A PROBLEM IN PARTITIONS: ENUMERATION OF ELEMENTS OF A GIVEN DEGREE IN THE FREE COMMUTATIVE ENTROPIC CYCLIC GROUPOID

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A *groupoid* is a set closed with respect to a binary operation. It is commutative and entropic if $xy = yx$ and $xy.zw = xz.yw$ hold for all its elements. It is cyclic if it is generated by one element. Let x be the generator of the free commutative entropic cyclic groupoid $\mathfrak{A}$. Then any element of $\mathfrak{A}$ can be written in the form x^P where $x^1 = x$ and $x^{Q+R} = x^Q x^R$. Two indices P, Q are equal (called " concordant " in (**3**)) if and only if $x^P = x^Q$. The groupoid of these indices, the free additive commutative entropic logarithmetic (cf. (**3**)), is clearly isomorphic to $\mathfrak{A}$.

We further define index θ-polynomials

$$\theta_1 = 0, \quad \theta_{P+Q} = (\theta_P + \theta_Q)\lambda + 1$$

where λ is an indeterminate in the domain of integers. It has been shown in (**3**) that these polynomials represent $\mathfrak{A}$ faithfully.

If δ_P is the degree of x^P, i.e. δ_P is the number of factors equal to x in x^P, we obviously have

$$\delta_1 = 1, \quad \delta_{P+Q} = \delta_P + \delta_Q.$$

The degree of x^P is therefore the value of P interpreted as an integer in ordinary arithmetic and is equal to $\theta_P(1) + 1$, i.e. to the sum of coefficients in $\theta_P(\lambda)$ increased by 1. It was called "potency of P" in (**2**), (**3**) and (**4**) and "degree of P" in (**1**). The degree of θ_P increased by 1 is called the altitude of P. A formula for enumeration of indices of a given altitude was given in (**3**). In the present paper we give a method for calculating the number of indices of given potency δ, i.e. the number of elements in $\mathfrak{A}$ of degree δ.

A non-zero polynomial $c_0 + c_1\lambda + c_2\lambda^2 + \ldots + c_n\lambda^n$, where the c_i are positive integers, is a θ-polynomial if and only if $c_0 = 1$ and $c_{i+1} \leqslant 2c_i$ $(i = 0, 1, 2, \ldots, n-1)$ (cf. (**3**)). Hence the problem of finding the number of elements in $\mathfrak{A}$ of degree $d+1$ is equivalent to the problem of finding the number of partitions of d such that $d = 1 + c_1 + c_2 + \ldots + c_n$ where $c_1 = 1$ or 2 and $c_{i+1} \leqslant 2c_i$. To solve it, consider the more general problem: given two positive integers c and d find the number of partitions of d such that $d = c + c_1 + c_2 + \ldots + c_n$ where $c_1 \leqslant 2c$ and $c_{i+1} \leqslant 2c_i$. Denote this number by $v(c, d)$.

Since c_1 can take any value between 1 and min $(2c, d-c)$ we have

$$v(c, d) = \sum_{i=1}^{2c} v(i, d-c)$$

where $v(x, y) = 0$ unless $x \leqslant y$. The formula expresses $v(c, d)$ in terms of values of the function for smaller values of the second argument. Since $v(x, x) = 1$

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for all positive integers x, we can calculate $v(c, d)$ for any given c and d by repeated use of the formula. Thus

	$d = 1$	2	3	4	5	6	7	8	9	10	11	12	13	14
	$v(c, d) =$													
$c = 1$	1	1	2	3	5	9	16	28	50	89	159	285	510	914
2	0	1	1	2	4	7	12	22	39	70	126	225	404	725
3	0	0	1	1	2	4	7	13	24	42	76	137	245	441
4	0	0	0	1	1	2	4	7	13	24	43	78	140	251
5	0	0	0	0	1	1	2	4	7	13	24	43	78	141
6	0	0	0	0	0	1	1	2	4	7	13	24	43	78
7	0	0	0	0	0	0	1	1	2	4	7	13	24	43
8	0	0	0	0	0	0	0	1	1	2	4	7	13	24

The first row ($c = 1$) in the above table gives the numbers of elements in $\mathfrak{A}$ of degree $d+1$.

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