{Y}$ having $\widetilde{X}_n, \widetilde{X}_{n+1}$ as the corresponding subcomplexes $q(\widetilde{X} \times \{0\}), q(\widetilde{X}) \subset M_{\widetilde{f}}$ respectively. Thus, we shall say that a given copy of $\widetilde{X}$ in $\widetilde{Y}$ is *at level n* if it is as the same level as $\widetilde{X}_n \subset \widetilde{Y}$. Let us fix a sequence $C_1 \subset C_2 \subset \cdots \subset \widetilde{X}$ of compact subcomplexes with $\widetilde{X} = \bigcup_{i \geqslant 1} C_i$. For any $n \geqslant 1$ and any chain γ within $\widetilde{Y}$ from a copy of $\widetilde{X}$ at level $-n$ to $\widetilde{X}_n$, we define a compact subcomplex $D_{n,\gamma} \subset \widetilde{Y}$ as follows. Let

$$\widetilde{X}^{(\gamma)}_{-n,i_0}, \widetilde{X}^{(\gamma)}_{-n+1,i_1}, \dots, \widetilde{X}^{(\gamma)}_{n-1,i_{2n-1}}, \widetilde{X}^{(\gamma)}_{n,i_{2n}} = \widetilde{X}_n$$

be those copies of $\widetilde{X}$ which occur in γ, and denote by $M^{(\gamma)}_{i_j,i_{j+1}} \subset \widetilde{Y}$ the copy of $M_{\widetilde{f}}$ containing $\widetilde{X}^{(\gamma)}_{-n+j,i_j}$ and $\widetilde{X}^{(\gamma)}_{-n+j+1,i_{j+1}}$. Next, using back and forth the cylinder structures, one can build inductively compact subcomplexes

$$K^n_{-n+j} \subset \widetilde{X}^{(\gamma)}_{-n+j,i_j} \ (0 \leqslant j \leqslant 2n)$$

satisfying:

(i) $K^n_{-n+j} \supset C_n$ (as subsets of a generic copy of $\widetilde{X}$), for each $0 \leqslant j \leqslant 2n$.

(ii) $\widetilde{f}^{-1}(K^n_{-n+j+1}) = K^n_{-n+j}$, for each $0 \leqslant j \leqslant 2n$, where $\widetilde{f}$ is regarded as a map $\widetilde{X}^{(\gamma)}_{-n+j,i_j} \longrightarrow \widetilde{X}^{(\gamma)}_{-n+j+1,i_{j+1}}$.

(iii) $K^m_\alpha \supset K^n_\beta$ (again as subsets of a generic copy of $\widetilde{X}$) whenever $m \geqslant n$ and for any indexes $-m \leqslant \alpha \leqslant m$, $-n \leqslant \beta \leqslant n$.

We define

$$D_{n,\gamma} = \bigcup_{j=0}^{2n-1} q\big(K^n_{-n+j} \times [0,1]\big) \cup K^n_{-n+j+1} \subset \widetilde{Y},$$

where

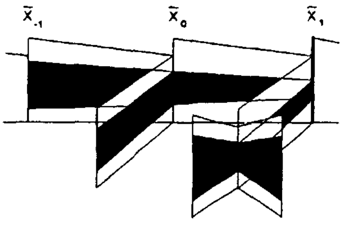
$$q\big(K^n_{-n+j} \times [0,1]\big) \subset M^{(\gamma)}_{i_j,i_{j+1}},$$

so that $D_{n,\gamma}$ deformation retracts (using the mapping cylinder structures) onto $K^n_n \subset \widetilde{X}_n$. Furthermore, the closure of $\bigcup\limits_{j=0}^{2n-1} M^{(\gamma)}_{i_j,i_{j+1}} - D_{n,\gamma}$ in $\bigcup\limits_{j=0}^{2n-1} M^{(\gamma)}_{i_j,i_{j+1}}$ deformation retracts onto a (non-compact) subcomplex of $\widetilde{X}_n - \operatorname{int}(K^n_n)$.

Finally, we shall build a particular sequence of compact subcomplexes $D_1 \subset D_2 \subset \cdots \subset \widetilde{Y}$ with $\widetilde{Y} = \bigcup\limits_{i\geqslant 1} D_1$, and study the inverse sequence of groups

$$\{1\} \longleftarrow \pi_1\big(\widetilde{Y} - \operatorname{int}(D_1)\big) \longleftarrow \pi_1\big(\widetilde{Y} - \operatorname{int}(D_2)\big) \longleftarrow \cdots$$

More precisely, we declare $D_n \subset \widetilde{Y}$ to be the compact subcomplex $D_n = \bigcup\limits_{\gamma} D_{n,\gamma}$, where γ ranges over all possible (finitely many) chains within $\widetilde{Y}$ from a copy of $\widetilde{X}$ at level $-n$ to $\widetilde{X}_n$. Thus, D_n contains all copies of $\widetilde{X}$ in $\widetilde{Y}$ at level $-n$. It is easy to check that the D_n's meet the required properties, by construction. Also, observe that each D_n deformation retracts onto $K^n_n \subset \widetilde{X}_n$. The picture below roughly describes what D_n may look like for $n = 1$.



Given $n \geqslant 1$, we define subcomplexes $V_n, W_n \subset \widetilde{Y}$ as follows. We take $V_n \subset \widetilde{Y}$ to be the union of those copies of $M_{\bar{f}}$ in $\widetilde{Y}$ involved in the construction of D_n; and $W_n \subset \widetilde{Y}$ to be the union of those copies M_k of $M_{\bar{f}}$ in $\widetilde{Y}$ containing $\widetilde{X}_k, \widetilde{X}_{k+1}$, for all $k \geqslant n$. Note that V_n deformation retracts onto $\widetilde{X}_n$, and both V_n and W_n are simply connected. One can check that $\widetilde{Y} - \operatorname{int}(D_n)$ deformation retracts onto

$$Z_n = \big(V_n - \operatorname{int}(D_n)\big) \cup W_n.$$

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Moreover, if $\widetilde{X}_{-n,1}, \dots, \widetilde{X}_{-n,j_n}$ are all copies of $\widetilde{X}$ in $\widetilde{Y}$ at level $-n$, then

$$Z'_n = cl\left(Z_n - \bigcup_{r=1}^{j_n} \widetilde{X}_{-n,r}\right) \subset Z_n$$

deformation retracts onto W_n (by construction of the D_n's), and hence Z'_n is simply connected. Thus, using an argument similar to that in ([**6**, Theorem 1.1]), one can show that $\pi_1(\widetilde{Y} - \mathrm{int}(D_n)) \cong \pi_1(Z_n)$ can be expressed as an iterated pseudo HNN-extension starting off from the fundamental group of the (simply connected) complex obtained by gluing each of the copies $\widetilde{X}_{-n,1}, \dots, \widetilde{X}_{-n,j_n}$ to Z'_n along the corresponding copy of a fixed connected component of $\widetilde{X} - K^n_{-n}$, where each stable letter represents a free generator for $\pi_1(Z_n)$. More precisely, if $\widetilde{X} - K^n_{-n}$ has l_n connected components then $\pi_1(Z_n)$ is a free group of rank $j_n(l_n - 1) \geqslant 0$. Furthermore, since $K^n_{-n} \subset K^{n+1}_{-n-1}$ (as subsets of a generic copy of $\widetilde{X}$) then

$$\mathrm{rank}\Big(\pi_1\big(\widetilde{Y} - \mathrm{int}(D_{n+1})\big)\Big) \geqslant \mathrm{rank}\Big(\pi_1\big(\widetilde{Y} - \mathrm{int}(D_n)\big)\Big).$$

Finally, note that the following towers of groups are pro-isomorphic

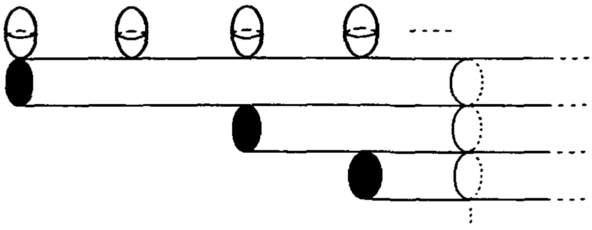
$$\begin{gathered}\{1\} \longleftarrow \pi_1\big(\widetilde{Y} - \mathrm{int}(D_1)\big) \stackrel{i_1}{\longleftarrow} \pi_1\big(\widetilde{Y} - \mathrm{int}(D_2)\big) \stackrel{i_2}{\longleftarrow} \cdots \\ \{1\} \longleftarrow \mathrm{Im}\ i_1 \stackrel{i_1}{\longleftarrow} \mathrm{Im}\ i_2 \stackrel{i_2}{\longleftarrow} \cdots\end{gathered}$$

and it is not hard to check that the latter may be regarded as a telescopic tower $\underline{P}$ (see the Appendix), as $\widetilde{f}$ maps connected components of $\widetilde{X} - K^{n+1}_{-n-1}$ into connected components of

$$\widetilde{X} - K^{n+1}_{-n} \subset \widetilde{X} - K^n_{-n} \ (n \geqslant 1),$$

by construction. Therefore, we conclude that the fundamental pro-group of $\widetilde{Y}$ is pro-isomorphic to a telescopic tower $\underline{P}$.

Next, let $p : \widetilde{Y} \longrightarrow Y$ be the covering projection, and pick a base ray ω in $\widetilde{Y}$. Since $\widetilde{Y}$ is 2-dimensional and one-ended, there exist spherical objects S^2_ω and $S^2_{\omega'}$ and a proper homotopy equivalence $\widetilde{Y} \vee S^2_\omega \simeq B(\underline{P}) \vee S^2_{\omega'}$, by Theorem 3.2. Let $V \subset \widetilde{K}$ be the set of vertices in $\omega([0,\infty))$, with $p(V) = \{v_1, \dots, v_r\} \subset Y$, and denote by $\widehat{Y}$ the polyhedron obtained from $\widetilde{Y} \vee S^2_\omega$ by attaching one sphere S^2 through every vertex in $p^{-1}(p(V)) - \omega([0,\infty))$. Thus, $\widehat{Y}$ is the universal cover of the compact 2-polyhedron obtained from Y by attaching one sphere S^2 at each of the vertices $v_1, \dots, v_r$ (which is homotopy equivalent to a wedge $Y \vee (\vee_{i=1}^r S^2)$). On the other hand, $\widehat{Y}$ is proper homotopy equivalent to a polyhedron Q obtained from $B(\underline{P}) \vee S^2_{\omega'}$ by attaching infinitely many spheres S^2 in a proper way (that is, via the corresponding proper homotopy equivalence given by Theorem 3.2). Finally, the proper homotopy type of the proper wedge $B(\underline{P}) \vee S^2_{\omega'}$ can be represented by the closed subpolyhedron in $\mathbf{R}^3$ shown in the figure below. It is then easy to check that the proper homotopy type of Q can also be represented by a closed subpolyhedron $\widehat{Q}$ in $\mathbf{R}^3$.



Therefore, the universal cover of the compact 2-polyhedron $Y \vee (\vee_{i=1}^{r} S^2)$ (with $\pi_1(Y \vee (\vee_{i=1}^{r} S^2)) \cong G*_{\varphi}$) turns out to be proper homotopy equivalent to the 3-manifold obtained by taking a regular neighbourhood of $\widehat{Q}$ in $\mathbf{R}^3$, and hence $G*_{\varphi}$ is properly 3-realisable. □

REMARK 2.1. Note that the argument used in the proof of Theorem 1.1 above gives an alternative proof of the fact that any ascending HNN-extension of a (finitely presented) group is semistable at infinity (see [**14**]).

PROOF OF THEOREM 1.2: Let X be a compact 2-polyhedron with $\pi_1(X) \cong G$ and having $\widetilde{X}$ as universal cover, and let $K_1 \subset K_2 \subset \cdots \subset \widetilde{X}$ be a sequence of compact subsets so that $\widetilde{X} = \bigcup_{i \geqslant 1} K_i$. The inverse system

$$\{1\} \longleftarrow \pi_1(\widetilde{X} - K_1) \xleftarrow{i_1} \pi_1(\widetilde{X} - K_2) \xleftarrow{i_2} \cdots$$

is then pro-isomorphic to a tower $\underline{F} = \{F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \longleftarrow \cdots\}$ of finitely generated free groups which is semistable, by hypothesis. It suffices to show that $\underline{F}$ is in fact pro-isomorphic to a telescopic tower $\underline{P}$, and then apply the same argument as in the proof of Theorem 1.1, as G is 1-ended.

For this, recall that $\underline{F}$ is said to be semistable if for each k there is $n \geqslant k$ so that for every $m \geqslant n$ we have $\mathrm{Im}(\phi_{k+1} \cdots \phi_m) = \mathrm{Im}(\phi_{k+1} \cdots \phi_n) \subset F_k$. It is well known that if $\underline{F}$ has this property then it is pro-isomorphic to a tower of groups

$$\underline{P} = \{P_0 \xleftarrow{\psi_1} P_1 \xleftarrow{\psi_2} P_2 \longleftarrow \cdots\}$$

where the bonding maps ψ_n are all epimorphisms. Moreover, there is a level morphism $\{i_n\}$ inducing this pro-isomorphism such that every $i_n : P_n \longrightarrow F_n$ is a monomorphism between finitely generated groups (see [**13**]). Thus, $\underline{P}$ is a tower of finitely generated free groups in which all bonding maps are epimorphisms. Finally, since $\psi_n : P_n \longrightarrow P_{n-1}$ is an epimorphism between free groups, it follows from [**8**] that P_n is a free product $P_n = P'_n * P''_n$, where $\psi_n(P''_n) = \{1\}$ and $\psi_n \mid P'_n$ is an isomorphism. Therefore, each ψ_n is a projection and hence $\underline{P}$ can be regarded as a telescopic tower, as each P''_n is finitely generated by the Grushko–Neumann theorem. □

Proof of Corollary 1.3: Let G be an infinite finitely presented group, and suppose G is simply connected at infinity. We show that G is properly 3-realisable. Of course, if G is 2-ended then it is properly 3-realisable by ([**1**, Corollary 1.2]). On the other hand, if G has infinitely many ends, then G splits as a non-trivial free product (as G is torsion-free, by hypothesis) by the Stallings' Structure theorem (see [**15**]), and one can easily check that each of the factors must be simply connected at infinity. Moreover, this splitting process must terminate after finitely many steps, by Dunwoody's accessibility result [**7**]. Thus, we may as well assume that G is 1-ended and simply connected at infinity, since the free product of properly 3-realisable groups is properly 3-realisable, by ([**1**, Lemma 3.2]). For this, note that in this case pro $-\pi_1(\widetilde{K})$ is pro-isomorphic to the trivial (telescopic) tower, where K is a compact 2-dimensional CW-complex with $\pi_1(K) \cong G$ and having $\widetilde{K}$ as universal cover (see [**9**]). Therefore, using an argument similar to that of Theorem 1.1 (as $\widetilde{K}$ is 1-ended), one shows that G is indeed properly 3-realisable. ∎

3. Appendix

This section is intended to provide the background and notation needed in the previous section, as well as to indicate what is behind those results. In what follows, we shall be working within the category tow $-Gr$ of *towers of groups* whose objects are inverse sequences of groups

$$\underline{A} = \{A_0 \xleftarrow{\phi_1} A_1 \xleftarrow{\phi_2} A_2 \longleftarrow \cdots\}$$

A morphism in this category will be called a *pro-morphism*. See [**2, 13**] for a general reference.

A tower $\underline{L}$ is a *free tower* if it is of the form

$$\underline{L} = \{L_0 \xleftarrow{i_1} L_1 \xleftarrow{i_2} L_2 \longleftarrow \cdots\}$$

where $L_i = \langle B_i \rangle$ are free groups of basis B_i such that $B_{i+1} \subset B_i$, the differences $B_i - B_{i+1}$ are finite and $\bigcap_{i=0}^{\infty} B_i = \emptyset$, and the bonding homomorphisms i_k are given by the corresponding basis inclusions. On the other hand, a tower $\underline{P}$ is a *telescopic tower* if it is of the form

$$\underline{P} = \{P_0 \xleftarrow{p_1} P_1 \xleftarrow{p_2} P_2 \longleftarrow \cdots\}$$

where $P_i = \langle D_i \rangle$ are free groups of basis D_i such that $D_{i-1} \subset D_i$, the differences $D_i - D_{i-1}$ are finite (possibly empty), and the bonding homomorphisms p_k are the obvious projections.

We shall also use the full subcategory $(Gr, \text{tow} -Gr)$ of Mor(tow $-Gr$) whose objects are arrows $\underline{A} \longrightarrow G$, where $\underline{A}$ is an object in tow $-Gr$ and G is a group regarded as a constant tower whose bonding maps are the identity. Morphisms in $(Gr, \text{tow} -Gr)$ will also be called pro-morphisms.

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From now on, X will be a (strongly) locally finite CW-complex. A proper map $\omega : [0,\infty) \longrightarrow X$ is called a *proper ray* in X. We say that two proper rays ω, ω' *define the same end* if their restrictions $\omega|_{\mathbf{N}}, \omega'|_{\mathbf{N}}$ are properly homotopic. Moreover, we say that they *define the same strong end* if ω and ω' are in fact properly homotopic.

Given a base ray ω in X and a collection of compact subsets $C_1 \subset C_2 \subset \cdots \subset X$ so that $X = \bigcup_{n=1}^{\infty} C_n$, the following tower

$$\text{pro}-\pi_1(X,\omega) = \Big\{\pi_1\big(X,\omega(0)\big) \leftarrow \pi_1\big(X - C_1, \omega(t_1)\big) \leftarrow \pi_1\big(X - C_2, \omega(t_2)\big) \leftarrow \cdots\Big\}$$

can be regarded as an object in $(Gr, \text{tow}-Gr)$ and it is called the *fundamental pro-group of* (X,ω), where $\omega\big([t_i,\infty)\big) \subset X - C_i$ and the bonding homomorphisms are induced by the inclusions. This tower does not depend (up to pro-isomorphism) on the sequence of subsets $\{C_i\}_i$. It is worth mentioning that if ω and ω' define the same strong end, then $\text{pro}-\pi_1(X,\omega)$ and $\text{pro}-\pi_1(X,\omega')$ are pro-isomorphic. In particular, we may always assume that ω is a cellular map. Moreover, if X is strongly connected at each end (that is, any two proper rays defining the same end define the same strong end), then $\pi_1^e(X,\omega) = \varprojlim \text{pro}-\pi_1(X,\omega)$ is a well-defined useful invariant which only depends (up to isomorphism) on the end determined by ω (see [**11**]). In a similar way, one can define objects in $(Gr, \text{tow}-Gr)$ corresponding to the higher homotopy pro-groups of (X,ω).

Definition 3.1: Given $n \geqslant 1$, a tree T and a proper ray $\omega : [0,\infty) \longrightarrow T$, a *spherical object* S^n_ω *under* T is a space obtained from T by attaching finitely n-spheres S^n at each vertex of $\omega\big([0,\infty)\big)$. Observe that any two of such spherical objects (along ω) are proper homotopy equivalent (under T), by ([**2**, Proposition 4.5(b)]).

The following result, which can be thought of as a special case of a proper version of Whitehead's theorem for compact 2-dimensional CW-complexes, will be crucial for the proof of those results in Section 2.

Theorem 3.2. ([**5**, Corollary 6.4]) *If X is a one-ended 2-dimensional locally finite CW-complex, then the following are equivalent*

(a) $\text{pro}-\pi_1(X,\omega)$ *is pro-isomorphic to a (coproduct) tower of the form* $\underline{L} \vee \underline{P}$.

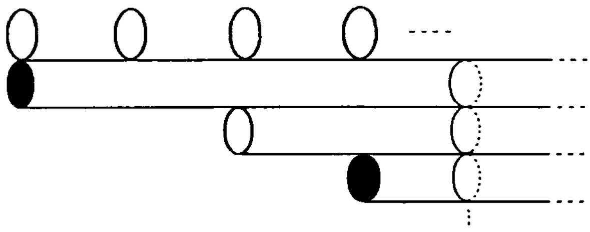
(b) *There exist spherical objects S^2_ω and $S^2_{\omega'}$ and a proper homotopy equivalence (under $[0,\infty)$) $X \vee S^2_\omega \simeq B(\underline{L} \vee \underline{P}) \vee S^2_{\omega'}$.*

Here, $\big(B(\underline{L} \vee \underline{P}), \omega'\big)$ is the properly based 2-polyhedron defined as the proper wedge (that is, along a base ray) of a one-ended spherical object S^1_ε, with $\text{pro}-\pi_1(S^1_\varepsilon, \omega') \cong \underline{L}$ ($\omega' : [0,\infty) \hookrightarrow S^1_\varepsilon$ the canonical inclusion), and a proper wedge C of a decreasing sequence (possibly infinite) of cylinders $C_n = S^1 \times [n,\infty)$ and/or Euclidean planes

$$\mathbf{R}^2_m = S^1 \times [m,\infty)/S^1 \times \{m\}$$

attached along the half line $[0,\infty)$ for which $\text{pro}-\pi_1(C,\omega') \cong \underline{P}$, with $\omega' : [0,\infty) \hookrightarrow C$ the canonical inclusion. Thus, $B(\underline{L} \vee \underline{P})$ can be seen as a "proper Eilenberg-MacLane space"

$K(\underline{L} \vee \underline{P}, 1)$ and its proper homotopy type can be represented by a closed subpolyhedron in $\mathbf{R}^3$ of the type as shown in the figure below.



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