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Comparisons of General Linear Groups and their Metaplectic Coverings I

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Abstract. We prepare for a comparison of global trace formulas of general linear groups and their metaplectic coverings. In particular, we generalize the local metaplectic correspondence of Flicker and Kazhdan and describe the terms expected to appear in the invariant trace formulas of the above covering groups. The conjectural trace formulas are then placed into a form suitable for comparison.

1 Introduction

This paper provides the framework for a comparison of global trace formulas [Mez00]. The trace formulas pertain to general linear groups and their metaplectic coverings. The framework consists of three broad topics: the description and generalization of the local metaplectic correspondence; the definition and grouping of the local terms occurring in the trace formulas; and the introduction of the invariant trace formula of Arthur for metaplectic coverings.

In Sections 2–4 we recapitulate and expand upon the local results of Flicker and Kazhdan [FK86]. For a number field F containing the n-th roots of unity, we define the orbit map,

$$\gamma \mapsto \gamma',$$

which is a map from the general linear group over a local completion F_v to its n-fold metaplectic covering $\widetilde{\mathrm{GL}}(r, F_v)$. This map preserves conjugacy classes and is the foundation of all our correspondences. There is a dual map

$$\tilde{f} \mapsto \tilde{f}',$$

from the Hecke space of $\widetilde{\mathrm{GL}}(r, F_v)$ to the Paley-Wiener space of $\mathrm{GL}(r, F_v)$, which matches orbital integrals compared under the orbit map. The local metaplectic correspondence amounts to a map,

$$\tilde{\pi} \mapsto \tilde{\pi}',$$

from certain tempered representations of $\widetilde{\mathrm{GL}}(r, F_v)$ to tempered representations of $\mathrm{GL}(r, F_v)$, which satisfies the character identity,

$$\mathrm{tr}\big(\tilde{\pi}(\tilde{f})\big) = \tilde{f}'(\tilde{\pi}').$$

We make two assumptions in these sections. The first is that the local metaplectic

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correspondence commutes with parabolic induction as claimed in [FK86, Proposition 26.2]. The second is that the trace Paley-Wiener theorem holds for $\widetilde{\mathrm{GL}}(r, F_v)$ (*cf.* [BDK86]). Metaplectic coverings of the general linear group over the real numbers are not considered although they are well-understood [AH97].

The local terms in Arthur's invariant trace formula depend on the normalization of intertwining operators between induced representations. This is the subject of Section 5. We show that the intertwining operators for the metaplectic coverings can be suitably normalized by following [AC89, Section 2 II]. The argument depends on the Plancherel formula for $\widetilde{\mathrm{GL}}(r, F_v)$. The proof of this formula has not been written out, but can be seen to follow from the proof for $\mathrm{GL}(r, F_v)$ once the observations at the beginning of Section 5 are taken into consideration.

The local terms of the geometric side of the invariant trace formula are considered in Sections 6–8. These terms are distributions parameterized by conjugacy classes. They are grouped according to the orbit map in Section 6, and the groupings $I_M^{\Sigma}(\gamma)$ are shown to preserve properties of descent and splitting. In Section 7 we assume that the construction and properties of the local geometric terms pass to metaplectic coverings. From these terms we define the distributions $I_M^{\mathcal{M}}(\gamma)$ that are expected to match $I_M^{\Sigma}(\gamma)$. Descent and splitting properties are proven for $I_M^{\mathcal{M}}(\gamma)$ as well. In order to compare $I_M^{\mathcal{M}}(\gamma)$ and $I_M^{\Sigma}(\gamma)$ using the invariant trace formula, we must show that the geometric distributions which occur in the trace formula for the metaplectic coverings, but are not of the form $I_M^{\mathcal{M}}(\gamma)$, are inconsequential. Ideally, every such distribution could be shown to vanish. In Section 8 we show that the ideal situation holds only if n is relatively prime to the positive integers less than or equal to r. Conversely, we show that if n is relatively prime to an additional integer, which depends on the metaplectic covering, then the ideal situation holds.

The conjectural invariant trace formula for a metaplectic covering of the general linear group is given in Section 9. This formula is conjectural not because of any expected complications in its proof; rather, because its proof is very long. The most formidable hurdle in its proof is quite possibly the trace Paley-Wiener theorem for metaplectic coverings (*cf.* [BDK86]). Due to the volume of the material, the reader is assumed to be familiar with the basic notions of the invariant trace formula for reductive groups. We close the paper by placing the geometric sides of the invariant trace formulas of $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ and $\mathrm{GL}(r, \mathbf{A})$ in a form that is suitable for comparison under the orbit map.

In the sequel to this work, the invariant trace formula for $\widetilde{\mathrm{GL}}(r, \mathbf{A})$ is placed into the form

$$\sum_{M \in \mathcal{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{\gamma \in (M(F))_{F,S}/\mu_n^M} a^{\tilde{M}}(S, \gamma') I_M^{\mathcal{M}}(\gamma, \tilde{f})$$
$$= \sum_{M \in \mathcal{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{t \geq 0} \int_{\Pi(\tilde{M}, t)} a^{\tilde{M}}(\tilde{\pi}) I_{\tilde{M}}(\tilde{\pi}, \tilde{f}) d\tilde{\pi},$$

and the invariant trace formula for $\mathrm{GL}(r, \mathbf{A})$ is placed into the form

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{F,S}/\mu_n^M} a^M(S,\gamma) I_M^\Sigma(\gamma,\tilde{f}) = \sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{t\geq 0} \int_{\Pi^\Sigma(M,t)} a^{M,\Sigma}(\tilde{\pi}) I_M^\Sigma(\tilde{\pi},\tilde{f}) d\tilde{\pi}.$$

The interesting terms in these formulas are of two types: local and global. The local terms, which are each identified by an "I", are distributions defined in terms of weighted orbital integrals and weighted characters. The global terms, which are each distinguished by an "a", are constants which depend on the automorphic nature of the representations and the rational geometry of the groups. In the sequel, we show that, under some conditions on γ, $\tilde{\pi}$ and the order of the covering, we have the equalities,

$$I_M^{\mathcal{M}}(\gamma,\tilde{f}) = I_M^\Sigma(\gamma,\tilde{f}),$$
$$I_{\tilde{M}}(\tilde{\pi},\tilde{f}) = I_M^\Sigma(\tilde{\pi},\tilde{f}),$$

of local distributions, and the equalities,

$$a^{\tilde{M}}(S,\gamma') = a^M(S,\gamma),$$
$$a^{\tilde{M}}(\tilde{\pi}) = a^{M,\Sigma}(\tilde{\pi}),$$

of global terms. Moreover, it is shown there that the final equality implies a correspondence between the automorphic representations of $\widetilde{\mathrm{GL}}(r,\mathbf{A})$ and those of the Levi subgroups of $\mathrm{GL}(r,\mathbf{A})$. This correspondence constitutes what is usually known as a weak lift of automorphic representations. However, in the case that a corresponding representation is a *cuspidal* automorphic representation of $\mathrm{GL}(r,\mathbf{A})$ the lifting is actually a strong lift.

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2 Preliminaries

We begin by briefly describing metaplectic coverings of the general linear group as in [FK86, Section 2]. We then fix Haar measures and define some vector spaces which are fundamental to the representation theory of our groups.

Let n be a positive integer and let F be a totally imaginary number field containing the group μ_n of n-th roots of unity. If $n \geq 3$ then the condition that F be totally imaginary is superfluous. The completion of F at a valuation v is denoted by F_v, and its absolute value, which is determined by Haar measure, is denoted by $|\cdot|_v$. Throughout, S denotes a finite set of valuations of F. Let F_S be the ring $\prod_{v\in S} F_v$ and set

$$|\gamma|_S = \prod_{v\in S} |\gamma_v|_v, \quad \gamma = \prod_{v\in S} \gamma_v \in F_S.$$

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We write $\mathbf{A}$ for the adele ring of F.

We take $r \geq 2$ to be an integer and write G for the algebraic group $\mathrm{GL}(r)$. Clearly, $\mathrm{GL}(r)$ is defined over F. Thus we may form the group of F- or F_v-valued points of G. They are $G(F) = \mathrm{GL}(r, F)$ and $G(F_v) = \mathrm{GL}(r, F_v)$ respectively. F also embeds diagonally into F_S and so we may form the group of F_S-valued points of G. This is simply $G(F_S) = \prod_{v \in S} \mathrm{GL}(r, F_v)$. The adele group of $\mathrm{GL}(r, F)$ is denoted by $G(\mathbf{A})$.

We now recall the construction of the metaplectic coverings as in [FK86, Section 2]. For each valuation v, there are two-cocycles,

$$\sigma_{mv}\colon G(F_v) \times G(F_v) \to \mu_n, \quad 0 \leq m \leq n-1.$$

If v is archimedean, that is $F_v = \mathbf{C}$, then σ_{mv} is the trivial cocycle.

Suppose v is nonarchimedean. Then these cocycles are "twists" of Matsumoto's cocycle σ_{0v}. In this case there exists a compact open subgroup K_v^0 and a Borel function,

$$\kappa_v\colon K_v^0 \to \mu_n,$$

such that

$$\sigma_{mv}(k_1, k_2) = \frac{\kappa_v(k_1 k_2)}{\kappa_v(k_1)\kappa_v(k_2)}, \quad k_1, k_2 \in K_v^0.$$

According to [FK86, property (2.2)], we have

$$\text{(1)} \qquad \sigma_{mv}(u_1\gamma_1, \gamma_2 u_2) = \sigma_{mv}(\gamma_1, \gamma_2), \quad \gamma_1, \gamma_2 \in G(F_v), \quad u_1, u_2 \in U(F_v),$$

where $U(F_v)$ is the subgroup of upper-triangular unipotent elements in $G(F_v)$. From (1) and the equation preceding it, it follows [FK86, paragraph 1, p. 59] that

$$\kappa_v(u\gamma) = \kappa_v(\gamma u) = \kappa_v(\gamma), \quad \gamma \in G(F_v), \quad u \in U(F_v) \cap K_v^0.$$

We use this equation to extend κ_v to the open set $U(F_v)K_v^0U(F_v)$. We then extend κ_v to a Borel function on $G(F_v)$ by setting κ_v equal to one outside of $U(F_v)K_v^0U(F_v)$.

Define the cocycle τ_{mv} by

$$\tau_{mv}(\gamma_1, \gamma_2) = \sigma_{mv}(\gamma_1, \gamma_2)\frac{\kappa_v(\gamma_1)\kappa_v(\gamma_2)}{\kappa_v(\gamma_1\gamma_2)}, \quad \gamma_1, \gamma_2 \in G(F_v),$$

if v is nonarchimedean, and by $\tau_{mv} = \sigma_{mv}$ otherwise. By definition, τ_{mv} is cohomologous to σ_{mv}. It follows immediately from our choice of κ_v and (1) that

$$\text{(2)} \qquad \tau_{mv}(u_1\gamma_1, \gamma_2 u_2) = \tau_{mv}(\gamma_1, \gamma_2), \quad \gamma_1, \gamma_2 \in G(F_v), \quad u_1, u_2 \in U(F_v).$$

Furthermore, τ_{mv} is trivial when restricted to $K_v^0 \times K_v^0$.

The cocycle τ_{mv} yields a central extension,

$$1 \to \mu_n \to \tilde{G}_m(F_v) \to G(F_v) \to 1,$$

called an n-fold metaplectic covering of $G(F_v)$. Note that $\tilde{G}_m(F_v)$ is not identical to the extension investigated in [KP84], [KP86] and [FK86], which is defined from just

σ_{mv}. Nevertheless, $\tilde{G}_m(F_v)$ shares all of the relevant properties of the extension used in these works. We fix m and write $\tilde{G}(F_v)$ in place of $\tilde{G}_m(F_v)$. The n-fold metaplectic covering,

$$1 \to \mu_n \xrightarrow{\mathbf{i}} \tilde{G}(F_S) \underset{\mathbf{s}}{\overset{\mathbf{p}}{\rightleftarrows}} G(F_S) \to 1,$$

is defined by way of the two-cocycle $\tau_S = \prod_{v \in S} \tau_{mv}$. It is comforting to keep in mind that, as a set, $\tilde{G}(F_S)$ is equal to $G(F_S) \times \mu_n$. The maps of the above sequence may therefore be expressed more concretely by

$$\mathbf{i}(\zeta) = (1, \zeta), \quad \mathbf{p}(\gamma, \zeta) = \gamma, \quad \text{and} \quad \mathbf{s}(\gamma) = (\gamma, 1),$$

for all $\gamma \in G(F_S)$ and $\zeta \in \mu_n$. Multiplication is given by

$$(\gamma_1, \zeta_1)(\gamma_2, \zeta_2) = \big(\gamma_1\gamma_2, \zeta_1\zeta_2\, \tau_S(\gamma_1, \gamma_2)\big), \quad \gamma_1, \gamma_2 \in G(F_S), \quad \zeta_1, \zeta_2 \in \mu_n.$$

Given a subgroup H of $G(F_S)$, we write $\tilde{H}$ for $\mathbf{p}^{-1}(H)$. We say that $\tilde{G}(F_S)$ splits over H if $\tilde{H}$ is isomorphic to $H \times \mu_n$ as a group. The group $\tilde{G}(\mathbf{C})$ splits over $G(\mathbf{C})$.

One can also form the product, $\tau = \prod_v \tau_{mv}$, over all valuations of F to obtain a two-cocycle, which gives rise to an n-fold metaplectic covering $\tilde{G}(\mathbf{A})$ of $G(\mathbf{A})$. The previous notions for $\tilde{G}(F_S)$ have obvious parallels for $\tilde{G}(\mathbf{A})$. It is important to realize that $\tilde{G}(\mathbf{A})$ splits over $G(F)$. The splitting homomorphism,

$$\mathbf{s_0}\colon G(F) \to \tilde{G}(\mathbf{A}),$$

is given by

$$\mathbf{s_0}(\gamma) = \mathbf{s}(\gamma)\mathbf{i}\Big(\prod_v \kappa_v(\gamma)^{-1}\Big), \quad \gamma \in G(F). \tag{3}$$

Suppose v is nonarchimedean. We define the orbit map,

$$G_0(F_v) \xrightarrow{\,\prime\,} \tilde{G}(F_v), \tag{4}$$

as the map given in [FK86, Section 4]. If n is odd then $G_0(F_v) = G(F_v)$ and the orbit map is given by

$$\gamma' = \mathbf{s}(\gamma)^n = (\gamma, 1)^n, \quad \gamma \in G(F_v).$$

If n is even then $G_0(F_v)$ is a dense open subset of $G(F_v)$ and

$$\gamma' = \mathbf{u}(\gamma)\mathbf{s}(\gamma)^n = (1, \pm 1)(\gamma, 1)^n, \quad \gamma \in G_0(F_v),$$

where $\mathbf{u}$ is a class function taking values in $\mathbf{i}(\mu_2)$. At the archimedean places, we define the orbit map by

$$\gamma \mapsto \gamma' = \mathbf{s}(\gamma)^n = \mathbf{s}(\gamma^n), \quad \gamma \in G(\mathbf{C}).$$

The orbit map is so called, because it preserves conjugacy classes, *i.e.*, orbits under the action of conjugation. The orbit map extends to maps on dense subsets of $G(F_S)$ and $G(\mathbf{A})$ by acting on each factor. We abuse notation by denoting these maps by $\prime$ and referring to them as "the orbit map" as well.

We shall concern ourselves primarily with the regular semisimple elements and conjugacy classes of $\tilde{G}(F_S)$. We therefore define G_{reg} to be the set of regular and semisimple elements in G. The sets $G_{\mathrm{reg}}(F)$ and $G_{\mathrm{reg}}(F_S)$ are the respective subsets of F- and F_S-valued points of G_{reg}. Set $\tilde{G}_{\mathrm{reg}}(F_S) = \mathbf{p}^{-1}\big(G_{\mathrm{reg}}(F_S)\big)$. Taking the orbit map into consideration we define the set,

$$G_{\mathrm{oreg}}(F_S) = \{\gamma \in G(F_S) : \gamma^n \in G_{\mathrm{reg}}(F_S)\}.$$

In the terminology of [Vig81, Section 1.i], the image of $G_{\mathrm{oreg}}(F_S)$ under the orbit map is contained in the set of ordinary regular elements of $\tilde{G}(F_S)$ [FK86, Proposition 3]. Flicker and Kazhdan use the word *good* in place of *ordinary* [FK86, Section 8].

Let M_0 be the diagonal subgroup of G. The set of Levi subgroups of G containing the minimal Levi subgroup M_0 is denoted by $\mathcal{L}$. We reserve the letter M for an element of $\mathcal{L}$. As usual, the center of M is denoted by A_M. The set of Levi subgroups of some element $L \in \mathcal{L}$, which also contain $M \subset L$, is written as $\mathcal{L}^L(M)$. Similarly, the set of parabolic subgroups of L with Levi component $M \subset L$ is written as $\mathcal{P}^L(M)$. We shall suppress the superscript "G" in these notations so that $\mathcal{L}(M) = \mathcal{L}^G(M)$ and $\mathcal{P}(M) = \mathcal{P}^G(M)$. Given a parabolic subgroup $P \in \mathcal{P}(M)$, we write U_P for its unipotent radical. Let P_0 be the subgroup of upper-triangular matrices in G. Its unipotent radical U_{P_0} is the subgroup of upper-triangular unipotent matrices. Clearly, we may adelize or take the F_S-valued points of these groups. We therefore write $M(F_S)$, $M(\mathbf{A})$, etc. without further comment.

The following decomposition of M will be useful in inductive arguments made later on. Recall that there exist positive integers, $r_1, \ldots, r_\ell$, and subgroups, $M(1) \cong \mathrm{GL}(r_1), \ldots, M(\ell) \cong \mathrm{GL}(r_\ell)$, of M such that $\sum_{i=1}^{\ell} r_i = r$ and

$$M = M(1) \times \cdots \times M(\ell). \tag{5}$$

For each valuation v we fix a maximal compact subgroup K_v of $G(F_v)$ as follows. If v is nonarchimedean set $K_v = \mathrm{GL}(r, R_v)$, where R_v is the ring of integers of F_v. If $F_v = \mathbf{C}$ then K_v is the unitary group $U(r, \mathbf{C})$. Set $K_S = \prod_{v \in S} K_v$.

In order to perform harmonic analysis on these groups without any ambiguity, we endow them with measures. Fix measures on $M(F_S)$ and $U_P(F_S)$, where $P \in \mathcal{P}^L(M)$, $L \in \mathcal{L}(M)$ as in [Art81, Section 1]. If H is a subgroup of $G(F_S)$ with a fixed Haar measure dh, we choose $d\tilde{h}$ to be the unique Haar measure on $\tilde{H}$ satisfying

$$d\tilde{h} \circ \mathbf{p}^{-1} = n \cdot dh.$$

We define quotient measures as follows. Given subgroups $H_1 \subset H_2$ of $G(F_S)$ with a fixed measure on $H_1 \setminus H_2$, we define the measure on $\tilde{H}_1 \setminus \tilde{H}_2$ to be the pull-back of the measure on $H_1 \setminus H_2$ via the obvious homeomorphism,

$$\tilde{H}_1 \setminus \tilde{H}_2 \to H_1 \setminus H_2.$$

We close this section with a description of some vector spaces which are basic to representation theory. Let $X(M)$ be the group of rational characters of M. Set $\mathfrak{a}_M = \mathrm{Hom}_{\mathbf{Z}}\big(X(M), \mathbf{R}\big)$, $\mathfrak{a}_M^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbf{R})$ and $\mathfrak{a}_{M,\mathbf{C}}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbf{C})$. The group $X(M)$ embeds into $\mathfrak{a}_{M,\mathbf{C}}^*$ as a lattice. Fix a Euclidean norm on $\mathfrak{a}_{M_0}$, which is invariant under W_0^G, the Weyl group of (G, A_{M_0}). There is an obvious embedding of $\mathfrak{a}_M$ into $\mathfrak{a}_{M_0}$ and we endow $\mathfrak{a}_M$ with the Euclidean norm obtained from this embedding by restriction. This norm provides a measure on $\mathfrak{a}_M$.

As usual, we define the homomorphism,

$$H_M \colon M(F_S) \to \mathfrak{a}_M,$$

by the equation,

$$e^{\langle H_M(\gamma), \xi \rangle} = \prod_{v \in S} |\xi(\gamma_v)|_v, \quad \gamma = \prod_{v \in S} \gamma_v \in M(F_S), \quad \xi \in X(M).$$

The composition of H_M with $\mathbf{p}$ is evidently a homomorphism of $\tilde{M}(F_S)$ into $\mathfrak{a}_M$. This product formula allows us to generalize the above homomorphisms to apply to $M(\mathbf{A})$ and $\tilde{M}(\mathbf{A})$ as well. These maps provide the spaces $M(\mathbf{A})^1 = \ker(H_M)$ and $\tilde{M}(\mathbf{A})^1 = \ker(H_M \circ \mathbf{p})$ with measures.

Suppose that γ belongs to either $M(F_S)$ or $M(\mathbf{A})$. Then

$$H_M\big(\mathbf{p}(\gamma')\big) = H_M(\gamma^n) = nH_M(\gamma). \tag{6}$$

This equation motivates the definition of the isomorphism of $\mathfrak{a}_M$ defined by

$$X \mapsto X' = n^{-1}X, \quad X \in \mathfrak{a}_M,$$

and the adjoint isomorphism of $\mathfrak{a}_{M,\mathbf{C}}^*$ defined by

$$\lambda \mapsto \lambda' = n\lambda.$$

3 The Local Metaplectic Correspondence

3.1 The General Case

The local metaplectic correspondence is a map from certain representations of $\tilde{G}(F_S)$ to representations of $G(F_S)$, which also preserves a specific character relation. It also has dual characterizations in terms of a map of Hecke spaces or Paley-Wiener spaces. In the case of Hecke spaces, the map preserves an identity of orbital integrals. We examine each of these characterizations, giving appropriate definitions along the way.

A representation $\tilde{\pi}$ of $\tilde{M}(F_S)$ or $\tilde{M}(\mathbf{A})$ is said to be genuine if

$$\tilde{\pi}(\gamma, \zeta) = \zeta \tilde{\pi}(\gamma, 1),$$

for all $\zeta \in \mu_n$ and γ in $M(F_S)$ or $M(\mathbf{A})$. Let $\Pi\big(\tilde{M}(F_S)\big)$ be the set of (equivalence classes of) genuine, irreducible and admissible representations of $\tilde{M}(F_S)$. Let

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$\Pi_{\text{temp}}\big(\tilde{M}(F_S)\big)$ and $\Pi_{\text{unit}}\big(\tilde{M}(F_S)\big)$ be the subsets of $\Pi\big(\tilde{M}(F_S)\big)$ which consist respectively of tempered and unitary representations.

Given $M_1 \in \mathcal{L}^M$, $P \in \mathcal{P}^M(M_1)$ and $\tilde{\pi}_1 \in \Pi\big(\tilde{M}_1(F_S)\big)$, we denote the unitarily induced representation $\operatorname{Ind}_P^M \tilde{\pi}_1$ by $\tilde{\pi}_1^{\tilde{M}}$. If $\lambda \in \mathfrak{a}^*_{M_1,\mathbf{C}}$ then the representation $\tilde{\pi}_{1,\lambda}$, given by

$$\tilde{\pi}_{1,\lambda}(\tilde{\gamma}) = \tilde{\pi}_1(\tilde{\gamma})e^{\lambda(H_{M_1}(\mathbf{p}(\tilde{\gamma})))}, \quad \tilde{\gamma} \in \tilde{M}_1(F_S),$$

belongs to $\Pi\big(\tilde{M}_1(F_S)\big)$ as well.

Given $\tilde{\pi} \in \Pi\big(\tilde{M}(F_S)\big)$, we denote its central character by $\omega_{\tilde{\pi}}$ and its distribution character by $\Theta_{\tilde{\pi}}$. For $\gamma = \prod_{v\in S}\gamma_v \in M(F_S)$, the Weyl discriminant is defined as

$$D^M(\gamma) = \prod_{v\in S}\det\big(1 - \operatorname{Ad}(\gamma_v)_{\mathfrak{m}/\mathfrak{m}_{\sigma_v}}\big) \in F_S.$$

Here σ_v is the semisimple component in the Jordan decomposition of γ_v, and $\mathfrak{m}$ and $\mathfrak{m}_{\sigma_v}$ are the respective Lie algebras of $M(F_v)$ and the centralizer of σ_v in $M(F_v)$.

Suppose v is nonarchimedean, $\tilde{\pi} \in \Pi\big(\tilde{G}(F_v)\big)$ and $\pi \in \Pi\big(G(F_v)\big)$. Let d equal $\gcd(n, r-1+2rm)$. Following [FK86, (26.1)], $\tilde{\pi}$ is said to correspond (or lift) to π if

$$\omega_{\tilde{\pi}}(\gamma') = \omega_\pi(\gamma), \quad \gamma \in A_G(F_v),$$

and for any $\gamma \in G_{\text{oreg}}(F_v)$,

$$|D^G(\gamma^n)|_v^{1/2}\Theta_{\tilde{\pi}}(\gamma') = \frac{n|d|_v^{1/2}}{d|n|_v^{r/2}} \sum_{\{\gamma_1,\gamma_2:\gamma_1'\gamma_2=\gamma'\}} |D^G(\gamma)|_v^{1/2}\Theta_\pi(\gamma_1)\omega_{\tilde{\pi}}(\gamma_2). \tag{7}$$

The sum on the right is parameterized by $\gamma_1 \in G_\gamma(F_v)/A_G(F_v)$ and $\gamma_2 \in \widetilde{A^{n/d}}(F_v)/A'_G(F_v)$. The group $G_\gamma(F_v)$ is the set of F_v-valued points of G_γ, the centralizer of γ in G. The set $A'_G(F_v)$ is equal to $\{\gamma_0' : \gamma_0 \in A_G(F_v)\}$.

Theorem 27.3 [FK86] implies that there is a map,

$$\Pi_{\text{temp}}\big(\tilde{G}(F_v)\big) \xrightarrow{\ \prime\ } \Pi_{\text{temp}}\big(G(F_v)\big), \tag{8}$$

such that $\tilde{\pi}$ corresponds to $\tilde{\pi}'$. Proposition 26.2 implies that (8) can be generalized to a map,

$$\Pi_{\text{temp}}\big(\tilde{M}(F_v)\big) \xrightarrow{\ \prime\ } \Pi_{\text{temp}}\big(M(F_v)\big), \tag{9}$$

which preserves a character relation analogous to (7). In addition, map (9) commutes with parabolic induction. These results depend on the method of induction delineated in [FK86, Section 26.2]. Unfortunately, a counterexample in [Sun97] shows that this method is faulty for an arbitrary covering number n. Despite this circumstance, we show in the Appendix that under some assumptions on n the method of Section 26.2 [FK86] is valid. Such assumptions on n shall be made in Part II of this work. For the time being, we shall assume that the above results are true for any n. We refer the reader to the Appendix for further analysis of this issue.

In Section 3.2 we construct a map,

$$\Pi_{\mathrm{temp}}\left(\tilde{M}(\mathbf{C})\right) \xrightarrow{\prime} \Pi_{\mathrm{temp}}\left(M(\mathbf{C})\right), \tag{10}$$

which preserves a character relation analogous to (7), and commutes with parabolic induction.

Combining (9) and (10), we define the local metaplectic correspondence (of tempered representations) by

$$\bigotimes_{v\in S}\tilde{\pi}_v \overset{\prime}{\mapsto} \bigotimes_{v\in S}\tilde{\pi}'_v, \quad \bigotimes_{v\in S}\tilde{\pi}_v \in \Pi_{\mathrm{temp}}\left(\tilde{M}(F_S)\right). \tag{11}$$

The set of genuine standard representations $\Sigma\big(\tilde{M}(F_S)\big)$ is defined to be the set of (equivalence classes of) representations of the form $\bigotimes_{v\in S}\tilde{\pi}^{\tilde{M}}_{v,\lambda_v}$, where $\tilde{\pi}_v \in \Pi_{\mathrm{temp}}\left(\tilde{M}_v(F_v)\right)$, $\lambda_v \in \mathfrak{a}^*_{M_v}$ is regular (that is $\lambda_v(\beta) \neq 0$ for every root β of (G, A_{M_v})) and $M_v \in \mathcal{L}^M$, for $v \in S$. We generalize the local metaplectic correspondence to $\Sigma\big(\tilde{M}(F_S)\big)$ by

$$\bigotimes_{v\in S}\tilde{\pi}^{\tilde{M}}_{v,\lambda_v} \overset{\prime}{\mapsto} \bigotimes_{v\in S}(\tilde{\pi}'_{v,\lambda'_v})^M.$$

It follows from equation (6) and some basic properties of distribution characters that this map preserves a character relation analogous to that of the local metaplectic correspondence.

We now extend the local metaplectic correspondence to $\Pi\big(\tilde{M}(F_S)\big)$. By using the Jacquet modules introduced in [FK86, Section 14], and following the proof of [Sil78], it can be shown that the Langlands quotient theorem holds for $\tilde{M}(F_S)$. More precisely, every $\tilde{\pi} \in \Pi\big(\tilde{M}(F_v)\big)$ may be written uniquely as the quotient of some induced representation $\tilde{\pi}^{\tilde{M}}_{1,\lambda}$, where $\tilde{\pi}_1 \in \Pi_{\mathrm{temp}}\left(\tilde{M}_1(F_v)\right)$, λ is in a fixed positive Weyl chamber of $\mathfrak{a}^*_{\tilde{M}_1}$ and $M_1 \in \mathcal{L}^M$. With the Langlands quotient theorem in place, we may extend map (11) by assigning the Langlands quotient of the representation $\tilde{\pi}^{\tilde{M}}_{1,\lambda}$ to the Langlands quotient of $(\tilde{\pi}'_{1,\lambda'})^M$. This prescribes a map

$$\Pi\big(\tilde{M}(F_S)\big) \xrightarrow{\prime} \Pi\big(M(F_S)\big). \tag{12}$$

The reader is warned that, unlike the local metaplectic correspondence on $\Pi_{\mathrm{temp}}\left(\tilde{M}(F_S)\right)$ or $\Sigma\big(\tilde{M}(F_S)\big)$, this map does not necessarily preserve any character relations.

The dual formulation of the local metaplectic correspondence is defined in terms of Hecke spaces or Paley-Wiener spaces. A function $\tilde{f}\colon \tilde{M}(F_S) \to \mathbf{C}$ is said to be antigenuine if

$$\tilde{f}(\gamma,\zeta) = \zeta^{-1}\tilde{f}(\gamma,1), \quad \gamma \in M(F_S), \quad \zeta \in \mu_n.$$

Let $C^\infty_c\left(\tilde{M}(F_S)\right)$ be the space of antigenuine, smooth and compactly supported functions on $\tilde{M}(F_S)$. The Hecke space $\mathcal{H}\big(\tilde{M}(F_S)\big)$ is defined to be the subspace of functions in $C^\infty_c\left(\tilde{M}(F_S)\right)$ which are $\left(\tilde{K}_S \cap \tilde{M}(F_S)\right)$-finite under left and right translation.

Given $M_1 \in \mathcal{L}^M$ and $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$, we define the map, $\tilde{f}_{\tilde{M}_1}$ on the set $\Pi_{\mathrm{temp}}\big(\tilde{M}_1(F_S)\big)$ by

$$\tilde{f}_{\tilde{M}_1}(\tilde{\pi}) = \mathrm{tr}\big(\tilde{\pi}^{\tilde{M}}(\tilde{f})\big) = \mathrm{tr}\left(\int_{\tilde{M}(F_S)} \tilde{\pi}^{\tilde{M}}(x)\tilde{f}(x)\,dx\right). \tag{13}$$

This definition is justified by the well-known fact that $\tilde{\pi}^{\tilde{M}}(\tilde{f})$ is a trace-class operator. Indeed, $\tilde{\pi}$ is admissible and $\tilde{f}$ has compact support.

The trace Paley-Wiener theorem holds for $\tilde{M}(\mathbf{C})$ by [CD84] and the fact that $\tilde{M}(\mathbf{C})$ splits over $M(\mathbf{C})$. We assume that the trace Paley-Wiener theorem of [BDK86] generalizes to $\tilde{M}(F_v)$ for nonarchimedean F_v. Under this assumption we may take the Paley-Wiener space of $\tilde{M}(F_S)$ to be

$$\mathcal{I}\big(\tilde{M}(F_S)\big) = \big\{\, \tilde{f}_{\tilde{M}} : \tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)\,\big\}.$$

We recall some ideas presented on [AC89, p. 79]. Suppose θ is a continuous linear map from $\mathcal{H}\big(\tilde{M}(F_S)\big)$ to a topological vector space $\mathcal{V}$. Then θ is said to be supported on characters if it vanishes on any function $\tilde{f}$ with $\tilde{f}_{\tilde{M}} = 0$. If θ is supported on characters then there exists a unique continuous map,

$$\hat{\theta}\colon \mathcal{I}\big(\tilde{M}(F_S)\big) \to \mathcal{V},$$

such that

$$\hat{\theta}(\tilde{f}_{\tilde{M}}) = \theta(f), \quad \tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big).$$

The map $\tilde{f} \mapsto \tilde{f}_{\tilde{M}_1}$ defined by (13) is supported on characters and factors through a map $\phi \mapsto \phi_{\tilde{M}_1}$ from $\mathcal{I}\big(\tilde{M}(F_S)\big)$ to $\mathcal{I}\big(\tilde{M}_1(F_S)\big)$.

The set of valuations S is said to have the closure property if

$$\mathfrak{a}_{M,S} = \{H_M(\gamma) : \gamma \in M(F_S)\}$$

is a closed subgroup of $\mathfrak{a}_M$. If S contains an archimedean valuation, it has the closure property. If not, S has the closure property if and only if it is comprised entirely of valuations which divide a fixed rational prime. For the remainder of this paper, S is assumed to have the closure property unless otherwise specified. Put $i\mathfrak{a}^*_{M,S} = i\mathfrak{a}^*_M / i\,\mathrm{Hom}(\mathfrak{a}_{M,S}, \mathbf{Z})$. The group $i\mathfrak{a}^*_{M,S}$ inherits a measure from the Euclidean measure defined previously on $\mathfrak{a}_M$. We identify $\phi \in \mathcal{I}\big(\tilde{M}(F_S)\big)$ with its Fourier transform,

$$\phi(\tilde{\pi}, X) = \int_{i\mathfrak{a}^*_{M,S}} \phi(\tilde{\pi}_\lambda) e^{-\lambda(X)}\,d\lambda.$$

Define a map $\tilde{f} \mapsto \tilde{f}'$ from $\mathcal{H}\big(\tilde{M}(F_S)\big)$ to functions in $\pi \in \Pi_{\mathrm{temp}}\big(M(F_S)\big)$ by

$$\tilde{f}'(\pi) = \begin{cases} \mathrm{tr}\big(\tilde{\pi}(\tilde{f})\big), & \text{if } \tilde{\pi}' = \pi \text{ for some } \tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}(F_S)\big) \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

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It is simple to check that $\tilde{f}'$ satisfies the axioms defining the Paley-Wiener space listed in [BDK86] and [CD84]. Thus $\tilde{f}'$ belongs to $\mathcal{I}\big(M(F_S)\big)$. We define a map,

$$\mathcal{I}\big(\tilde{M}(F_S)\big) \xrightarrow{\ '\ } \mathcal{I}\big(M(F_S)\big),$$

which is compatible with (14), by

$$\phi'(\pi, X') = \begin{Bmatrix} n^{\dim(A_M)}\phi(\tilde{\pi}, X), & \text{if } \tilde{\pi}' = \pi \\ 0, & \text{otherwise} \end{Bmatrix}, \quad \pi \in \Pi_{\text{temp}}\big(M(F_S)\big).$$

Once again, it is a simple matter to check that the image of this map lies in $\mathcal{I}\big(M(F_S)\big)$. To see that this map is compatible with (14), suppose $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$, $\tilde{\pi} \in \Pi_{\text{temp}}\big(\tilde{M}(F_S)\big)$ and $X \in \mathfrak{a}_{M,S}$. Then we have

$$\begin{aligned} (\tilde{f}_{\tilde{M}})'(\tilde{\pi}', X) &= n^{\dim(A_M)} \tilde{f}_{\tilde{M}}(\tilde{\pi}, X) \\ &= n^{\dim(A_M)} \int_{i\mathfrak{a}^*_{M,S}} \operatorname{tr}\big(\tilde{\pi}_\lambda(\tilde{f})\big)\, e^{-\lambda(X)}\, d\lambda \\ &= n^{\dim(A_M)} \int_{i\mathfrak{a}^*_{M,S}} \tilde{f}'\big((\tilde{\pi}_\lambda)'\big)\, e^{-\lambda(X)}\, d\lambda \\ &= \int_{i\mathfrak{a}^*_{M,S}} \tilde{f}'(\tilde{\pi}'_{\lambda'}) e^{-\lambda'(X')}\, d\lambda' \\ &= \tilde{f}'(\tilde{\pi}', X') \end{aligned}$$

as to be desired.

One would hope that map (14) commutes with the map (13). This is indeed the case, but in order to prove this fact we must describe map (14) purely in terms of Hecke spaces. The trace Paley-Wiener theorem tells us that for each $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$ there exists $f \in \mathcal{H}\big(M(F_S)\big)$ such that $f_M = \tilde{f}'$. The relationship between the Hecke functions, $\tilde{f}$ and f, is expressed in terms of orbital integrals.

Suppose $\tilde{\gamma} \in \tilde{M}(F_S)$ such that $\mathbf{p}(\tilde{\gamma})$ is semisimple. Let $\tilde{M}_{\tilde{\gamma}}(F_S)$ denote the centralizer of $\tilde{\gamma}$ in $\tilde{M}(F_S)$. We define

$$I^{\tilde{M}}_{\tilde{M}}(\tilde{\gamma}, \tilde{h}) = \big|D^M\big(\mathbf{p}(\tilde{\gamma})\big)\big|_S^{1/2} \int_{\tilde{M}_{\tilde{\gamma}}(F_S)\backslash \tilde{M}(F_S)} \tilde{h}(x^{-1}\tilde{\gamma}x)\, dx, \quad \tilde{h} \in C_c^\infty\big(\tilde{M}(F_S)\big).$$

These integrals converge ([FK86, Section 6], [Art88c, Section 2]) and are called orbital integrals. Now suppose that $\tilde{\gamma}$ has Jordan decomposition $\tilde{\gamma} = \tilde{\sigma}\mathbf{s}(u)$, where $u = \prod_{v\in S} u_v$ is a unipotent element of $M(F_S)$. A general orbital integral is given by

$$I^{\tilde{M}}_{\tilde{M}}(\tilde{\gamma}, \tilde{h}) = \lim_{a\to 1} I^{\tilde{M}}_{\tilde{M}}\big(\tilde{\sigma}\mathbf{s}(a^n), \tilde{h}\big),$$

where $a = \prod_{v\in S} a_v$ and a_v is regular, lying in the center of a parabolic subgroup for which u_v is a Richardson element. It is obvious that $I^{\tilde{M}}_{\tilde{M}}(\tilde{\gamma}, \tilde{h})$ depends only on the conjugacy class of $\tilde{\gamma}$ in $\tilde{M}(F_S)$.

For any valuation v set

$$\Lambda_v^M(\gamma) = |n|_v^{\dim(M_\gamma)/2}, \quad \gamma \in M(F_v),$$

and

$$\Lambda^M(\gamma) = \prod_{v \in S} \Lambda_v^M(\gamma_v), \quad \gamma = \prod_{v \in S} \gamma_v \in M(F_S).$$

It follows from [FK86, Proposition 27.3], [FK86, Section 24], Section 3.2 and the normalization of our measures that

$$I_M^M(\gamma, f) = \Lambda^M(\gamma) I_{\tilde{M}}^{\tilde{M}}(\gamma', \tilde{f}), \quad \gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S), \tag{15}$$

where $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$ and $f \in \mathcal{H}\big(M(F_S)\big)$ are as above. Any two functions, $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$ and $f \in \mathcal{H}\big(M(F_S)\big)$, satisfying (15) are said to match.

Using this terminology, we can characterize the image of the local metaplectic correspondence on $\Pi_{\mathrm{temp}}\big(\tilde{M}(F_S)\big)$. An admissible representation π of $M(F_S)$ is called metic if there exist matching functions $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$ and $f \in \mathcal{H}\big(M(F_S)\big)$ such that $\mathrm{tr}\big(\pi_1(f)\big) \neq 0$ for every subquotient π_1 of π. Theorem 27.3 of [FK86] and Proposition 3.1 imply that the image of the local metaplectic correspondence on $\Pi_{\mathrm{temp}}\big(\tilde{M}(F_S)\big)$ is the subset of metic representations in $\Pi_{\mathrm{temp}}\big(M(F_S)\big)$. (We remark in passing, that hitherto there has been no direct characterization of arbitrary metic representations. Flicker and Kazhdan do however provide a characterization of the metic representations which are induced from irreducible elliptic representations in [FK86, Section 27.3].)

With our characterization of the image in hand, we can prove that map (13) and map (14) commute. Suppose $\tilde{f} \in \mathcal{H}\big(\tilde{M}(F_S)\big)$, $\pi \in \Pi_{\mathrm{temp}}\big(M_1(F_S)\big)$ and $M_1 \in \mathcal{L}^M$. Then, by definition

$$(\tilde{f}_{\tilde{M}_1})'(\pi) = \begin{cases} \tilde{f}_{\tilde{M}}(\tilde{\pi}^{\tilde{M}}), & \text{if } \pi = \tilde{\pi}', \tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}_1(F_S)\big) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(\tilde{f}')_{M_1}(\pi) = \begin{cases} \tilde{f}_{\tilde{M}}(\tilde{\pi}), & \text{if } \pi^M = \tilde{\pi}', \tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}(F_S)\big) \\ 0, & \text{otherwise} \end{cases}$$

Thus, in order to show that $(\tilde{f}_{\tilde{M}_1})' = (\tilde{f}')_{M_1}$ we need only show that π is metic if and only if π^M is metic. If π is metic then π^M is metic since the local metaplectic correspondence commutes with parabolic induction. Conversely, if π^M is metic then there exist matching $\tilde{f}_1 \in \mathcal{H}\big(\tilde{M}(F_S)\big)$ and $f_1 \in \mathcal{H}\big(M(F_S)\big)$ such that $\mathrm{tr}\big(\pi^M(f_1)\big) \neq 0$. Applying a well-known descent property for orbital integrals [FK86, Section 7], there exist matching functions $f_{1,\tilde{P}} \in \mathcal{H}\big(\tilde{M}_1(F_S)\big)$ and $f_{1,P} \in \mathcal{H}\big(M_1(F_S)\big)$ for $P \in \mathcal{P}^M(M_1)$ such that

$$\mathrm{tr}\big(\pi(f_{1,P})\big) = \mathrm{tr}\big(\pi^M(f_1)\big) \neq 0.$$

This means that π is metic.

3.2 The Complex Metaplectic Correspondence

In this subsection we describe the metaplectic correspondence at a complex valuation quite concretely. This is done partly as there is only a sketch of the complex metaplectic correspondence in the literature [FK86, proof of Theorem 19], and partly to help with computations in Section 4.

It is well-known [Duf75, 4.4 and 4.5] that

$$\Pi_{\mathrm{temp}}\big(\tilde{M}(\mathbf{C})\big) = \left\{ \tilde{\pi}^{\tilde{M}} : \tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}_0(\mathbf{C})\big) \right\}.$$

In other words, $\Pi_{\mathrm{temp}}\big(\tilde{M}(\mathbf{C})\big)$ is the set of unitary principal series representations. A consequence of this is that any map,

$$\Pi_{\mathrm{temp}}(\tilde{M}_0) \longrightarrow \Pi_{\mathrm{temp}}(M_0),$$

which is equivariant under the Weyl group W_0^G of (G, A_{M_0}), gives rise to a map,

$$\Pi_{\mathrm{temp}}(\tilde{M}) \longrightarrow \Pi_{\mathrm{temp}}(M),$$

which commutes with parabolic induction. We define an injection,

$$\Pi_{\mathrm{temp}}(\tilde{M}_0) \xrightarrow{\;\prime\;} \Pi_{\mathrm{temp}}(M_0),$$

by sending $\pi \in \Pi_{\mathrm{temp}}(\tilde{M}_0)$ to its composition with the orbit map. Otherwise stated,

$$\tilde{\pi}'(\gamma) = \tilde{\pi}(\gamma') = \tilde{\pi}^n\big(\mathbf{s}(\gamma)\big), \quad \gamma \in M_0(\mathbf{C}).$$

The resulting injection,

$$\Pi_{\mathrm{temp}}(\tilde{M}) \xrightarrow{\;\prime\;} \Pi_{\mathrm{temp}}(M),$$

is also denoted by $\prime$. By using the character formula for induced representations [Kna86, Proposition 10.18], the reader may readily verify that $\tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}(\mathbf{C})\big)$ corresponds to $\tilde{\pi}'$. Therefore, the previous map is the complex metaplectic correspondence.

Lemma 3.1 *A representation $\pi \in \Pi_{\mathrm{temp}}\big(M_0(\mathbf{C})\big)$ lies in the image of the complex metaplectic correspondence if and only if π is trivial on $\mu_n^{M_0}$.*

Proof Suppose $\pi = \tilde{\pi}'$ for some $\tilde{\pi} \in \Pi_{\mathrm{temp}}\big(\tilde{M}_0(\mathbf{C})\big)$. Then

$$\pi(\gamma) = \tilde{\pi}\big(\mathbf{s}(\gamma)^n\big) = \tilde{\pi}(1) = 1, \quad \gamma \in \mu_n^{M_0}.$$

Conversely, suppose π is trivial on $\mu_n^{M_0}$. Suppose $\omega_1, \dots, \omega_r$ are unitary characters of $\mathbf{C}^\times$. There are numbers $\lambda_j \in \mathbf{R}$ and $k_j \in \mathbf{Z}$, $1 \le j \le r$, such that

$$\omega_j(r_0 e^{i\theta}) = e^{2\pi i(\lambda_j \log(r_0) + k_j\theta)}, \quad r_0 > 0, \quad \theta \in [0, 2\pi).$$

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We may represent π as $\bigotimes_{j=1}^{r} \omega_j$. Since π is trivial on $\mu_n^{M_0}$, it follows that ω_j is trivial on μ_n. This in turn implies that k_j must be a multiple of n. Thus the n-th root of ω_j is also a unitary character of $\mathbf{C}^\times$. It is readily verified that the genuine representation given by the tensor product of the n-th roots of $\omega_1, \ldots, \omega_r$ is the preimage of π under the metaplectic correspondence. ■

It is not necessarily the case that $\pi \in \Pi_{\text{temp}}\big(M(\mathbf{C})\big)$ is metic if it is trivial on $\mu_n^M \subset \mu_n^{M_0}$. However, Lemma 3.1 tells us that if $\pi = \pi_1^M$ for some $\pi_1 \in \Pi_{\text{temp}}\big(M_0(\mathbf{C})\big)$, then π is in the image if and only if π_1 is trivial on $\mu_n^{M_0}$. In particular, if π lies in the image, then it is trivial on μ_n^M.

We continue by constructing for every $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{C})\big)$ a matching function $f \in \mathcal{H}\big(G(\mathbf{C})\big)$. Suppose that $\gamma \in G_{\text{oreg}}(\mathbf{C})$ has eigenvalues $z_1, \ldots, z_r$. Then

$$\frac{D^G(\gamma^n)}{D^G(\gamma)} = \frac{\psi(\gamma)^2}{\big(\det(\gamma)\big)^{(n-1)(r-1)}},$$

where $\psi(\gamma) = \prod_{i<j}(z_i^n - z_j^n)/(z_i - z_j)$. Since $\psi(\gamma)$ is a symmetric polynomial in the eigenvalues of γ, it can be expressed polynomially in terms of the coefficients of the characteristic polynomial of γ. In particular, ψ extends to a rational function on the space of $r \times r$ matrices. Therefore

$$\left|\frac{D^G(\gamma^n)}{D^G(\gamma)}\right|_{\mathbf{C}}^{1/2} = \frac{\psi(\gamma)\overline{\psi(\gamma)}}{|\det(\gamma)|_{\mathbf{C}}^{(n-1)(r-1)/2}}$$

extends to a smooth function on $G(\mathbf{C})$.[1] We define a function $f \in C_c^\infty\big(G(\mathbf{C})\big)$ by

$$f(\gamma) = n^r \left|\frac{D^G(\gamma^n)}{D^G(\gamma)}\right|_{\mathbf{C}}^{1/2} \tilde{f}(\gamma'), \quad \gamma \in G(\mathbf{C}).$$

It is a simple exercise to check that $\tilde{f}$ and f have matching orbital integrals. The Weyl integration formula provides a means of converting matching orbital integrals into an identity of characters.

Proposition 3.1 *Suppose $\tilde{\pi} \in \Pi_{\text{temp}}\big(\tilde{M}_0(\mathbf{C})\big)$ and $\pi \in \Pi_{\text{temp}}\big(M_0(\mathbf{C})\big)$. Furthermore suppose $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{C})\big)$ and $f \in C_c^\infty\big(G(\mathbf{C})\big)$ are as above. Then $\operatorname{tr}\big(\pi^G(f)\big)$ vanishes unless π is in the image of the metaplectic correspondence. Moreover*

$$\operatorname{tr}\big(\tilde{\pi}^{\tilde{G}}(\tilde{f})\big) = \operatorname{tr}\big((\tilde{\pi}^{\tilde{G}})'(f)\big).$$

Proof According to [Kna86, 10.18], the value of the character of π^G at $\gamma \in M_0(\mathbf{C})$ is

$$|D^G(\gamma)|_{\mathbf{C}}^{-1/2} \sum_{w \in W_0^G} \pi(\gamma^w).$$

[1] I owe this tidy smoothness argument to the referee of an earlier manuscript.

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Therefore, using the Weyl integration formula [Kna86, 10.27],

$$\begin{aligned}\operatorname{tr}\big(\pi^G(f)\big) &= |W_0^G|^{-1}\int_{M_0(\mathbf{C})} |D^G(\gamma)|_{\mathbf{C}}^{1/2}\,|D^G(\gamma)|_{\mathbf{C}}^{-1/2}\sum_{w\in W_0^G}\pi(\gamma^w)I_G(\gamma,f)\,d\gamma\\ &= \int_{M_0(\mathbf{C})}\pi(\gamma)I_G(\gamma,f)\,d\gamma\\ &= n^r\int_{M_0(\mathbf{C})}\pi(\gamma)I_{\tilde{G}}(\gamma',\tilde{f})\,d\gamma.\end{aligned}$$

In particular, if $\eta\in\mu_n^{M_0}$ then

$$\begin{aligned}\operatorname{tr}\big(\pi^G(f')\big) &= n^r\int_{M_0(\mathbf{C})}\pi(\eta\gamma)I_{\tilde{G}}\big((\eta\gamma)',\tilde{f}\big)\,d\gamma\\ &= \pi(\eta)n^r\int_{M_0(\mathbf{C})}\pi(\gamma)I_{\tilde{G}}(\gamma',\tilde{f})\,d\gamma\\ &= \pi(\eta)\operatorname{tr}\big(\pi(f)\big).\end{aligned}$$

The first assertion of the lemma follows from this equation and Lemma 3.1. The second assertion follows from

$$\begin{aligned}\operatorname{tr}\big((\tilde{\pi}^{\tilde{G}})'(f)\big) &= n^r\int_{M_0(\mathbf{C})}\tilde{\pi}'(\gamma)I_{\tilde{G}}(\gamma',\tilde{f})\,d\gamma\\ &= n^{2r}\int_{M_0(\mathbf{C})}\tilde{\pi}(\gamma')I_{\tilde{G}}(\gamma',\tilde{f})\,d\gamma\\ &= \int_{\tilde{M}_0(\mathbf{C})}\tilde{\pi}(\tilde{\gamma})I_{\tilde{G}}(\tilde{\gamma},\tilde{f})\,d\tilde{\gamma}\\ &= \operatorname{tr}\big(\tilde{\pi}^{\tilde{G}}(\tilde{f})\big).\end{aligned}$$

∎

It follows from Proposition 3.1 that the function,

$$\pi\mapsto\operatorname{tr}\big(\pi(f)\big),\quad \pi\in\Pi_{\text{temp}}\big(G(\mathbf{C})\big),$$

belongs to $\mathcal{I}\big(G(\mathbf{C})\big)$. Thus, by the trace Paley-Wiener theorem [CD84], f belongs to $\mathcal{H}\big(G(\mathbf{C})\big)$. Defining $\tilde{f}'=f_G$, we see that $\tilde{f}'$ satisfies all of the properties appealed to in Section 3.1.

We conclude the discussion of the metaplectic correspondence over the complex field with a short lemma.

Lemma 3.2 *Suppose $\tilde{f}\in\mathcal{H}\big(\tilde{G}(\mathbf{C})\big)$ and $f\in\mathcal{H}\big(G(\mathbf{C})\big)$ are as above. Then*

$$f(1)=|n|_{\mathbf{C}}^{r^2/2}\tilde{f}(1).$$

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Proof It is a straightforward consequence of the definition of the Weyl discriminant that

$$f(\gamma) = n^r \Big| \det\Big(\sum_{k=0}^{n-1} \mathrm{Ad}(\gamma)_{\mathfrak{g}/\mathfrak{g}_\gamma} \Big) \Big|_{\mathbf{C}}^{1/2} \tilde{f}(1), \quad \gamma \in G_{\mathrm{oreg}}(\mathbf{C}).$$

We may take the limit of this equation as $\gamma \in G_{\mathrm{oreg}}(\mathbf{C})$ approaches the identity to obtain

$$f(1) = n^r \Big| \det\Big(\sum_{k=0}^{n-1} \mathrm{Ad}(1)_{\mathfrak{g}/\mathfrak{g}_\gamma} \Big) \Big|_{\mathbf{C}}^{1/2} \tilde{f}(1),$$

where $\gamma \in G_{\mathrm{oreg}}(\mathbf{C})$ is arbitrary. The map $\mathrm{Ad}(1)$ acts as the identity on the complex vector space $\mathfrak{g}/\mathfrak{g}_\gamma$ of dimension $r^2 - r$. We therefore have

$$f(1) = n^r |n|_{\mathbf{C}}^{(r^2-r)/2} \tilde{f}(1) = |n|_{\mathbf{C}}^{r/2} |n|_{\mathbf{C}}^{(r^2-r)/2} \tilde{f}(1) = |n|_{\mathbf{C}}^{r^2/2} \tilde{f}(1). \qquad \blacksquare$$

4 Comparison of Orbital Integrals at Singular Elements

In this section we show how the identity of orbital integrals (15) may be generalized to some singular elements of $G(F_S)$.

Lemma 4.1 *Suppose E is a field and u is a unipotent element of* $\mathrm{GL}(k, E)$*. Then u is conjugate to u^n in* $\mathrm{GL}(k, E)$*.*

Proof Set u_s to be the $s \times s$ matrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Jordan canonical form tells us that any unipotent element u of $\mathrm{GL}(k, E)$ is conjugate to

$$\begin{pmatrix} u_{s_1} & & 0 \\ & \ddots & \\ 0 & & u_{s_j} \end{pmatrix},$$

for some positive integers, $s_1, \ldots, s_j$, satisfying $\sum_{i=1}^{j} s_i = k$. It therefore suffices to prove the lemma in the case that $j = 1$ and $u = u_k$.

Observe that $u - 1$ is a nilpotent matrix of order $k - 1$. Now

$$u^n - 1 = \big(1 + (u-1)\big)^n - 1 = \sum_{i=0}^{n} \binom{n}{i} (u-1)^i - 1 = \sum_{i=1}^{n} \binom{n}{i} (u-1)^i.$$

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From this it is apparent that $u^n - 1$ is a nilpotent matrix of order $k-1$. Another application of Jordan canonical form tells us that u^n is conjugate to

$$\begin{pmatrix} u_{b_1} & & 0 \\ & \ddots & \\ 0 & & u_{b_t} \end{pmatrix},$$

where $\sum_{i=1}^{t} b_i = k$. Clearly,

$$\begin{pmatrix} u_{b_1} & & 0 \\ & \ddots & \\ 0 & & u_{b_t} \end{pmatrix} - \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is a nilpotent matrix of order $\max_{1\leq i \leq t}(b_i - 1)$. This implies that $t = 1$ and $b_1 = k$, whence the lemma. ■

Given a nonarchimedean valuation v and a semisimple element $\sigma \in M(F_v)$, let $\mathcal{U}_{M_\sigma}(F_v)$ be the set of unipotent elements in $M_\sigma(F_v)$ and let $\big(\mathcal{U}_{M_\sigma}(F_v)\big)$ be the corresponding set of $M_\sigma(F_v)$-conjugacy classes. The latter set is the index set in the Shalika germ expansion of an orbital integral at σ. Such germ expansions exist for $\tilde{M}(F_v)$ as well [Vig81]. We shall use these germ expansions to compare orbital integrals at elements which are not regular. As preparation for this, we show that $\big(\mathcal{U}_{M_\sigma}(F_v)\big)$ parameterizes the germs in the Shalika germ expansion for $\tilde{M}(F_v)$.

Lemma 4.2 *Suppose v is nonarchimedean, $\tilde{\gamma} \in \tilde{G}(F_v)$, $\mathbf{p}(\tilde{\gamma})$ has Jordan decomposition σu and $\tilde{\gamma} = \tilde{\sigma}\mathbf{s}(u)$. Suppose further that σu_0 belongs to the closure of the $G(F_v)$-conjugacy class of $\mathbf{p}(\tilde{\gamma})$. Then $\tilde{\sigma}\mathbf{s}(u_0)$ belongs to the closure of the $\tilde{G}(F_v)$-conjugacy class of $\tilde{\gamma}$.*

Proof We prove the lemma in a special case and leave the details of the general case to the interested reader. Suppose $r = 2$, $\sigma_1 \in F_v^\times$, $\zeta \in \mu_n$, $\sigma = \left(\begin{smallmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{smallmatrix}\right)$, $\tilde{\sigma} = \mathbf{i}(\zeta)\mathbf{s}(\sigma)$, $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and u_0 is the identity element. Let $x \in F_v^\times$ such that $|x|_v < 1$. Denote the n-th Hilbert symbol of F_v by $(\cdot,\cdot)_{F_v}$. Using equation (2) we can show that

$$\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{s}(u)\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \mathbf{s}\begin{pmatrix} 1 & x^{kn} \\ 0 & 1 \end{pmatrix}.$$

Together with Proposition 0.1.5 [KP86], this implies that

$$\begin{aligned} &\lim_{k\to\infty} \mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}\tilde{\sigma}\mathbf{s}(u)\mathbf{s}\begin{pmatrix} x^{kn} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &\qquad = \mathbf{i}(\zeta)\mathbf{s}(\sigma)\lim_{k\to\infty}(\sigma_1^2, x^{kn})_{F_v}^{(1+2m)}(\sigma_1, x^{kn})_{F_v}^{-1}(\sigma_1, 1)_{F_v}^{-1}\mathbf{s}\begin{pmatrix} 1 & x^{kn} \\ 0 & 1 \end{pmatrix} \\ &\qquad = \tilde{\sigma}. \end{aligned}$$

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In the general case, $G_\sigma(F_v)$ is isomorphic to $\prod_{i=1}^k \mathrm{GL}(m_i, F_i)$, where $F_1, \dots, F_k$ are extensions of F_v [KP86, Section 1]. One can imitate the above computation in $\mathrm{GL}(m_i, F_i)$ for each $1 \le i \le k$ to obtain the lemma. ■

In the notation of the previous lemma, Section 1.j [Vig81] tells us that if the closure of the $G(F_v)$-conjugacy class of σu is equal to the disjoint union of the $G(F_v)$-conjugacy classes of $\sigma u_1, \dots, \sigma u_k$ then there exist $\zeta_1, \dots, \zeta_k \in \mu_n$ such that the closure of the $\tilde{G}(F_v)$-conjugacy class of $\tilde{\sigma}\mathbf{s}(u)$ is equal to the disjoint union of the $\tilde{G}(F_v)$-conjugacy classes of $\mathbf{i}(\zeta_1)\mathbf{s}(\sigma)\mathbf{s}(u_1), \dots, \mathbf{i}(\zeta_k)\mathbf{s}(\sigma)\mathbf{s}(u_k)$. Lemma 4.2 implies that $\zeta_1 = \cdots = \zeta_k$ and $\tilde{\sigma} = \mathbf{i}(\zeta_1)\mathbf{s}(\sigma)$. In particular, the closure of the two conjugacy classes may be identified by way of the map $\mathbf{s}$ acting on the unipotent classes. Bearing in mind decomposition (5), it follows that, $\big(\mathcal{U}_{M_\sigma}(F_v)\big)$ is an index set for the germs in the Shalika germ expansion for $\tilde{M}(F_v)$ via the map $\mathbf{s}$.

Lemma 4.3 *Suppose v is nonarchimedean, and $\sigma \in G(F_v)$ is a semisimple element which lies in the domain of the orbit map and satisfies $G_\sigma = G_{\sigma^n}$. Suppose further that $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_v)\big)$ and $f \in \mathcal{H}\big(G(F_v)\big)$ match. Then*

$$I_G(\sigma, f) = \Lambda^G(\sigma) I_{\tilde{G}}(\sigma', \tilde{f}).$$

Proof According to [Vig81, Corollaire 2.4], [Vig81, Section 3.3] and the remarks preceding this lemma, we have the following two germ expansions:

$$I_{\tilde{G}}(\gamma', \tilde{f}) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u) I_{\tilde{G}}(\sigma'\mathbf{s}(u), \tilde{f}),$$

$$I_G(\gamma, f) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma, f_u) I_G(\sigma u, f).$$

In these equations, the element γ belongs to $G_{\mathrm{oreg}}(F_v)$ and is taken to be in a very small open neighborhood of σ depending on $\tilde{f}$ and f. The functions $f_u, h_u \in C_c^\infty\big(G(F_v)\big)$ have the following properties:

$$I_G(\sigma^n u_1, h_u) = \begin{Bmatrix} 1, & \text{if } u_1 = u \\ 0, & \text{otherwise} \end{Bmatrix},$$

$$I_G(\sigma u_1, f_u) = \begin{Bmatrix} 1, & \text{if } u_1 = u \\ 0, & \text{otherwise} \end{Bmatrix}, \quad u, u_1 \in \big(\mathcal{U}_{G_\sigma}(F_v)\big).$$

We may assume that

$$f_u(\gamma_1) = \left| \frac{D^G(\sigma^n)}{D^G(\sigma)} \right|_v^{1/2} h_u(\gamma_1^n) = |n|_v^{(r^2 - \dim(G_\sigma))/2} h_u(\gamma_1^n), \quad \gamma_1 \in G(F_v)$$

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[Vig81, Application 1, Section 4.1]. Indeed, for $u_1 \in \big(\mathcal{U}_{G_\sigma}(F_v)\big)$, we have

$$\begin{aligned} I_G(\sigma u_1, f_u) &= |D^G(\sigma)|_v^{1/2} \int_{G_\sigma(F_v)\backslash G(F_v)} f_u(x^{-1}\sigma u_1 x)\,dx \\ &= |D^G(\sigma^n)|_v^{1/2} \int_{G_{\sigma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\sigma^n u_1^n x)\,dx \\ &= I_G(\sigma^n u_1^n, h_u) \\ &= \begin{cases} 1, & \text{if } u_1^n = u \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and by Lemma 4.1, $u_1^n = u_1$, so f_u satisfies the required property. As a consequence $I_G(\gamma, f_u)$ equals

$$\begin{aligned} &|D^G(\gamma)|_v^{1/2} \int_{G_\gamma(F_v)\backslash G(F_v)} f_u(x^{-1}\gamma x)\,dx \\ &\quad = |D^G(\gamma^n)|_v^{1/2} \left|\frac{D^G(\gamma)}{D^G(\gamma^n)}\right|_v^{1/2} |n|_v^{(r^2-\dim(G_\sigma))/2} \int_{G_{\gamma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\gamma^n x)\,dx \\ &\quad = |n|_v^{(-r^2+r+r^2-\dim(G_\sigma))/2} |D^G(\gamma^n)|_v^{1/2} \int_{G_{\gamma^n}(F_v)\backslash G(F_v)} h_u(x^{-1}\gamma^n x)\,dx \\ &\quad = |n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\gamma^n, h_u). \end{aligned}$$

The germ expansion for G is thus

$$I_G(\gamma, f) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u)\big(|n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\sigma u, f)\big).$$

After applying Lemma 4.1 to the germ expansion for $\tilde{G}$, we obtain

$$I_{\tilde{G}}(\gamma', \tilde{f}) = \sum_{u \in (\mathcal{U}_{G_\sigma}(F_v))} I_G(\gamma^n, h_u) I_{\tilde{G}}\big(\sigma' \mathbf{s}(u), \tilde{f}\big).$$

Let us examine the germ $I_G(\gamma^n, h_u)$ in more detail. According to [Vig81, Section 2.5],

$$I_G(\gamma^n, h_u) = \left|\frac{D^G(\sigma^n u) D^G(\gamma^n)}{D^{G_\sigma}(\sigma^n u) D^{G_\sigma}(\gamma^n)}\right|_v^{1/2} I_{G_\sigma}(\gamma^n, h_u). \tag{16}$$

Since σ is central in $G_\sigma(F_v)$, it follows that $I_{G_\sigma}(\gamma, h_u)$ satisfies a homogeneity property [Vig81, Application 2, Section 4.1], namely

$$I_{G_\sigma}(\gamma^{k^2}, h_u) = |k|_v^{\dim(G_{\sigma u})-r} I_{G_\sigma}(\gamma, h_u), \quad k \in \mathbf{Z}.$$

Together with (16), this implies that

$$I_G(\gamma^{nk^2}, h_u) = |k|_v^{((r^2-\dim G_\sigma)/2)+\dim(G_{\sigma u})-r} I_G(\gamma^n, h_u).$$

Hence the germs, $I_G(\gamma^n, h_u)$, for $u \in (\mathcal{U}_{G_\sigma}(F_v))$, may be said to be homogeneous of degree $\dim(G_{\sigma u})$ respectively. It is clear that nonzero germs of different degree are linearly independent. Now suppose $\gamma \in G_{\mathrm{oreg}}(F_v)$ is F_v-elliptic in $G_\sigma(F_v)$ and is close to σ. The difference,

$$\Lambda_v^G(\gamma) I_{\tilde G}(\gamma', \tilde f) - I_G(\gamma, f)$$
$$= \sum_{u\in(\mathcal{U}_{G_\sigma}(F_v))} \Big(\Lambda_v^G(\gamma) I_{\tilde G}\big(\sigma'\mathbf{s}(u), \tilde f\big) - |n|_v^{-(\dim(G_\sigma)-r)/2} I_G(\sigma u, f)\Big) I_G(\gamma^n, h_u),$$

vanishes since $\tilde f$ matches f. It follows from [Rog83, Proposition 2.1] and [Rog83, Theorem 2.2] that the germ $I_G(\gamma^n, h_1)$ is not zero. It is also the only germ in this linear combination of degree $\dim(G_\sigma)$. Its coefficient must vanish by linear independence. This proves the lemma. ■

Lemma 4.4 *Suppose $\sigma \in G(\mathbf{C})$ is semisimple and $G_\sigma = G_{\sigma^n}$. Suppose further that $\tilde f \in \mathcal{H}\big(\tilde G(\mathbf{C})\big)$ matches $f \in \mathcal{H}\big(G(\mathbf{C})\big)$. Then*

$$I_G(\sigma, f) = \Lambda^G(\sigma) I_{\tilde G}(\sigma', \tilde f).$$

Proof We first consider the case $\sigma = 1$. In this case we must show that

$$f(1) = |n|_{\mathbf{C}}^{r^2/2} \tilde f(1).$$

This equation was shown in Lemma 3.2 for a specific choice of f matching $\tilde f$. However, it follows from the Weyl integration formula and the Plancherel formula that the value at the identity of any function matching $\tilde f$ is equal to $f(1)$. Now suppose that $\sigma \in G(\mathbf{C})$ is any semisimple element satisfying $G_\sigma = G_{\sigma^n}$. Since we will only be considering orbital integrals at semisimple elements, Jordan canonical form allows us to assume that our orbital integrals taken are taken at elements in $M_0(\mathbf{C})$. Clearly, we may assume that $\sigma \in A_M(\mathbf{C})$ for some $M \in \mathcal{L}$, so that $G_\sigma = M$. Fix $P \in \mathcal{P}(M)$ and set $\tilde f_{\tilde P}^{\sigma^n}(\gamma)$ equal to

$$\big|\det\big(1 - \mathrm{Ad}(\gamma\sigma^n)\big)_{|\mathfrak{g}/\mathfrak{m}}\big|_{\mathbf{C}}^{1/2} \int_{U_P(\mathbf{C})}\int_{K_{\mathbf{C}}} \tilde f\big(\mathbf{s}(k)^{-1}\mathbf{s}(u)^{-1}\mathbf{s}(\gamma\sigma^n)\mathbf{s}(u)\mathbf{s}(k)\big)\, du\, dk,$$

where $\gamma \in M(\mathbf{C})$ and $\mathfrak{m}$ is the (real) Lie algebra of $M(\mathbf{C})$. For $\gamma \in M_0(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C})$ in a sufficiently small neighborhood $V_\sigma \subset M_0(\mathbf{C})$ of the identity, we have $\gamma\sigma \in M_0(\mathbf{C}) \cap G_{\mathrm{oreg}}(\mathbf{C})$. It then follows from a standard descent formula [FK86, Section 7] that

$$\Lambda^G(\gamma\sigma) I_{\tilde G}\big((\gamma\sigma)', \tilde f\big) = \Lambda^M(\gamma) I_{\tilde M}^{\tilde M}(\gamma', \tilde f_{\tilde P}^{\sigma^n}), \quad \gamma \in V_\sigma \cap G_{\mathrm{oreg}}(\mathbf{C}).$$

After making similar definitions, we find that

$$I_G(\gamma\sigma, f) = I_M^M(\gamma, f_P^\sigma), \quad \gamma \in V_\sigma \cap G_{\mathrm{oreg}}(\mathbf{C}).$$

Combining the last two equations with the matching property, we obtain

$$\Lambda^M(\gamma) I_{\tilde{M}}^{\tilde{M}}(\gamma, \tilde{f}_{\tilde{P}}^{\sigma^n}) = I_M^M(\gamma, f_P^\sigma), \quad \gamma \in V_\sigma \cap G_{\mathrm{oreg}}(\mathbf{C}). \tag{17}$$

The functions $\tilde{f}_{\tilde{P}}^{\sigma^n}$ and f_P^σ also satisfy the identities,

$$\begin{aligned}\Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1) &= \Lambda^G(\sigma^n)\Big|\det\big(1 - \mathrm{Ad}(\sigma^n)\big)_{|\mathfrak{g}/\mathfrak{g}_{\sigma^n}}\Big|_{\mathbf{C}}^{1/2} \int_{\tilde{G}_{\sigma'}(\mathbf{C})\backslash \tilde{G}(\mathbf{C})} \tilde{f}(x^{-1}\sigma' x)\, dx \\ &= \Lambda^G(\sigma^n) I_{\tilde{G}}(\sigma', \tilde{f}),\end{aligned}$$

and $f_P^\sigma(1) = I_G(\sigma, f)$. The lemma therefore follows if we show that

$$f_P^\sigma(1) = \Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1).$$

According to Section 3.2, there exists a function $h \in \mathcal{H}\big(M(\mathbf{C})\big)$ matching $\tilde{f}_{\tilde{P}}^{\sigma^n} \in \mathcal{H}\big(\tilde{M}(\mathbf{C})\big)$. From our proof of the case $\sigma = 1$ it follows that

$$h(1) = \Lambda^M(1)\tilde{f}_{\tilde{P}}^{\sigma^n}(1).$$

Thus, if $h(1) = f_P^\sigma(1)$ the lemma is complete. This last equation follows from the fact that $h(1)$ is determined by the values of $I_M^M(\cdot, h)$ restricted to any neighborhood of the identity in $M_0(\mathbf{C})$. More precisely, there exists a differential operator $\partial(\varpi)$ on $M_0(\mathbf{C})$ such that $h(1) = \partial(\varpi) I_M^M(1, h)$ [Kna86, Theorem 11.17]. Notice that equation (17) and the matching property imply

$$I_M^M(\gamma, h) = I_M^M(\gamma, f_P^\sigma), \quad \gamma \in V_\sigma \cap G_{\mathrm{oreg}}(\mathbf{C}).$$

Since $V_\sigma \cap G_{\mathrm{oreg}}(\mathbf{C})$ contains an open subset of V_σ, the previous two equations imply that

$$h(1) = \partial(\varpi) I_M^M(1, h) = \partial(\varpi) I_M^M(1, f_P^\sigma) = f_P^\sigma(1). \qquad \blacksquare$$

The reader might wonder whether one might be able to remove the hypothesis $G_\sigma = G_{\sigma^n}$ from the previous two lemmas. The following example shows that, at least for Lemma 4.4, this hypothesis cannot be removed. Suppose $F_v = \mathbf{C}$, $n = r = 2$ and $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{C})\big)$ such that $\tilde{f}(1) \neq 0$. Let $f \in \mathcal{H}\big(G(\mathbf{C})\big)$ be the function defined in Section 3.2. In particular, if $\gamma \in G(\mathbf{C})$ has eigenvalues $z_1, z_2 \in \mathbf{C}$ then

$$f(\gamma) = \big|(z_1 + z_2)^2/z_1 z_2\big|_{\mathbf{C}}^{1/2} \tilde{f}(\gamma').$$

We know that $\tilde{f}$ matches f. However, for the semisimple element $\sigma = \mathrm{diag}(1, -1) \in G(\mathbf{C})$ the equation of Lemma 4.4 does not hold. Indeed, the equation,

$$f(x^{-1}\sigma x) = 0, \quad x \in G_\sigma(\mathbf{C}) \setminus G(\mathbf{C}),$$

implies that $I_G(\sigma, f) = 0$, but $\Lambda^G(\sigma) I_{\tilde{G}}(\sigma', \tilde{f})$ is equal to $n^4 \tilde{f}(1)$ (Lemma 3.2), which by assumption is not zero.

This example suggests that we should restrict ourselves to the comparison of orbital integrals at elements in the subset $M_{\mathrm{comp}}(F_S)$ of $M(F_S)$, defined by

$$M_{\mathrm{comp}}(F_S) = \Big\{ \gamma = \prod_{v \in S} \gamma_v \in M(F_S) : M_{\sigma_v} = M_{\sigma_v^n},\ \sigma_v \text{ semisimple part of } \gamma_v \Big\}.$$

The set $M_{\mathrm{comp}}(F_S)$ is a dense open subset of $M(F_S)$.

Lemma 4.5 *Suppose $\gamma \in G(F_S)$ lies in the domain of the orbit map and has Jordan decomposition σu. Suppose further that $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big)$ matches $f \in \mathcal{H}\big(G(F_S)\big)$. Then*

$$I_G(\gamma, f) = \Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}), \quad \gamma \in G_{\mathrm{comp}}(F_S).$$

Proof It suffices to prove the lemma in the case that S consists of a single valuation v. The unipotent element u is contained in the Richardson orbit of some parabolic subgroup $P_{\sigma^n} = M_\sigma U_\sigma$ of G_σ. Let A_σ be the center of M_σ. By equation (2) and definition (1.3) II [AC89], we have

$$I_{\tilde{G}}(\gamma', \tilde{f}) = I_{\tilde{G}}\big(\sigma' \mathbf{s}(u^n), \tilde{f}\big) = \lim_{a \to 1} I_{\tilde{G}}\big(\mathbf{s}(a^n)\sigma', \tilde{f}\big),$$

where a is regular in $A_\sigma(F_v)$. We also have the parallel definition,

$$I_G(\gamma, f) = I_G(\sigma u, f) = \lim_{a \to 1} I_G(a\sigma, f).$$

The lemma now follows from these two definitions and the matching of $\tilde{f}$ and f. ∎

5 Normalization of Intertwining Operators and the Plancherel Formula

Our goal here is to normalize the intertwining operators between induced representations. This is necessary for the definition of the distributions occurring in the invariant trace formula of Arthur ([Art89] and [Art88a]).

Our method of normalization follows [AC89, Section 2 II]. In particular, we make use of the Plancherel formula [HC84]. The proof of this formula has not been carried out for $\tilde{G}(F_v)$. We nevertheless assert that the Plancherel formula holds in this case as well. In what follows, we justify this assertion by listing the properties of reductive algebraic groups used in the unpublished published proof of Harish-Chandra's Plancherel formula [CW] and show that they also hold for $\tilde{G}(F_v)$. In this justification we assume that v is nonarchimedean.

Suppose $P \in \mathcal{P}(M)$ such that $P \supset P_0$. Then $\tilde{G}(F_v)$ splits over $U_P(F_v)$ by (2). The splitting homomorphism is $\mathbf{s}$. In other words,

$$\mathbf{s}\big(U_P(F_v)\big) = \{(u, 1) : u \in U_P(F_v)\}$$

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is a subgroup of $\tilde{G}(F_v)$. Every parabolic subgroup in $\mathcal{P}(M)$ is of the form $P^w = w^{-1}Pw$ for some representative w of the Weyl group W_0^G. It is easy to check that $\tilde{G}(F_v)$ splits over $U_{P^w}(F_v) = w^{-1}U_P(F_v)w$ with splitting homomorphism $\mathbf{s}_w$ defined by

$$w^{-1}uw \mapsto \mathbf{s}(w)^{-1}\mathbf{s}(u)\mathbf{s}(w).$$

Clearly $\tilde{P^w}(F_v) = \tilde{M}(F_v)\mathbf{s}_w\big(U_{P^w}(F_v)\big)$ as $\tilde{M}(F_v)$ is stable under conjugation by $\mathbf{s}(w)$.

We define the Jacquet module of an admissible representation $(\tilde{\pi}, V)$ of finite length with respect to $U_{P^w}(F_v)$ in the following way. Let $V_{U_{P^w}}$ be the linear span of

$$\left\{ \tilde{\pi}\big(\mathbf{s}_w(w^{-1}uw)\big)\, v - v : u \in U_P(F_v), v \in V \right\}.$$

It is a consequence of equation (2) that $\tilde{M}(F_v)$ normalizes $\mathbf{s}_w\big(U_{P^w}(F_v)\big)$. Thus $\tilde{M}(F_v)$ stabilizes $V_{U_{P^w}}$. We define the Jacquet module of $\tilde{\pi}$ with respect to U_{P^w} to be the representation obtained by twisting the quotient representation $V/V_{U_{P^w}}$ with the modular function $\delta_{P^w}^{-1/2}$. We denote this representation by $\tilde{\pi}_{U_{P^w}}$. This is a mild generalization of [FK86, Section 14]. It is left to the reader to check that this definition yields the expected properties of Jacquet modules.

Another consequence of the splitting of $\tilde{G}(F_v)$ over $U_{P^w}(F_v)$ is the decomposition,

$$\tilde{G}(F_v) = \tilde{M}(F_v)\mathbf{s}_w\big(U_{P^w}(F_v)\big)\, K_v = \tilde{M}(F_v)\mathbf{s}(w)^{-1}\mathbf{s}\big(U_P(F_v)\big)\, \mathbf{s}(w)K_v.$$

The associated integration formula follows in the usual fashion.

Suppose, for this paragraph only, that w is the representative of W_0^G such that the opposite parabolic subgroup $\bar{P}$ is equal to P^w. Then we obtain the Gelfand-Naimark decomposition, which is a decomposition of an open dense subset of $\tilde{G}(F_v)$ as

$$\mathbf{s}_w\big(U_{P^w}(F_v)\big)\, \tilde{M}(F_v)\mathbf{s}\big(U_P(F_v)\big).$$

Its associated integration formula is given by

$$\int_{\tilde{G}(F_v)} \tilde{f}(x)\, dx = \gamma_{\tilde{M}} \int_{\mathbf{s}_w(U_{P^w}(F_v))} \int_{\tilde{M}(F_v)} \int_{\mathbf{s}(U_P(F_v))} \tilde{f}(\bar{u}\tilde{m}u)\, du\, d\tilde{m}\, d\bar{u},$$

where

$$\gamma_{\tilde{M}} = \int_{\mathbf{s}_w(U_{P^w}(F_v))} \delta_P\Big(m_P\big(\mathbf{p}(\bar{u})\big)\Big)\, d\bar{u} = \gamma_M \tag{18}$$

$\bar{u} = u_P(\bar{u})m_P(\bar{u})k_P(\bar{u})$, $u_P(\bar{u}) \in U_P(F_v)$, $m_P(\bar{u}) \in M(F_v)$ and $k_P(\bar{u}) \in K_v$. The results of this paragraph do not rely on the assumption that $P \supset P_0$.

The abelian group,

$$\widetilde{A_M^{n/d}}(F_v) = \{\tilde{\gamma} \in \tilde{M}(F_v) : \mathbf{p}(\tilde{\gamma}) = \gamma^{n/d}, \gamma \in A_M(F_v)\},$$

lies in the center of $\tilde{M}(F_v)$ [KP84, Proposition 0.1.1]) and provides a decomposition of $\mathbf{s}_w\big(U_{P^w}(F_v)\big)$ by way of the adjoint action. Using this decomposition we may define a subset $\widetilde{A_M^{n/d}}^{-}(F_v)$ of $\widetilde{A_M^{n/d}}(F_v)$ as in [Cas93, Section 1.4]. This subset may be used to examine the asymptotic behavior of matrix coefficients (*cf.* [Cas93, Section 4]). The only other ingredient needed to prove the results concerning the asymptotic behavior of matrix coefficients for $\tilde{G}(F_v)$ is the Iwahori factorization for arbitrarily small compact open subgroups. These Iwahori factorizations exist in $\tilde{G}(F_v)$, because there exist arbitrarily small compact open subgroups of $G(F_v)$ over which $\tilde{G}(F_v)$ splits [FK86, Section 2], and Iwahori factorizations hold in $G(F_v)$ [Cas93, Proposition 1.4.4].

The only decomposition which still needs to be addressed for $\tilde{G}(F_v)$ is the Cartan decomposition. This may be recast as

$$\tilde{G}(F_v) = \bigcup_{\gamma} K_v \gamma \widetilde{A_{M_0}^{n/d}}(F_v) K_v,$$

where γ runs over a set of representatives of $\widetilde{A_{M_0}}(F_v)/\widetilde{A_{M_0}^{n/d}}(F_v)$. This union is finite and disjoint. It is the finiteness which allows us to restrict our attention to $K_v\widetilde{A_{M_0}^{n/d}}(F_v)K_v$ when proving the convergence of integrals or bounds of certain functions.

We make one further remark concerning bounding functions on $\tilde{G}(F_v)$. If $\tilde{f}$ is a genuine or antigenuine function on $\tilde{G}(F_v)$ then clearly

$$\sup_{\tilde{\gamma}\in\tilde{G}(F_v)} |\tilde{f}(\tilde{\gamma})| = \sup_{\tilde{\gamma}\in\tilde{G}(F_v)} \left|\tilde{f}\big(\mathbf{p}(\tilde{\gamma}),1\big)\right| = \sup_{\gamma\in G(F_v)} \left|\tilde{f}\big(\mathbf{s}(\gamma)\big)\right|.$$

Therefore, in cases where one is interested in finding uniform bounds of such functions, the techniques of the non-metaplectic groups may be used.

This concludes the discussion of the properties necessary for the proof of the Plancherel formula. The unpublished proof of [CW] may now be imitated after making some apparent definitions.

The normalization of intertwining operators amounts to the definition of functions,

$$r_{\tilde{Q}|\tilde{P}}\colon \Pi\big(\tilde{M}(F_S)\big) \times \mathfrak{a}^*_{M,\mathbf{C}} \to \mathbf{C}, \quad Q,P \in \mathcal{P}(M),$$

which satisfy the conditions of [Art89, Theorem 2.1]. These functions are called normalizing factors. Such normalizing factors exist for general linear groups [Sha84], [Art89, Section 4]. Since we may take the normalizing factors for $\Pi\big(\tilde{M}(F_S)\big)$ to be the product of the normalizing factors for $\Pi\big(\tilde{M}(F_v)\big)$, $v \in S$, it suffices to consider the case that S consists of a single valuation. As mentioned above, we already have normalizing factors of the form $r_{Q|P}$. We define candidates for normalizing factors of metaplectic coverings by setting

$$r_{\tilde{Q}|\tilde{P}}(\tilde{\pi}_\lambda) = \Lambda^M(1)^{1/2}\Lambda^G(1)^{-1/2} r_{Q|P}(\tilde{\pi}'_{\lambda'}),$$

for all $\tilde{\pi} \in \Pi\big(\tilde{M}(F_S)\big)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. As indicated in [AC89, Section 2 II] and [Art89, Sections 2 and 4], there is only one required property which $r_{\tilde{Q}|\tilde{P}}$ does not obviously satisfy. Let $\Pi_{\mathrm{disc}}\big(\tilde{M}(F_v)\big)$ be the subset of representations in $\Pi_{\mathrm{temp}}\big(\tilde{M}(F_v)\big)$ which are square-integrable modulo the center of $\tilde{M}(F_v)$. The property which remains to be verified is

$$r_{\tilde{P}|\bar{\tilde{P}}}(\tilde{\pi}_\lambda) r_{\bar{\tilde{P}}|\tilde{P}}(\tilde{\pi}_\lambda) = J_{\tilde{P}|\bar{\tilde{P}}}(\tilde{\pi}_\lambda) J_{\bar{\tilde{P}}|\tilde{P}}(\tilde{\pi}_\lambda), \quad \tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}(F_v)\right), \quad \lambda \in \mathfrak{a}^*_{M,\mathbf{C}}.$$

The right-hand side is, by definition, the inverse of the Harish-Chandra μ-function $\mu_{\tilde{M}}$ (not to be confused with the group μ_n^M).

***Lemma 5.1* (2.1)** *Suppose $\tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}(F_v)\right)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$. Then*

$$r_{\tilde{P}|\bar{\tilde{P}}}(\tilde{\pi}_\lambda) r_{\bar{\tilde{P}}|\tilde{P}}(\tilde{\pi}_\lambda) = \mu_{\tilde{M}}(\tilde{\pi}_\lambda)^{-1}.$$

Proof The normalizing factors for P, as defined in [Sha84], already satisfy the condition of the lemma. Furthermore, $\mu_{\tilde{M}}$ is meromorphic in $\mathfrak{a}^*_{M,\mathbf{C}}$. Therefore it suffices to show that

$$\mu_{\tilde{M}}(\tilde{\pi}) = \Lambda^M(1)\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'), \quad \tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}(F_v)\right).$$

Suppose $\tilde{f} \in \mathcal{H}\left(\tilde{G}(F_v)\right)$ such that

$$\tilde{f}_{\tilde{M}_1}(\tilde{\pi}) = 0, \quad \tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}_1(F_v)\right), \quad M_1 \in \mathcal{L}, \quad M_1 \neq M.$$

It follows from the Plancherel formula for $\tilde{G}(F_v)$ that

$$\tilde{f}(1) = \gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^{\tilde{M}}(\tilde{\pi}_\lambda)\mu_{\tilde{M}}(\tilde{\pi}_\lambda)\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)\, d\lambda,$$

where $d^{\tilde{M}}(\tilde{\pi}_\lambda)$ is the formal degree of $\tilde{\pi}_\lambda$. Suppose $f \in \mathcal{H}\left(G(F_v)\right)$ such that $f_G = \tilde{f}'$ [FK86, Proposition 27.3]). Then, by Lemma 4.3 and Lemma 4.4, we have

$$\tilde{f}(1) = I_{\tilde{G}}(1, \tilde{f}) = \Lambda^G(1)^{-1} I_G(1, f).$$

Combined with the Plancherel formula and equation (18), this equation implies that $\tilde{f}(1)$ is equal to the product of $\Lambda^G(1)^{-1}$ with

$$\begin{aligned}
&\gamma_M \sum_{\Pi_{\mathrm{disc}}(M(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M(\pi_{\lambda'})\mu_M(\pi_{\lambda'})\tilde{f}'_M(\pi_{\lambda'})\, d\lambda' \\
&\quad = \gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M\left((\tilde{\pi}_\lambda)'\right)\mu_M\left((\tilde{\pi}_\lambda)'\right)\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)\, d\lambda' \\
&\quad = n^{\dim(A_M)}\gamma_{\tilde{M}} \sum_{\Pi_{\mathrm{disc}}(\tilde{M}(F_v))/i\mathfrak{a}^*_{M,\{v\}}} \int_{i\mathfrak{a}^*_{M,\{v\}}} d^M\left((\tilde{\pi}_\lambda)'\right)\mu_M\left((\tilde{\pi}_\lambda)'\right)\tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)\, d\lambda
\end{aligned}$$

Taking the difference of this equation from the earlier Plancherel expansion of $\tilde{f}(1)$ we find that

$$\int_{i\mathfrak{a}^*_{M,\{v\}}} \left(d^{\tilde{M}}(\tilde{\pi})\mu_{\tilde{M}}(\tilde{\pi}_\lambda) - d^M(\tilde{\pi}')n^{\dim(A_M)}\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'_{\lambda'}) \right) \tilde{f}_{\tilde{M}}(\tilde{\pi}_\lambda)\, d\lambda$$

vanishes for all $\tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}(F_v)\right)$. An application of the trace Paley-Wiener formula (for metaplectic coverings) to this equation implies that

$$d^{\tilde{M}}(\tilde{\pi})\mu_{\tilde{M}}(\tilde{\pi}) = d^M(\tilde{\pi}')n^{\dim(A_M)}\Lambda^G(1)^{-1}\mu_M(\tilde{\pi}'), \quad \tilde{\pi} \in \Pi_{\mathrm{disc}}\left(\tilde{M}(F_v)\right).$$

Notice that $\mu_{\tilde{G}} = \mu_G = 1$ and so

$$d^{\tilde{G}}(\tilde{\pi}_0) = n\Lambda^G(1)^{-1}d^G(\tilde{\pi}'_0), \quad \tilde{\pi}_0 \in \Pi_{\mathrm{disc}}\left(\tilde{G}(F_v)\right).$$

It follows in general that

$$d^{\tilde{M}}(\tilde{\pi}) = \prod_{i=1}^{\ell} d^{\tilde{M}(i)}(\tilde{\pi}_i) = \prod_{i=1}^{\ell} n\Lambda^{M(i)}(1)^{-1}d^{M(i)}(\tilde{\pi}'_i) = n^{\dim(A_M)}\Lambda^M(1)^{-1}d^M(\tilde{\pi}'),$$

where $M = M(1) \times \cdots \times M(\ell)$ is decomposition (5) and $\tilde{\pi}_1, \ldots, \tilde{\pi}_\ell$ are as in the Appendix. The lemma is now complete. ∎

6 The Distribution $I^{\Sigma}_M(\gamma)$

The invariant trace formula for G contains invariant distributions of the form $I_M(\gamma)$ parameterized by $\gamma \in M(F_S)$. These distributions were introduced in [Art88a, Section 2]. The reader is assumed to be familiar with their definitions and properties. Our goal is to compare these distributions with the analogous distributions on $\tilde{G}(F_S)$ using the orbit map. The orbit map is not injective. Consequently we must group the distributions $I_M(\gamma)$ in terms of the fibers of the orbit map. Since the orbit map raises elements of $\tilde{G}(F_S)$ to the n-th power, a good guess would be that we should group the distributions in terms of the n-th roots of unity μ_n. According to decomposition (5),

$$M(F_S) \cong \mathrm{GL}(r_1, F_S) \times \cdots \times \mathrm{GL}(r_\ell, F_S).$$

The center of $\mathrm{GL}(r_i, F_S)$ is isomorphic to $\mathrm{GL}(1, F_S)$. In consequence,

$$A_M(F_S) \cong \prod_{i=1}^{\ell} \mathrm{GL}(1, F_S).$$

Let μ_n^M be the finite subgroup of $A_M(F_S)$ corresponding to the subgroup generated by the diagonal embedding of $\prod_{i=1}^{\ell} \mu_n$ into $\prod_{i=1}^{\ell} \mathrm{GL}(1, F_S)$. Suppose $\gamma \in M(F_S)$, $\eta \in \mu_n^M$ and that $(\eta\gamma)'$ is defined. Then by [KP86, Theorem 4.1 (iii)] we have

$$(\eta\gamma)' = \mathbf{s}\left((\eta\gamma)^n\right)\kappa_v\left((\eta\gamma)^n\right)^{-1} = \gamma'. \tag{19}$$

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This equation suggests that we ought to group our distributions into μ_n^M-invariant sums. The most obvious method is to take the sum,

$$\sum_{\eta\in\mu_n^M} I_M(\eta\gamma, f), \quad f \in \mathcal{H}\big(M(F_S)\big).$$

This grouping has the shortcoming that if $M = G$ and $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big)$ then

$$\sum_{\eta\in\mu_n^G} \hat{I}_G(\eta\gamma, \tilde{f}') = n\Lambda^G(\gamma)I_{\tilde{G}}(\gamma', \tilde{f}) \neq \Lambda^G(\gamma)I_{\tilde{G}}(\gamma', \tilde{f}), \quad \gamma \in \cap G_{\mathrm{oreg}}(F_S).$$

It would be convenient to correct this shortcoming by replacing the sum over μ_n^M with the sum over μ_n^M/μ_n^G. The following two lemmas show that this is indeed possible. We assume the reader is familiar with the spaces $\mathcal{H}_{\mathrm{ac}}\big(M(F_S)\big)$ and $\mathcal{I}_{\mathrm{ac}}\big(M(F_S)\big)$ of [Art89, Section 11] and their respective subspaces of moderate functions.

Lemma 6.1 *Suppose $L \in \mathcal{L}(M)$ and ϕ is a moderate function in $\mathcal{I}_{\mathrm{ac}}\big(M(F_S)\big)$ such that $\phi(\pi)$ vanishes for all $\pi \in \Pi_{\mathrm{temp}}\big(M(F_S)\big)$ whose central character is not trivial on μ_n^L. Then there exists a moderate function $f \in \mathcal{H}_{\mathrm{ac}}\big(M(F_S)\big)$ such that $f_M = \phi$ and f is invariant under translation by μ_n^L.*

Proof By [Art88a, Lemma 6.1] there exists a moderate function $h \in \mathcal{H}_{\mathrm{ac}}\big(M(F_S)\big)$ such that $h_M = \phi$. Given $\eta \in \mu_n^L$, set

$$h^\eta(\gamma) = h(\eta\gamma), \quad \gamma \in M(F_S).$$

Since μ_n^L is contained in the center of $M(F_S)$ and the nonzero matrix entries of $\eta \in \mu_n^L$ have absolute value one with respect to any valuation in S, the K_S-finiteness, support and growth conditions of h^η are identical with those of h. Thus, the function given by

$$f(\gamma) = |\mu_n^L|^{-1} \sum_{\eta\in\mu_n^L} h^\eta(\gamma), \quad \gamma \in M(F_S),$$

is a moderate function in $\mathcal{H}_{\mathrm{ac}}\big(M(F_S)\big)$. It is simple to see that f satisfies the remaining properties asserted by the lemma. ■

Lemma 6.2 *Suppose $L \in \mathcal{L}(M)$, $\eta \in \mu_n^L$ and $\tilde{f} \in \mathcal{H}\big(\tilde{L}(F_S)\big)$. Then*

$$\hat{I}_M^L(\eta\gamma, \tilde{f}') = \hat{I}_M^L(\gamma, \tilde{f}'), \quad \gamma \in M(F_S).$$

Proof Suppose $\gamma \in M(F_S)$ such that $M_\gamma(F_S) = G_\gamma(F_S)$. By Lemma 6.1 we may choose $f \in \mathcal{H}\big(L(F_S)\big)$ such that $f_L = \tilde{f}'$ and f is invariant under μ_n^L. If $L = M$ then the lemma follows immediately from the μ_n^L-invariance of f and $\mu_n^L \subset A_L(F_S)$.

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Assume inductively that the lemma holds if L is replaced by $L_1 \in \mathcal{L}^L(M)$ such that $L_1 \subsetneq L$. By definition

$$\hat{I}^L_M(\eta\gamma, \tilde{f}') = I^L_M(\eta\gamma, f)$$
$$= J^L_M(\eta\gamma, f) - \sum_{\{L_1 \in \mathcal{L}^L(M): L_1 \subsetneq L\}} \hat{I}^{L_1}_M\big(\eta\gamma, \phi^L_{L_1}(f)\big).$$

The first term on the right-hand side satisfies

$$J^L_M(\eta\gamma, f) = |D^L(\eta\gamma)|_S^{1/2} \int_{L_{\eta\gamma}(F_S)\backslash L(F_S)} f(x^{-1}\eta\gamma x) v^L_M(x)\, dx = J^L_M(\gamma, f),$$

as f is μ^L_n-invariant and $\mu^L_n \subset A_L(F_S)$. As for the remaining terms on the right-hand side, we know by [Art89, Theorem 12.1] that $\phi^L_{L_1}(f)$ is a moderate function in $\mathcal{I}_{\mathrm{ac}}\big(L_1(F_S)\big)$. Furthermore $\phi^L_{L_1}(f)$ is defined by

$$\phi^L_{L_1}(f)(\pi) = \mathrm{tr}\big(\mathcal{R}^L_{L_1}(\pi)\pi^L(f)\big), \quad \pi \in \Pi_{\mathrm{temp}}\big(L_1(F_S)\big),$$

for a certain linear operator $\mathcal{R}^L_{L_1}(\pi)$. Obviously,

$$\phi^L_{L_1}(f)(\pi) = \mathrm{tr}\left(\mathcal{R}^L_{L_1}(\pi) \int_{\mu^L_n\backslash L(F_S)} f(x) \sum_{\eta \in \mu^L_n} \pi^L(\eta x)\, dx\right), \quad \pi \in \Pi_{\mathrm{temp}}\big(L_1(F_S)\big).$$

It follows that $\phi^L_{L_1}(f)$ satisfies the hypotheses of Lemma 6.1 and so there exists a moderate function $h \in \mathcal{H}_{\mathrm{ac}}\big(L_1(F_S)\big)$ such that $h_{L_1} = \phi^L_{L_1}(f)$ and h is invariant under μ^L_n. Apparently, $\mu^L_n \subset \mu^{L_1}_n$, so by induction

$$\hat{I}^{L_1}_M\big(\eta\gamma, \phi^L_{L_1}(f)\big) = I^{L_1}_M(\eta\gamma, h) = I^{L_1}_M(\gamma, h) = \hat{I}^{L_1}_M\big(\gamma, \phi^L_{L_1}(f)\big).$$

In consequence, the lemma holds if $M_\gamma(F_S) = G_\gamma(F_S)$. For arbitrary $\gamma \in M(F_S)$ we have [Art88a, (2.2)]

$$I^L_M(\eta\gamma, f) = \lim_{\{a\to 1: a \in A_{M,\mathrm{reg}}(F_S)\}} \sum_{L_1 \in \mathcal{L}^L(M)} r^{L_1}_M(\eta\gamma, a) I^L_{L_1}(a\eta\gamma, f)$$
$$= \lim_{\{a\to 1: a \in A_{M,\mathrm{reg}}(F_S)\}} \sum_{L_1 \in \mathcal{L}^L(M)} r^{L_1}_M(\eta\gamma, a) I^L_{L_1}(a\gamma, f),$$

where $A_{M,\mathrm{reg}}(F_S)$ is the subset of elements in $A_M(F_S)$ whose centralizer in G is contained in M. The function $r^L_M(\gamma, a)$, defined in [Art88c, Section 5], is easily seen to be invariant under translation by $A_L(F_S)$ in the first variable. In particular we have $r^L_M(\eta\gamma, a) = r^L_M(\gamma, a)$. The lemma now follows. ■

Suppose $\gamma \in M(F_S)$. By Lemma 6.2, it makes sense to define the distribution $I^\Sigma_M(\gamma)$ on $\tilde{G}(F_S)$ by

$$I^\Sigma_M(\gamma, \tilde{f}) = \sum_{\eta \in \mu^M_n/\mu^G_n} \hat{I}_M(\eta\gamma, \tilde{f}'), \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big).$$

More generally, if $L \in \mathcal{L}(M)$ then

$$I_M^{L,\Sigma}(\gamma, \tilde{f}) = \sum_{\eta \in \mu_n^M/\mu_n^L} \hat{I}_M^L(\eta\gamma, \tilde{f}'), \quad \tilde{f} \in \mathcal{H}\big(\tilde{L}(F_S)\big).$$

Two important properties that $I_M(\gamma)$ satisfies are descent [Art88a, Section 8] and splitting [Art88a, Section 9]. We show that $I_M^\Sigma(\gamma)$ satisfies these properties as well.

Lemma 6.3 *Suppose M and M_1 belong to $\mathcal{L}$ and $M_1 \subset M$. Suppose further that $L \in \mathcal{L}(M_1)$ such that $d_{M_1}^G(M, L) \neq 0$. Then the map,*

$$\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G \to \mu_n^{M_1}/\mu_n^G,$$

given by $(\eta_1\mu_n^G, \eta_2\mu_n^G) \mapsto \eta_1\eta_2\mu_n^G$, is an isomorphism.

Proof If $d_{M_1}^G(M, L) \neq 0$ as above, then by definition, $\mathfrak{a}_{M_1}^M \oplus \mathfrak{a}_{M_1}^L \cong \mathfrak{a}_{M_1}^G$ [Art88a, Section 7]. The vector spaces $\mathfrak{a}_M^G$ and $\mathfrak{a}_L^G$ may be regarded as the respective orthogonal complements of $\mathfrak{a}_{M_1}^M$ and $\mathfrak{a}_{M_1}^L$ in $\mathfrak{a}_{M_1}^G$. As a consequence we also have $\mathfrak{a}_M^G \oplus \mathfrak{a}_L^G \cong \mathfrak{a}_{M_1}^G$. Consider the homomorphism

$$H_{M_1} : M_1(F_S) \to \mathfrak{a}_{M_1}.$$

It is readily verified that it passes to a homomorphism

$$H_{A_{M_1}} : A_{M_1}(F_S)/A_G(F_S) \to \mathfrak{a}_{M_1}^G$$

such that $H_{A_{M_1}}\big(A_M(F_S)/A_G(F_S)\big) \subset \mathfrak{a}_M^G$ and $H_{A_{M_1}}\big(A_L(F_S)/A_G(F_S)\big) \subset \mathfrak{a}_L^G$. Accordingly

$$H_{A_{M_1}}\Big(\big(A_M(F_S) \cap A_L(F_S)\big)/A_G(F_S)\Big) \subset \mathfrak{a}_M^G \cap \mathfrak{a}_L^G = 0.$$

In other words, $|\xi(\gamma)| = 1$ for all γ belonging to the split torus $A_M(F_S) \cap A_L(F_S)$, and all characters $\xi \in X(M_1)$ which are trivial when restricted to G. This implies that $A_L(F_S) \cap A_M(F_S) \subset A_G(F_S)$. As a result, the multiplication map

$$A_M(F_S)/A_G(F_S) \times A_L(F_S)/A_G(F_S) \to A_{M_1}(F_S)/A_G(F_S)$$

is injective. It now follows from the commutative diagram,

$$\begin{array}{ccc} \mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G & \hookrightarrow & A_M(F_S)/A_G(F_S) \times A_L(F_S)/A_G(F_S) \\ \downarrow & & \downarrow \\ \mu_n^{M_1}/\mu_n^G & \hookrightarrow & A_{M_1}(F_S)/A_G(F_S) \end{array}$$

that the map of the lemma is injective. The surjectivity of the map can be seen from the following equalities.

$$|\mu_n^M/\mu_n^G \times \mu_n^L/\mu_n^G| = n^{\dim(\mathfrak{a}_M^G)} n^{\dim(\mathfrak{a}_L^G)} = n^{\dim(\mathfrak{a}_{M_1}^G)} = |\mu_n^{M_1}/\mu_n^G|. \qquad \blacksquare$$

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Lemma 6.4 *Let M, M_1 and L be as in Lemma 6.3. The map*

$$\mu_n^M/\mu_n^G \to \mu_n^{M_1}/\mu_n^L,$$

given by $\eta\mu_n^G \mapsto \eta\mu_n^L$ for $\eta \in \mu_n^M$, is an isomorphism.

Proof The proof of this lemma follows from arguments similar to those of Lemma 6.3. ■

The following proposition proves a descent property for $I_M^\Sigma(\gamma)$. For this we need the notion of an induced space of orbits. Given $\gamma \in M(F_S)$ define the induced space γ^G to be the union of the $G(F_S)$-conjugacy classes which intersect $\gamma U_P(F_S)$ in an open set. Here $P \in \mathcal{P}(M)$ is arbitrary [Art88c, Section 6]. Since γ^G is contained in a single geometric conjugacy class and $\mathrm{GL}(r)$ is stable, the induced space γ^G is a single conjugacy class which we call the induced conjugacy class of γ.

Proposition 6.1 *Suppose M and M_1 belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$. Then*

$$I_M^\Sigma(\gamma^M, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}), \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big).$$

Proof Suppose $\eta \in \mu_n^M$. Since η lies in the center of $M(F_S)$,

$$\eta\gamma^M = \{\eta\gamma_1 : \gamma_1 \in \gamma^M\}$$

is a conjugacy class in $M(F_S)$. Suppose $P \in \mathcal{P}^M(M_1)$. Clearly, left multiplication by η is a homeomorphism between $\gamma U_P(F_S)$ and $\eta\gamma U_P(F_S)$. It follows that $\eta\gamma^M$ is a conjugacy class of $M(F_S)$ which intersects $\eta\gamma U_P(F_S)$ in an open set. In other words $\eta\gamma^M = (\eta\gamma)^M$. The descent property for $I_M(\gamma)$ [Art88a, Theorem 8.1] together with Lemma 6.4 imply that

$$\begin{aligned}
I_M^\Sigma(\gamma^M, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma^M, \tilde{f}') \\
&= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M\big((\eta\gamma)^M, \tilde{f}'\big) \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_{M_1}^L(\eta\gamma, \tilde{f}_L') \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \sum_{\eta \in \mu_n^{M_1}/\mu_n^L} \hat{I}_{M_1}^L(\eta\gamma, \tilde{f}_{\tilde{L}}') \\
&= \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L)\hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}).
\end{aligned}$$

■

Corollary 6.1 *Suppose M and M_1 belong to $\mathcal{L}$ with $M_1 \subset M$. Moreover suppose $\gamma \in M_1(F_S)$ and $M_{1,\gamma} = M_\gamma$. Then*

$$I_M^\Sigma(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \hat{I}_{M_1}^{L,\Sigma}(\gamma, \tilde{f}_{\tilde{L}}), \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big).$$

Proposition 6.2 *Suppose S is the disjoint union of nonempty sets S_1 and S_2, and that $\tilde{f} = \tilde{f}_1 \tilde{f}_2 \in \mathcal{H}\big(\tilde{G}(F_S)\big)$, $\gamma = \gamma_1\gamma_2 \in M(F_S)$ are corresponding decompositions. Then*

$$I_M^\Sigma(\gamma, \tilde{f}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \hat{I}_M^{L_1,\Sigma}(\gamma_1, \tilde{f}_{1,\tilde{L}_1}) \hat{I}_M^{L_2,\Sigma}(\gamma_2, \tilde{f}_{2,\tilde{L}_2}).$$

Proof We begin by applying the splitting property to the summands of $I_M^\Sigma(\gamma)$.

$$\begin{aligned} I_M^\Sigma(\gamma, \tilde{f}) &= \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M(\eta\gamma, \tilde{f}') \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta \in \mu_n^M/\mu_n^G} \hat{I}_M^{L_1}(\eta\gamma_1, \tilde{f}'_{1,L_1}) \hat{I}_M^{L_2}(\eta\gamma_2, \tilde{f}'_{2,L_2}). \end{aligned}$$

Suppose $d_M^G(L_1, L_2)$ is not zero. By using arguments similar to those of the proof of Lemma 6.3, it may be established that

$$\mu_n^M/\mu_n^G \cong \mu_n^M/\mu_n^{L_1} \times \mu_n^M/\mu_n^{L_2},$$
$$\mu_n^M/\mu_n^G \cong \mu_n^{L_1}/\mu_n^G \times \mu_n^{L_2}/\mu_n^G,$$

and $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$. From $\mu_n^{L_1} \cap \mu_n^{L_2} = \mu_n^G$ it follows that the homomorphism

$$\mu_n^{L_1}/\mu_n^G \to \mu_n^M/\mu_n^{L_2},$$

given by $\eta\mu_n^G \mapsto \eta\mu_n^{L_2}$, is injective. This homomorphism is also surjective as

$$|\mu_n^{L_1}/\mu_n^G| = n^{\dim(\mathfrak{a}_M^{L_1}) - \dim(\mathfrak{a}_M^G)} = n^{\dim(\mathfrak{a}_M^{L_2})} = |\mu_n^M/\mu_n^{L_2}|.$$

It may be deduced in the same manner that $\mu_n^{L_2}/\mu_n^G \cong \mu_n^M/\mu_n^{L_1}$. Thus the previous sum is equal to

$$\begin{aligned} &\sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^M/\mu_n^{L_1}} \sum_{\eta_2 \in \mu_n^M/\mu_n^{L_2}} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, \tilde{f}'_{1,L_1}) \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, \tilde{f}'_{2,L_2}) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^{L_2}/\mu_n^G} \sum_{\eta_2 \in \mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, \tilde{f}'_{1,L_1}) \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, \tilde{f}'_{2,L_2}) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^{L_2}/\mu_n^G} \hat{I}_M^{L_1}(\eta_1\eta_2\gamma_1, \tilde{f}'_{1,L_1}) \sum_{\eta_2 \in \mu_n^{L_1}/\mu_n^G} \hat{I}_M^{L_2}(\eta_1\eta_2\gamma_2, \tilde{f}'_{2,L_2}) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \sum_{\eta_1 \in \mu_n^M/\mu_n^{L_1}} \hat{I}_M^{L_1}(\eta_1\gamma_1, \tilde{f}'_{1,L_1}) \sum_{\eta_2 \in \mu_n^M/\mu_n^{L_2}} \hat{I}_M^{L_2}(\eta_2\gamma_2, \tilde{f}'_{2,L_2}) \\ &= \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \hat{I}_M^{L_1,\Sigma}(\gamma_1, \tilde{f}_{1,\tilde{L}_1}) \hat{I}_{M,\Sigma}^{L_2}(\gamma_2, \tilde{f}_{2,\tilde{L}_2}). \end{aligned}$$

■

7 The Distribution $I_M^{\mathcal{M}}(\gamma)$

From here on we assume that the invariant trace formula of [Art88b] is valid for metaplectic coverings of the general linear group. In particular, we are assuming that there exist invariant distributions $I_{\tilde{M}}(\tilde{\gamma})$ parameterized by $\tilde{\gamma} \in \tilde{M}(F_S)$, which are defined in the same fashion as $I_M(\gamma)$ of the previous section, and also satisfy splitting and descent properties. We would like to compare these distributions to $I_M^{\Sigma}(\gamma)$. To this end, we define

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \Lambda^M(\gamma) I_{\tilde{M}}(\gamma', \tilde{f}), \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big),$$

for all $\gamma \in M(F_S)$ in the domain of the orbit map such that $M_\gamma(F_S) = G_\gamma(F_S)$. This definition is very similar to the definition of $I_M^{\mathcal{E}}(\gamma)$ of (3.8) II [AC89]. Indeed $I_M^{\mathcal{M}}(\gamma)$ shares most of the properties of $I_M^{\mathcal{E}}(\gamma)$ listed in [AC89, Section 3 II]. We list them below and invite the reader to verify that the proofs of that section apply to $I_M^{\mathcal{M}}(\gamma)$ also. From now on we shall tacitly assume that γ belongs to the domain of the orbit map whenever γ' appears.

A mild paraphrase of Lemma 3.1 and Corollary 3.2 II [AC89] implies that

$$\lim_{\{a \to 1 : a \in A_{M,\mathrm{reg}}(F_S)\}} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma^n, a) I_L^{\mathcal{M}}(a\gamma, \tilde{f}), \quad \gamma \in M(F_S) \tag{20}$$

exists and is equal to

$$\Lambda^M(\gamma) \sum_{L \in \mathcal{L}(M)} c_M^L(\gamma^n, n) I_{\tilde{L}}\big((\gamma^L)', \tilde{f}\big). \tag{21}$$

The functions $c_M^L(\gamma, n)$ are derived from the (G, M) family [Art81, Section 6],

$$c_P(\nu, \gamma, n) = \prod_{v \in S} \prod_{\beta} |n|_v^{-\rho(\beta, u_v)\nu(\beta^\vee)/2}, \quad P \in \mathcal{P}(M),$$

where $\gamma = \prod_{v \in S} \gamma_v$ has Jordan decomposition $\prod_{v \in S} \sigma_v u_v$, and the remaining terms stem from the (G, M) family $r_P(\nu, \gamma, a)$ defined in [Art88c, Section 5]. We define $I_M^{\mathcal{M}}(\gamma, \tilde{f})$ to be equal to (20) for general $\gamma \in M(F_S)$. It is clear from the proof of Corollary 3.4 II [AC89] that if $\gamma \in M(F)$ is embedded diagonally into $M(F_S)$ and S contains $\{v : |n|_v \neq 1\}$ then

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = I_{\tilde{M}}(\gamma', \tilde{f}), \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big). \tag{22}$$

The descent property,

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L \in \mathcal{L}(M_1)} d_{M_1}^G(M, L) \hat{I}_{M_1}^{L, \mathcal{M}}(\gamma, \tilde{f}_{\tilde{L}}), \tag{23}$$

for $\gamma \in M_1(F_S)$ with $M_1 \in \mathcal{L}^M$ and $M_{1,\gamma} = M_\gamma$; and the splitting property

$$I_M^{\mathcal{M}}(\gamma, \tilde{f}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \hat{I}_M^{L_1, \mathcal{M}}(\gamma_1, \tilde{f}_{1, \tilde{L}_1}) \hat{I}_M^{L_2, \mathcal{M}}(\gamma_2, \tilde{f}_{2, \tilde{L}_2}), \tag{24}$$

for $\gamma = \gamma_1 \gamma_2 \in M(F_S)$ and $\tilde{f} = \tilde{f}_1 \tilde{f}_2$ as in Proposition 6.2 also hold. This is immediate for $\gamma \in M(F_S) \cap G_{\mathrm{oreg}}(F_S)$ and can be established for general $\gamma \in M(F_S)$ by using [Art88a, Corollary 3.2] (see also the end of the proof of Theorem 8.1 [Art88a]).

8 Geometric Vanishing Properties

As alluded to earlier, we wish to compare $I_M^{\mathcal{M}}(\gamma)$ with $I_M^{\Sigma}(\gamma)$. At least in the case $M = G$, our wish is partially fulfilled as

$$I_G^{\Sigma}(\gamma, \tilde{f}) = \hat{I}_G(\gamma, \tilde{f}') = \Lambda^G(\gamma) I_{\tilde{G}}(\gamma', \tilde{f}) = I_G^{\mathcal{M}}(\gamma, \tilde{f}),$$

for $\gamma \in G_{\mathrm{oreg}}(F_S)$ and $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big)$. One might wonder whether, in our definition of $I_G^{\mathcal{M}}(\gamma)$, we are neglecting to take into account the distributions $I_{\tilde{G}}(\tilde{\gamma})$ for which $\tilde{\gamma} \in \tilde{G}_{\mathrm{reg}}(F_S)$ does not lie in the image of the orbit map. In this regard, Proposition 3 [FK86] tells us that if $\tilde{\gamma} \in \tilde{G}_{\mathrm{reg}}(F_S)$ then $I_{\tilde{G}}(\tilde{\gamma})$ is zero unless $\mathbf{p}(\tilde{\gamma}) = \gamma_0^{n/r}\gamma^n$ for some $\gamma_0 \in A_G(F_S)$ and $\gamma \in G_{\mathrm{oreg}}(F_S)$. This is a local vanishing property for $I_{\tilde{G}}(\tilde{\gamma})$. We shall extend this local vanishing property to the case $M \neq G$ and arbitrary $\tilde{\gamma} \in \tilde{M}(F_S)$ in Lemma 8.3, under an assumption on the order of the metaplectic covering. However, our real interest lies in proving the global vanishing property, Proposition 8.2, for which this local vanishing property is just a prelude.

In the context of cyclic base change for GL_r, Arthur has proven analogous vanishing properties [Art88b, Section 8]. The informed reader should be mindful that the norm map of orbits in base change maps from the restriction of GL_r over a cyclic extension to GL_r, whereas the orbit map used here maps from GL_r to one of its metaplectic coverings. Thus, in a loose sense, the two maps map in opposite directions.

Lemma 8.1 *Suppose v is nonarchimedean, $\tilde{\sigma}$ is an element of $\tilde{M}(F_v)$ such that $\sigma = \mathbf{p}(\tilde{\sigma})$ is semisimple, and $I_{\tilde{M}}(\tilde{\sigma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_v)\big)$. Then there exist elements $\gamma_0 \in A_G(F_v)$ and $\gamma \in M(F_v)$ such that $\sigma = \gamma_0^{n/d}\gamma^n$.*

Proof We will first prove the lemma for $M = G$. After inverting the germ expansion for orbital integrals at $\tilde{\sigma}$ [Vig81, Corollaire 2.4, Section 3.3], we find that $I_{\tilde{G}}(\tilde{\sigma}, \tilde{f})$ is a finite linear combination of nonzero orbital integrals of $\tilde{f}$ at elements in $\tilde{G}_{\tilde{\sigma}}(F_v) \cap \tilde{G}_{\mathrm{reg}}(F_v)$ close to $\tilde{\sigma}$. Let $\tilde{\delta}$ be one of these elements. By Proposition 3 [FK86], $\mathbf{p}(\tilde{\delta})$ is equal to $\gamma_0^{n/d}\delta^n$ for some $\gamma_0 \in A_G(F_v)$ and $\delta \in G_{\mathrm{oreg}}(F_v)$. By a standard argument [KP86, Section 1] there exist extensions $E_1, \dots, E_k$ of F_v such that $G_\delta(F_v)$ is isomorphic to $\prod_{i=1}^k E_i^\times$. Let σ_i and δ_i, $1 \leq i \leq k$, be the respective i-th components of σ and δ with respect to this isomorphism and regard γ_0 as an element of $F_v^\times \subset E_i^\times$. Applying Hensel's lemma to the polynomial $X^n - \gamma_0^{-n/d}\sigma_i \in E_i[X]$, we obtain an element $\gamma_i \in E_i$ such that $|\gamma_i - \delta_i|_{E_i} < 1$ and $\gamma_0^{n/d}\gamma_i^n = \sigma_i$, $1 \leq i \leq k$. It is now clear that the element $\gamma \in G_\delta(F_v)$ which maps to $(\gamma_1, \dots, \gamma_k)$ under the above isomorphism and γ_0 are the desired elements of the lemma in the case $M = G$. For the general case, note that we are assuming that the metaplectic version of (2.3) [Art88a] holds. Namely, if $\tilde{M}_{\tilde{\sigma}}(F_v) = \tilde{G}_{\tilde{\sigma}}(F_v)$ then there exists $\tilde{h} \in C_c^\infty\big(\tilde{M}(F_v)\big)$ such that for all $\tilde{\gamma} \in \tilde{M}_{\tilde{\sigma}}(F_v)$ close to $\tilde{\sigma}$

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = I_{\tilde{M}}^{\tilde{M}}(\tilde{\gamma}, \tilde{h}).$$

Since the K_v-finiteness of $\tilde{f}$ did not play role in the proof of the case $M = G$, that case together with the substitution $\tilde{\gamma} = \tilde{\sigma}$ in this equation imply the lemma whenever

$\tilde{M}_{\tilde{\sigma}}(F_v) = \tilde{G}_{\tilde{\sigma}}(F_v)$. The lemma now follows for general $\tilde{\sigma}$ from the definition [Art88a, (2.2*)],

$$I_{\tilde{M}}(\tilde{\sigma}, \tilde{f}) = \lim_{\{a \to 1 : a \in A_{M,\mathrm{reg}}(F_v)\}} \sum_{L \in \mathcal{L}(M)} r_M^L\big(\mathbf{p}(\tilde{\sigma}), a^n\big)\, I_{\tilde{L}}\big(\mathbf{s}(a^n)\tilde{\sigma}, \tilde{f}\big),$$

and the fact that

$$\tilde{L}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v) = \tilde{M}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v) = \tilde{G}_{\mathbf{s}(a^n)\tilde{\sigma}}(F_v)$$

for $a \in A_{M,\mathrm{reg}}(F_v)$ close to the identity. ■

It is unfortunate that Lemma 8.1 can not be extended to general elements in $\tilde{M}(F_v)$ without some additional hypotheses. To see this, suppose F_v is nonarchimedean, $n = r = 2$, $m = 0$, $x, y \in F_v^\times$, $x \notin F_v^{\times 2}$ and $\tilde{\gamma} = \mathbf{s}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. Using Proposition 0.1.5 [KP84] and equation (2), one can compute $\tilde{G}_{\tilde{\gamma}}(F_v)$ and show that it is equal to

$$\mathbf{p}^{-1}\big(G_{\mathbf{p}(\tilde{\gamma})}(F_v)\big) = \mathbf{p}^{-1}\left(\left\{\begin{pmatrix} z_1 & z_2 \\ 0 & z_1 \end{pmatrix} : z_1 \in F_v^\times, z_2 \in F_v\right\}\right).$$

It follows from [Vig81, Section 1.i] that $I_{\tilde{G}}(\tilde{\gamma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_v)\big)$. On the other hand, it is simple to show that if $\mathbf{p}(\tilde{\gamma}) = \gamma^2$ for some $\gamma \in G(F_v)$ then γ must lie in $P_0(F_v)$ and in turn x must lie in $F_v^{\times 2}$. Our assumption on x therefore excludes this possibility.

The pathology in this example stems from the fact that r divides n. In fact, one can construct similar examples for any n and r for which $\gcd(i, n) \neq 1$, for some $1 < i \leq r$. Thus, n must be relatively prime to the positive integers less than or equal to r if Lemma 8.1 is to be generalized. The following lemma converts this condition on n into properties which are compatible with the structure of $G(F_v)$.

Lemma 8.2 *Suppose that v is nonarchimedean and that E/F_v is a finite extension such that n is relatively prime to $[E : F_v]$. If $x \in E^\times$ such that $N_{E/F_v}(x) \in F_v^{\times n}$ then $x \in E^{\times n}$.*

Proof We first show that $E^\times/E^{\times n} \cong F_v^\times/F_v^{\times n}$ and that we may take coset representatives of $E^\times/E^{\times n}$ to lie in $F_v^\times$. The homomorphism, $F_v^\times/F_v^{\times n} \to E^\times/E^{\times n}$, given by

$$zF_v^{\times n} \mapsto zE^{\times n}, \quad z \in F_v^\times,$$

is injective. Indeed, suppose z does not belong to $F^{\times n}$, but does belong to $E^{\times n}$. By Theorem 10 (b), VIII, Section 6 [Lan84], $[F(z^{1/n}) : F_v]$ divides n. Moreover $z^{1/n} \in E$, so

$$[E : F_v] = [E : F_v(z^{1/n})][F_v(z^{1/n}) : F_v].$$

This contradicts $\gcd(n, [E : F_v]) = 1$. The surjectivity of this map follows at once from the fact that [KP84, Lemma 0.3.2]

$$|E^\times/E^{\times n}| = |F_v^\times/F_v^{\times n}| = n^2/|n|_v. \tag{25}$$

The enunciation of the lemma amounts to showing the injectivity of the homomorphism given by

$$xE^{\times n} \mapsto \big(N_{E/F_v}(x)\big)\, F_v^{\times n} = x^{[E:F_v]} F_v^{\times n}, \quad x \in F_v^{\times}.$$

If $x^{[E:F_v]} F_v^{\times n} = F_v^{\times n}$ and $x \notin F_v^{\times n}$, then $[E : F_v]$ must divide $n^2/|n|_v$ by (25), thereby contradicting $\gcd([E : F_v], n) = 1$. Thus this map is injective. ■

Lemma 8.3 *Suppose v is nonarchimedean and n is relatively prime to $r - 1 + 2rm$ and all positive integers less than or equal to r. Moreover suppose that $\tilde{\gamma} \in \tilde{M}(F_v)$ such that $I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \neq 0$ for some $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_v)\big)$. Then there exists $\gamma \in M(F_v)$ such that $\mathbf{p}(\tilde{\gamma}) = \gamma^n$.*

Proof Let σu be the Jordan decomposition of $\mathbf{p}(\tilde{\gamma})$. Suppose $M = G$. In this case, by the arguments of [KP86, Section 1], we may take σ to be a block diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_1 & & & & \\ & & & \ddots & & & \\ & & & & \sigma_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \sigma_k \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F_v)$, $1 \leq i \leq k$ are pairwise distinct, generate respective extensions F_i/F_v of degree m_i and each appear b_i times. We proceed by showing that σ_i is an n-th power in $\mathrm{GL}(m_i, F_v)$. Our hypothesis on $\tilde{\gamma}$ is equivalent to $\tilde{G}_{\tilde{\gamma}}(F_v) = \mathbf{p}^{-1}(G_{\mathbf{p}(\tilde{\gamma})})(F_v)$ (*cf.* [Vig81, 1.i]). The set $\mathbf{p}^{-1}(G_{\mathbf{p}(\tilde{\gamma})})(F_v)$ contains the diagonal matrices of the form

$$\tilde{\delta} = \mathbf{s} \begin{pmatrix} \delta_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \delta_1 & & & & \\ & & & \ddots & & & \\ & & & & \delta_k & & \\ & & & & & \ddots & \\ 0 & & & & & & \delta_k \end{pmatrix},$$

where $\delta_i \in F_v^{\times}$ appears $m_i b_i$ times, $1 \leq i \leq k$. In particular, $\tilde{\delta}$ commutes with $\mathbf{s}(\sigma)$. Let $(\cdot,\cdot)_{F_v}$ and $(\cdot,\cdot)_{F_i}$ be the n-th Hilbert symbol of F_v and F_i respectively, $1 \leq i \leq k$. Suppose that $\delta_1 = \cdots = \delta_k$. Then Proposition 0.1.5 [KP84], the fact that $\tilde{\delta}$ commutes

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with $\mathbf{s}(\sigma)$ and the properties of the Hilbert symbol imply that

$$\begin{aligned}1 &= \big(\det(\sigma), \delta_1^r\big)_{F_v}^{1+2m} \prod_{i=1}^{k} (\sigma_i, \delta_1^{b_i})_{F_i}^{-1} \\ &= \big(\det(\sigma)^{r(1+2m)}, \delta_1\big)_{F_v} \Big(\prod_{i=1}^{k} N_{F_i/F_v}(\sigma_i)^{b_i}, \delta_1\Big)_{F_v}^{-1} \\ &= \big(\det(\sigma)^{r-1+2rm}, \delta_1\big)_{F_v}.\end{aligned}$$

As $\delta_1 \in F_v^\times$ is arbitrary, $\det(\sigma)^{r-1+2rm}$ must lie in $F_v^{\times n}$. It follows from $\gcd(n, r-1+2rm) = 1$ and (25) that $\det(\sigma) \in F_v^{\times n}$. Now fix $1 \le j \le k$ and let $\tilde{\delta}$ be such that $\delta_i = 1$ for $1 \le i \le k$ and $i \ne j$. Then, repeating the above procedure and noting that n is relatively prime to b_j, we find that

$$\begin{aligned}1 &= \Big(\det(\sigma), \det\big(\mathbf{p}(\tilde{\delta})\big)\Big)_{F_v}^{1+2m} (\sigma_j, \delta_j^{b_j})_{F_j}^{-1} \\ &= \big(N_{F_j/F_v}(\sigma_j), \delta_j\big)_{F_v}^{-b_j} \\ &= \big(N_{F_j/F_v}(\sigma_j), \delta_j\big)_{F_v}.\end{aligned}$$

Consequently $N_{F_j/F_v}(\sigma_j) \in F_v^{\times n}$. By Lemma 8.2 and our hypotheses on n we have $\sigma_j \in F_j^{\times n}$. Therefore there exists $\sigma_{0j} \in \mathrm{GL}(m_j, F_v)$ such that $\sigma_{0j}^n = \sigma_j$, $1 \le j \le k$. Set σ_0 to be the obvious block diagonal matrix composed of the blocks $\sigma_{01}, \dots, \sigma_{0k}$ such that $\sigma_0^n = \sigma$. It is evident from the inclusion $G_{\sigma_0} \subset G_\sigma$ that we have

$$G_{\sigma_0}(F_v) = G_\sigma(F_v) \cong \prod_{i=1}^{k} \mathrm{GL}(b_i, F_i).$$

Now set $u_0 = \exp\big(\frac{1}{n}\log(u)\big)$. By construction, $u_0^n = u$ and $u_0 \in G_\sigma(F_v) = G_{\sigma_0}(F_v)$. The lemma now follows in the case $M = G$ by setting $\gamma = \sigma_0 u_0$. The lemma follows in the general case by using the arguments near the end of the proof of Lemma 8.1. ■

The assumption placed on n in Lemma 8.3 will remain to be a restriction in the comparison of the trace formulas later on.

We can use the local vanishing properties, namely Lemmas 8.1 and 8.3, to prove global vanishing properties by way of some local-global results on n-th roots in F and $G(F)$.

Lemma 8.4 *Suppose $x \in F^\times$ such that $x \in F_v^{\times n}$ for almost all valuations v. Then $x \in F^{\times n}$.*

Proof We will prove this lemma by contradiction. To this end, suppose that σ is a non-trivial element in the Galois group of the abelian extension $F(x^{1/n})$. (Observe that $F(x^{1/n})$ is well-defined since $\mu_n \subset F$.) It is immediate from the hypothesis of the lemma that the density of the valuations for which σ is the Frobenius automorphism is zero. However, the Tchebotarev density theorem [Lan70, VIII, Section 4 Theorem 10] tells us that this density is $1/[F(x^{1/n}) : F]$. ■

Lemma 8.5 *Suppose that $\gamma \in M(F)$ and that for almost every valuation v of F, there exists $\gamma_v \in M(F_v)$ such that $\gamma_v^n = \gamma$. Then there exists $\gamma_0 \in M(F)$ such that $\gamma_0^n = \gamma$.*

Proof By decomposition (5) of M it suffices to prove the case $M = G$. Let $\gamma = \sigma u$ be the Jordan decomposition of γ. By[KP86, Section 1], we may take the semisimple element σ to be a block diagonal matrix of the form

$$\begin{pmatrix} \sigma_1 & & & & & 0 \\ & \ddots & & & & \\ & & \sigma_1 & & & \\ & & & \ddots & & \\ & & & & \sigma_k & \\ & & & & & \ddots \\ 0 & & & & & \sigma_k \end{pmatrix},$$

where $\sigma_i \in \mathrm{GL}(m_i, F)$, $1 \le i \le k$, are pairwise distinct, generate respective extensions F_i/F of degree m_i and each appear b_i times. Clearly, $G_\sigma(F_v)$ may be identified with a subgroup of $\prod_{i=1}^k \mathrm{GL}(m_i b_i, F_v)$. Since $\gamma_v \in G_\sigma(F_v)$, we may assume that $k = 1$ without any loss of generality. Let $\sigma_v u_v$ be the Jordan decomposition of γ_v. We obviously have $\sigma_v^n = \sigma$ and $u_v^n = u$. The latter equation implies that $u_v = u_0 = \exp\bigl(\frac{1}{n}\log(u)\bigr) \in G(F)$ so we need only produce an appropriate semisimple factor. We first consider the case $u \ne 1$. Using Jordan canonical form in $G_\sigma(F) \cong \mathrm{GL}(b_1, F_1)$ we may assume that γ is of the form

$$\begin{pmatrix} \sigma_1 & 1 & 0 & 0 \\ 0 & \sigma_1 & * & 0 \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \sigma_1 \end{pmatrix} \in G(F),$$

where the 1 in the $(1, 2)$ entry represents the $m_1 \times m_1$ identity matrix. Since σ_v commutes with $u_v = u_0$, it also commutes with u. This implies that σ_v is of the form

$$\begin{pmatrix} \sigma_{v0} & * & * \\ 0 & \sigma_{v0} & * \\ 0 & * & * \end{pmatrix} \in G(F_v),$$

for some $\sigma_{v0} \in \mathrm{GL}(m_1, F_v)$. Consequently, $\sigma_{v0}^n = \sigma_1$ for almost every v. If $F_1 = F$ then $m_1 = 1$ and we may apply Lemma 8.4 to σ_1 to conclude that there exists $\sigma_0 \in F^\times$

such that the n-th power of $\gamma_0 = \sigma_0 u_0$ is equal to γ. Otherwise, suppose $w_1, \dots, w_t$ are the valuations of F_1 which divide v. Then σ_1 is $\mathrm{GL}(m_1, F_v)$-conjugate to

$$\begin{pmatrix} \sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \sigma_{1t} \end{pmatrix},$$

where σ_{1i} generates the completion F_{1,w_i} over F_v, $1 \le i \le t$. This implies that the centralizer of σ_1 in $\mathrm{GL}(m_1, F_v)$ is isomorphic to $\prod_{i=1}^t F_{1,w_i}^\times$. In accordance with this last isomorphism, we decompose σ_{v0} as $(\sigma_{v1}, \dots, \sigma_{vt})$, where $\sigma_{vi} \in F_{1,w_i}^\times$, $1 \le i \le t$. Observe that the map $\sigma_1 \mapsto \sigma_{1i}$ corresponds to the embedding $F_1 \hookrightarrow F_{1,w_i}$, $1 \le i \le t$. Therefore σ_1, regarded as an element of F_1, has an n-th root in almost every completion of F_1. Applying Lemma 8.4 (with F replaced by F_1) to σ_1 proves the case $u \neq 1$. Now suppose $u = 1$ and $F_1 = F$. Then

$$\sigma_1^r = \det(\sigma) = \det(\sigma_v^n) \in F_v^{\times n},$$

for almost every v. Set $c = n/\gcd(n, r)$. Since F_v contains the $\gcd(n, r)$-th roots of unity, we have $\sigma_1^{r/\gcd(n,r)} \in F_v^{\times c}$. Furthermore, since $r/\gcd(n, r)$ is relatively prime to $c^2/|c|_v$, the order of $F_v^\times/F_v^{\times c}$ (*cf.* equation (25)), we have $\sigma_1 \in F_v^{\times c}$. Lemma 8.4 (with n replaced by c) implies that $\sigma_1 \in F^{\times c}$. The lemma now follows in this case by setting γ_0 to be the block diagonal matrix in $G(F)$ with blocks of the form

$$\begin{pmatrix} 0 & 0 & 0 & \sigma_1^{1/c} \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}\big(\gcd(n, r), F\big).$$

The remaining case that $u = 1$ and $F_1 \neq F$, is proven similarly, but with the added complication of using the embeddings $F_1 \hookrightarrow F_{1,w_i}$, $1 \le i \le t$, mentioned earlier. We leave the details to the interested reader. ■

***Corollary 8.1* (Kazhdan-Patterson)** *Suppose $\sigma \in G(F)$ is semisimple and for almost every valuation v there exist elements $\delta_v \in A_G(F_v)$ and $\sigma_v \in G(F_v)$ such that $\sigma = \delta_v^{n/d}\sigma_v^n$. Then there exist $\delta_0 \in A_G(F)$ and $\sigma_0 \in G(F)$ such that $\sigma = \delta_0^{n/d}\sigma_0^n$.*

Proof Our proof follows the sketch on [KP86, p. 226]. Let $x = \det(\sigma)$, $y_v = \det(\sigma_v)$ and $z_v \in F_v^\times$ be the scalar corresponding to δ_v for almost every v. Clearly $x = z_v^{rn/d} y_v^n \in F_v^{n/d}$. Applying Lemma 8.4 (with n/d in place of n) and recalling that $F_v \supset \mu_n \supset \mu_{n/d}$, we find that there exists $x_0 \in F^\times$ such that $x_0 = z_v^r y_v^d$ for almost every v. It is simple to verify that r and d are relatively prime. In turn, r and d^2, the order of $F_v^\times/F_v^{\times d}$ (*cf.* equation (25)) whenever $|d|_v = 1$, are relatively prime. Combining this fact with the equality $x_0 F_v^{\times d} = z_v^r F_v^{\times d}$, we conclude that $z_v = x_0^t w_v^d$

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for some $t \in \mathbf{Z}$ and $w_v \in F_v^\times$ for almost every v. We may therefore assume that $\delta_v = \delta_0 \in A_G(F)$. The application of Lemma 8.5 to $\delta_0^{-1}\sigma$ completes the proof. ■

The following two lemmas are global vanishing properties (*cf.* [Art88b, Proposition 8.1]) which follow from the local vanishing properties, Lemma 8.1 and Lemma 8.3.

Proposition 8.1 *Suppose $\tilde{\sigma} \in \tilde{G}(F)$ is embedded diagonally into $\tilde{G}(F_S)$, for some large finite set of valuations S. Suppose further that $\mathbf{p}(\tilde{\sigma})$ is semisimple. Then*

$$I_{\tilde{G}}(\tilde{\sigma}, \tilde{f}) = 0, \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big),$$

unless $\mathbf{p}(\tilde{\sigma}) = \gamma_0^{n/d}\gamma^n$ for some $\gamma_0 \in A_G(F)$ and $\gamma \in G(F)$.

Proof Suppose $\mathbf{p}(\tilde{\sigma})$ is not of the form $\gamma_0^{n/d}\gamma^n$ for any $\gamma_0 \in A_G(F)$ and $\gamma \in G(F)$. Then by Corollary 8.1 we may assume that there exists a nonarchimedean valuation $v_1 \in S$ such that $\mathbf{p}(\tilde{\gamma})$ is not of the form $\delta_{v_1}^{n/d}\sigma_{v_1}^n$ for any $\delta_{v_1} \in A_G(F_{v_1})$ and $\sigma_{v_1} \in G(F_{v_1})$. The proposition now follows from Lemma 8.1. ■

Proposition 8.2 *Suppose $\tilde{\gamma} \in \tilde{M}(F)$ is embedded diagonally into $\tilde{M}(F_S)$, for some large finite set of valuations S. Suppose further that the assumption of Lemma 8.3 on n holds. Then*

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = 0, \quad \tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big),$$

unless $\mathbf{p}(\tilde{\gamma})$ is an n-th power in $M(F)$.

Proof Suppose $\mathbf{p}(\tilde{\gamma})$ is not an n-th power in $M(F)$. Then by Lemma 8.5 we may assume that there exists a nonarchimedean valuation $v_1 \in S$ such that $\mathbf{p}(\tilde{\gamma})$ is not an n-th power in $M(F_{v_1})$. Set $S_1 = \{v_1\}$, $S_2 = S - \{v_1\}$ and let $\tilde{f} = \tilde{f}_1\tilde{f}_2$ be the corresponding decomposition of $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big)$. It follows from Lemma 8.1 that

$$\hat{I}_{\tilde{M}}^{\tilde{L}}(\tilde{\gamma}, \tilde{f}_{1,\tilde{L}}) = 0, \quad L \in \mathcal{L}(M).$$

The proposition follows by combining this equation with the splitting property [Art88a, Proposition 9.1],

$$I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2)\hat{I}_{\tilde{M}}^{\tilde{L}_1}(\tilde{\gamma}, \tilde{f}_{1,\tilde{L}_1})\hat{I}_{\tilde{M}}^{\tilde{L}_2}\Big(\mathbf{s}\big(\mathbf{p}(\tilde{\gamma})\big), \tilde{f}_{2,\tilde{L}_2}\Big).$$ ■

9 The Invariant Trace Formula

We set forth the invariant trace formula of a function $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{A})\big)$ as the equality of

$$\sum_{M \in \mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\tilde{\gamma} \in (\mathbf{s_0}(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S, \tilde{\gamma}) I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}) \tag{26}$$

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with

$$\sum_{t}\sum_{M\in\mathcal{L}}|W_0^M|\,|W_0^G|^{-1}\int_{\Pi(\tilde{M},t)} a^{\tilde{M}}(\tilde{\pi})I_{\tilde{M}}(\tilde{\pi},\tilde{f})\,d\tilde{\pi}. \tag{27}$$

It will be convenient to denote (26) as $I(\tilde{f})$. This trace formula is extrapolated from the trace formula given in [Art88b] for reductive algebraic groups. Since non-trivial metaplectic coverings of algebraic groups are not algebraic, one ought to verify the results of Arthur for metaplectic groups in order to rigorously assert the existence of a trace formula as we have done above. There are, lamentably, too many results to be checked to be included here. Those results which have been checked (Section 5 for example) follow in a straightforward manner. There is no reason to doubt that the other results do not follow in the same way. We therefore assume that the invariant trace formula is correct as stated.

Expansion (27) is known as the spectral side of the trace formula, as its terms depend on representations of $\tilde{M}(\mathbf{A})$. It will be further elaborated upon in [Mez00]. Expansion (26) is known as the geometric side of the trace formula, as its terms depend (in our case) on conjugacy classes in $\tilde{M}(F_S)$. Both sides of the trace formula contain terms which are local, *i.e.*, determined by $\tilde{M}(F_S)$, and global, *i.e.*, determined by the subgroup $\mathbf{s_0}\big(M(F)\big)$ of $\tilde{M}(\mathbf{A})$.

The local terms of the geometric side have already been introduced in Sections 6–7. In order to introduce some of the global terms of the geometric side we perform a familiar calculation. Recall that the map $\mathbf{s_0}$ given by (3) is a splitting homomorphism for $\tilde{G}(\mathbf{A})$ over $G(F)$. Let $L^2\Big(\mathbf{s_0}\big(G(F)\big)\setminus\tilde{G}(\mathbf{A})\Big)$ be the space of genuine square-integrable functions on $\tilde{G}(\mathbf{A})$, which are left-invariant under $\mathbf{s_0}\big(G(F)\big)$. This space admits a theory of automorphic representations through the decomposition of the (right-)regular representation R of $\tilde{G}(\mathbf{A})$. The metaplectic version of the geometric side of the trace formula originates from the following calculation. Let $\varphi\in L^2\Big(\mathbf{s_0}\big(G(F)\big)\setminus\tilde{G}(\mathbf{A})\Big)$. We then have

$$\begin{aligned}\big(R(\tilde{f})\varphi\big)(y) &= \int_{\tilde{G}(\mathbf{A})}\tilde{f}(x)\varphi(yx)\,dx\\ &= \int_{\mathbf{s_0}(G(F))\setminus\tilde{G}(\mathbf{A})}\sum_{\gamma\in\mathbf{s_0}(G(F))}\tilde{f}(y^{-1}\gamma x)\varphi(\gamma x)\,dx\\ &= \int_{\mathbf{s_0}(G(F))\setminus\tilde{G}(\mathbf{A})}\Big(\sum_{\gamma\in\mathbf{s_0}(G(F))}\tilde{f}(y^{-1}\gamma x)\Big)\varphi(x)\,dx.\end{aligned}$$

Roughly speaking, the trace of the operator $R(\tilde{f})$ is obtained by integrating the integral kernel,

$$(x,y)\mapsto\sum_{\gamma\in\mathbf{s_0}(G(F))}\tilde{f}(y^{-1}\gamma x),\quad x,y\in\mathbf{s_0}\big(G(F)\big)\setminus\tilde{G}(\mathbf{A}),$$

over the diagonal.

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Suppose S contains $\{v : |n|_v \neq 1\}$ and $\tilde{f} \in \mathcal{H}\big(\tilde{G}(F_S)\big)$. For nonarchimedean valuations v define $\tilde{f}_v^0$ by

$$\tilde{f}_v^0(\gamma, \zeta) = \begin{Bmatrix} \zeta^{-1}, & \text{if } \gamma \in K_v \\ 0, & \text{otherwise} \end{Bmatrix}, \quad \gamma \in G(F_v), \quad \zeta \in \mu_n. \tag{28}$$

We embed $\tilde{f}$ into $\mathcal{H}\big(\tilde{G}(\mathbf{A})\big)$ by taking its product with $\prod_{v \notin S} \tilde{f}_v^0$. If S satisfies some additional properties, which are given in [Art88b, Section 3], then $I(\tilde{f})$ equals

$$\sum_{M \in \mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\tilde{\gamma} \in (\mathbf{s_0}(M(F)))_{\tilde{M},S}} a^{\tilde{M}}(S, \tilde{\gamma}) I_{\tilde{M}}(\tilde{\gamma}, \tilde{f}).$$

The set $\Big(\mathbf{s_0}\big(M(F)\big)\Big)_{\tilde{M},S}$ denotes the set of $(\tilde{M}, S)$-equivalence classes of elements in $\mathbf{s_0}\big(M(F)\big)$ [Art86, Section 8]. In the present case $\Big(\mathbf{s_0}\big(M(F)\big)\Big)_{\tilde{M},S}$ is simply the set of $\tilde{M}(F_S)$-conjugacy classes of elements in $\mathbf{s_0}\big(M(F)\big)$. Since conjugacy classes play such an prominent role we shall, for the sake of convenience, often abuse notation by identifying an element $\gamma \in M(F_S)$ with its conjugacy class.

The coefficient $a^{\tilde{M}}(S, \tilde{\gamma})$ requires more explanation. Let σu be the Jordan decomposition of $\mathbf{p}(\tilde{\gamma}) \in M(F)$. Set $i^M(S, \sigma) = 1$ if σ is F-elliptic in $M(F)$, and the $M(F_v)$-orbit of σ meets $K_v \cap M(F_v)$ for every valuation $v \notin S$. Otherwise set $i^M(S, \sigma) = 0$. It follows from the nature of the conjugacy classes of $M(F_S)$ and [Art88b, (3.2)] that

$$a^{\tilde{M}}(S, \gamma) = i^M(S, \sigma) a^{\tilde{M}_{\mathbf{s_0}(\sigma)}}(S, u). \tag{29}$$

For a description of $a^{M_\sigma}(S, u)$ see [Art86, Section 7]. The term $a^{\tilde{M}_{\mathbf{s_0}(\sigma)}}(S, u)$ is defined analogously.

Let us now contrast the geometric sides of the trace formulas. Consider the summand of (26) indexed by $M = G$, namely

$$\sum_{\tilde{\gamma} \in (\mathbf{s_0}(G(F)))_{\tilde{G},S}} a^{\tilde{G}}(S, \tilde{\gamma}) I_{\tilde{G}}(\tilde{\gamma}, \tilde{f}).$$

If we are to have any hope in comparing this term with its counterpart,

$$\sum_{\gamma \in (G(F))_{G,S}} a^G(S, \gamma) I_{\tilde{G}}(\gamma, \tilde{f}),$$

by using the orbit map, then we must eliminate those $\tilde{\gamma}$ from the former sum such that $\tilde{\gamma} \neq \gamma'$ for all $\gamma \in G(F)$. The global vanishing properties, Proposition 8.1 and Proposition 8.2, were proven with this in mind. However, there is also the additional concern that the orbit map is not injective. The following lemma determines the extent to which it is not injective on the elliptic set.

Lemma 9.1 *Suppose γ_1 and γ_2 are F-elliptic in $M(F)$. Then γ_1^n is $M(F_v)$-conjugate to γ_2^n if and only if there exists $\eta \in \mu_n^M$ such that γ_1 is $M(F_v)$-conjugate to $\eta\gamma_2$.*

Proof The "if" direction is trivial to prove. Suppose therefore that γ_1^n is $M(F_v)$-conjugate to γ_2^n. Using decomposition (5) it suffices to prove the lemma in the case that $M = G$ and γ_1^n is actually equal to γ_2^n. In this case, there exist extensions E_1, E_2 of F such that the elliptic torus of $G(F)$ containing γ_i is isomorphic to $E_i^\times$, $1 = 1, 2$. Let $E = E_1 \cap E_2$. Regarding γ_1 and γ_2 as n-th roots of the elements $\gamma_1^n, \gamma_2^n \in E$, it is obvious that $E(\gamma_1)$ is isomorphic to $E(\gamma_2)$ over E. In particular these fields are isomorphic over F and so any two embeddings of these fields into $G(F)$ must be conjugate over $G(F)$. Thus a $G(F)$-conjugate of γ_2 lies in the torus isomorphic to $E(\gamma_1)$. Since this element may also be regarded as a n-th root of γ_1 it must be of the form $\eta\gamma_1$ for some $\eta \in \mu_n^G \cong \mu_n$. ■

***Proposition 9.1* (5.1)** *Suppose the hypotheses of Proposition 8.2 hold and* $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{A})\big)$. *Then* $I(\tilde{f})$ *is equal to*

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^{\tilde{M}}(S,\gamma')I_{\tilde{M}}^{\mathcal{M}}(\gamma,\tilde{f})$$

for a sufficiently large finite set of valuations S.

Proof According to Proposition 8.2, the distribution $I_{\tilde{M}}\big(\mathbf{s_0}(\gamma)\big)$, $\gamma \in M(F)$, vanishes unless γ is an n-th power in $M(F)$. From Lemma 9.1, we see that the map,

$$\big(M(F)\big)_{M,S}/\mu_n^M \xrightarrow{\;\prime\;} \Big(\mathbf{s_0}\big(M(F)\big)\Big)_{\tilde{M},S},$$

given by (equation (19), equation (3))

$$\gamma\mu_n^M \mapsto \gamma' = \mathbf{s}(\gamma)^n\mathbf{i}\Big(\prod_v \kappa_v(\gamma)^{-1}\Big)^n = \mathbf{s_0}(\gamma)^n = \mathbf{s_0}(\gamma^n),$$

is injective when restricted to the conjugacy classes of F-elliptic elements in $M(F)$. The proposition now follows from equation (22) and the fact that $a^{\tilde{M}}\big(S,\mathbf{s_0}(\gamma)\big)$ vanishes unless γ is F-elliptic in $M(F)$ (equation (29)). ■

The trace formula for $G(\mathbf{A})$, which we expect to match $I(\tilde{f})$, is

$$I(\tilde{f}') = \sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)\hat{I}_M(\gamma,\tilde{f}').$$

The following proposition is parallel to Proposition 9.1.

Proposition 9.2 *Suppose* $\tilde{f} \in \mathcal{H}\big(\tilde{G}(\mathbf{A})\big)$. *Then* $I(\tilde{f}')$ *is equal to*

$$n\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma)I_M^{\Sigma}(\gamma,\tilde{f}).$$

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Proof Obviously, $I(\tilde{f}')$ is equal to

$$\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} \sum_{\eta\in\mu_n^M} a^M(S,\eta\gamma)\hat{I}_M(\eta\gamma,\tilde{f}').$$

Equation (29) specializes to

$$a^M(S,\gamma) = i^M(S,\sigma)a^{M_\sigma}(S,u), \tag{30}$$

for the Jordan decomposition σu of $\gamma \in M(F)$. Since every element of $\mu_n^M \subset A_M(F)$ is F-elliptic in $M(F)$ and also lies in $K_v \cap M(F_v)$ for all valuations v, it is immediate that $i^M(S,\eta\sigma) = i^M(S,\sigma)$ for all $\eta \in \mu_n^M$ and semisimple $\sigma \in M(F)$. This implies that $a^M(S,\eta\gamma) = a^M(S,\gamma)$ in the above expansion of $I(\tilde{f}')$. Explicitly, we have

$$\begin{aligned} I(\tilde{f}') &= \sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) \sum_{\eta\in\mu_n^M} \hat{I}_M(\eta\gamma,\tilde{f}') \\ &= n\sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}/\mu_n^M} a^M(S,\gamma) I_M^{\Sigma}(\gamma,\tilde{f}). \end{aligned}$$ ∎

10 Appendix: Tensor Products of Metaplectic Representations

Suppose v is a nonarchimedean valuation of F. In [FK86, Section 26.2] a method of induction from parabolic subgroups of $\tilde{G}(F_v)$ is delineated. Recall decomposition (5),

$$M = M(i) \times \cdots \times M(\ell).$$

This method of induction relates tensor products of representations of $\tilde{M}(i)(F_v)$ to representations of $\tilde{M}(F_v)$. We describe this relationship and prove all of the claims made in [FK86, Section 26.2] concerning it, under the assumption that n is relatively prime to $r_i(1+2m)-1$, $1 \le i \le \ell$. As mentioned earlier, it seems that the claims are not true in general [Sun97].

Let $(\cdot,\cdot)_{F_v}\colon F_v^\times \times F_v^\times \to \mu_n$ be the n-th Hilbert symbol on F_v and let B be a maximal subgroup of $F_v^\times$ with respect to the property that $(x_1,x_2)_{F_v} = 1$ for all $x_1,x_2 \in B$. For $1 \le i \le \ell$, set

$$\tilde{M}^B(i)(F_v) = \left\{ \tilde{\gamma} \in \tilde{M}(i)(F_v) : \det\bigl(\mathbf{p}(\tilde{\gamma})\bigr) \in B \right\}.$$

It is a simple matter to check that $\tilde{M}^B(i)(F_v)$ is a normal subgroup of finite index in $\tilde{M}(i)(F_v)$. Let $\tilde{\pi}_i$ be a genuine irreducible admissible representation of $\tilde{M}(i)(F_v)$. The restriction of $\tilde{\pi}_i$ to $\tilde{M}^B(i)(F_v)$ is the sum of conjugates of some irreducible representation $\tilde{\rho}_i$ of $\tilde{M}^B(i)(F_v)$. More precisely,

$$\tilde{\pi}_i|_{\tilde{M}^B(i)} = \sum_{\gamma} \tilde{\rho}_i^{\gamma},$$

where the sum runs over representatives γ of cosets in $\tilde{M}(i)(F_v)/\tilde{M}^B(i)(F_v)$ and

$$\tilde{\rho}_i^{\gamma}(\gamma_1) = \tilde{\rho}_i(\gamma\gamma_1\gamma^{-1}), \quad \gamma_1 \in \tilde{M}^B(i)(F_v).$$

Lemma 10.1 *Suppose that n is relatively prime to $r_i(1+2m)-1$, $1 \le i \le \ell$, and that γ is as above. Then $\tilde{\rho}_i$ is not equivalent to $\tilde{\rho}_i^{\gamma}$ unless $M(i) \cong \mathrm{GL}(1)$ or $\gamma \in \tilde{M}^B(i)(F_v)$.*

Proof If $M(i) = \mathrm{GL}(1)$ then $\tilde{M}(i)(F_v) \cong F_v^{\times} \times \mu_n$. In particular $\tilde{M}(i)(F_v)$ is abelian and $\tilde{\rho}_i = \tilde{\rho}_i^{\gamma}$. Suppose that $M(i)$ is not isomorphic to $\mathrm{GL}(1)$. By using the Iwasawa decomposition, it is easy to see that representatives of the quotient $\tilde{M}(i)(F_v)/\tilde{M}^B(i)(F_v)$ may be taken to be diagonal matrices. Let γ be such a representative corresponding to the diagonal element

$$\begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_{r_i} \end{pmatrix} \in \mathrm{GL}(r_i, F_v),$$

and suppose that $\tilde{\rho}_i^{\gamma}$ is equivalent to $\tilde{\rho}_i$. In other words, suppose that there exists a linear isomorphism T such that

$$T \circ \tilde{\rho}_i^{\gamma}(\tilde{\gamma}) = \tilde{\rho}_i'(\tilde{\gamma}) \circ T, \quad \tilde{\gamma} \in \tilde{M}^B(i)(F_v).$$

Suppose $x \in B$ and choose $\tilde{\gamma} \in \tilde{M}^B(i)(F_v)$ such that $\mathbf{p}(\tilde{\gamma})$ corresponds to the scalar matrix

$$\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} \in \mathrm{GL}(r_i, F_v).$$

By [KP84, Proposition 0.1.5] and the bilinearity of the Hilbert symbol, we have

$$\begin{aligned} \tilde{\rho}_i(\tilde{\gamma}) &= T \circ \tilde{\rho}_i^{\gamma}(\tilde{\gamma}) \circ T^{-1} \\ &= T \circ \tilde{\rho}_i(\gamma\tilde{\gamma}\gamma^{-1}) \circ T^{-1} \\ &= \left(\left(\det(\gamma), \det\big(\mathbf{p}(\tilde{\gamma})\big) \right)_{F_v}^{1+2m} \Big/ \prod_{j=1}^{r_i} (\gamma_i, x)_{F_v} \right) T \circ \tilde{\rho}_i(\tilde{\gamma}) \circ T^{-1}. \end{aligned}$$

It may be verified by following [KP84, 0.1.1] that $\tilde{\gamma}$ is in the center of $\tilde{M}^B(i)(F_v)$ and so, by Schur's lemma, $\tilde{\rho}_i(\tilde{\gamma})$ is a nonzero scalar operator. Consequently the above identity reduces to

$$\big(\det(\gamma), x\big)_{F_v}^{r_i(1+2m)-1} = 1.$$

As n and $r_i(1+2m)-1$ are relatively prime we have $\big(\det(\gamma), x\big)_{F_v} = 1$. The element $x \in B$ was chosen arbitrarily so this means that $\gamma \in \tilde{M}^B(i)(F_v)$. ■

Continuing with the discussion on tensor products, we set $\tilde{\rho} = \bigotimes_{i=1}^{\ell} \tilde{\rho}_i$. This representation passes to an irreducible representation of the subgroup

$$\tilde{M}^B(F_v) = \left\{ \tilde{\gamma} \in \tilde{M}(F_v) : \det\big(\mathbf{p}(\tilde{\gamma})\big) \in B \right\}.$$

If $M \neq M_0$ then Lemma 10.1 implies that $\tilde{\rho}$ is inequivalent to any of its conjugates by elements in $\tilde{M}(F_v) - \tilde{M}^B(F_v)$. If $M = M_0$, we obtain the same result by [FK86, Proposition 3]. Applying Mackey's criterion, we find that the representation of $\tilde{M}(F_v)$ induced from $\tilde{\rho}$ is irreducible. This process may be reversed without difficulty. Hence, every genuine irreducible admissible representation of $\tilde{M}(F_v)$ corresponds to a unique set of genuine irreducible admissible representations of $\tilde{M}(i)(F_v)$, for $1 \leq i \leq \ell$.

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