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The freeness of some projective metabelian groups

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The question whether there exist non-free projective groups of rank r in the variety $\underline{\underline{AA}}_n$ has been answered in the affirmative for $n \geq 2$, $r \geq 2$, except for $n = r = 2$, by V.A. Artamonov. This paper consists in a proof that a projective group G of rank 2 in $\underline{\underline{AA}}_2$ is free. If x and y are any two elements which generate G modulo $\underline{\underline{A}}_2(G)$, then the group F generated by x and y is free in $\underline{\underline{AA}}_2$, and the index of F in G is finite and not divisible by 2 . One wishes to replace x by xu and y by yv , where u and v lie in $\underline{\underline{A}}_2(G)$, so that $\langle xu, yv\rangle$ is the whole of G . This can be done: first, on general grounds, it is sufficient that $\langle xu, yv\rangle$ contain every $C(a)$, where $C(a)$ is the centralizer in the $G/\underline{\underline{A}}_2(G)$-module $\underline{\underline{A}}_2(G)$ of an element a in $G/\underline{\underline{A}}_2(G)$ (and moreover choices of u and v for each $C(a)$ can be combined to give a single choice good for all $C(a)$); second, for the particular small numbers involved, the structure of $C(a)$ is sufficiently simple for one to pick suitable u and v without trouble.

Artamonov exhibits, in [1], non-free projective groups of rank r in the variety $\underline{\underline{AA}}_n$, for any r and n with $r \geq 2$ and $n \geq 2$, except for

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$r = n = 2$. There, as here, $\underline{\underline{A}}$ is the variety of all abelian groups, $\underline{\underline{A}}_n$ is the variety of abelian groups of exponent n , and the rank of a projective group P in $\underline{\underline{AA}}_n$ is the rank of the free abelian group P/P' This note deals with the only case not covered by Artamonov, namely, we show, by examining the group structure of such projectives as there are, that any projective group of rank 2 in $\underline{\underline{AA}}_2$ is free. The methods are very different from those of Artamonov; they spring from the theorem of Hall, [2], which asserts that projective groups in nilpotent varieties of exponent zero or a prime power are free. This theorem is applied when a projective group in $\underline{\underline{AA}}_n$ is reduced, by factoring out a verbal subgroup, to a projective group in a smaller variety.

We prove the

THEOREM. *In the variety* $\underline{\underline{AA}}_2$ *all projective groups of rank* 2 *are free.*

First, however, we introduce a lemma which holds in more generality. Suppose G is projective of finite rank r in $\underline{\underline{AA}}_{p^n}$, where p is prime and $n \geq 1$, and let $\{x_1, \ldots, x_r\}$ be a set of elements of G . Write $F = \langle x_1, \ldots, x_r \rangle$, the subgroup of G generated by $\{x_1, \ldots, x_r\}$.

LEMMA 1. *If* $FA_p(G) = G$, *then* F *is free of rank* r *in* $\underline{\underline{AA}}_{p^n}$, *and the index of* F *in* G *is finite and not divisible by* p .

Proof. That F is freely generated by the x_i follows from Lemma 1 of [2], as the free groups in $\underline{\underline{AA}}_{p^n}$ are residually finite p-groups.

The inclusion $F \to G$ induces a homomorphism

$$\phi : F/A_pA_{p^n}(F) \to G/A_pA_{p^n}(G) .$$

As $\underline{\underline{AA}}_{p^n}$ is a locally finite variety of prime power exponent, $G/A_pA_{p^n}(G)$, being projective in $\underline{\underline{AA}}_{p^n}$, is free in $\underline{\underline{A}}_p\underline{\underline{A}}_{p^n}$, by Theorem 1 of [2]. So $G/A_pA_{p^n}(G)$ is isomorphic to $F/A_pA_{p^n}(F)$. As $FA_p(G) = G$, and

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$A_p(G)/A_pA_{p^n}(G)$ is the Frattini subgroup of $G/A_pA_{p^n}(G)$, ϕ is an epimorphism. As the groups in question are finite and isomorphic, ϕ must be an isomorphism.

As $F/A_{p^n}(F)$ and $G/A_{p^n}(G)$ both have order p^{nr}, and $F/A_pA_{p^n}(F)$ and $G/A_pA_{p^n}(G)$ have the same order, $A_{p^n}(F)$ and $A_{p^n}(G)$ must have the same Z-rank. So $A_{p^n}(F)$ is of finite index in $A_{p^n}(G)$. Moreover, if the index of $A_{p^n}(F)$ in $A_{p^n}(G)$ were divisible by p, there would be an element of $A_{p^n}(F)$ which was in $A_pA_{p^n}(G)$ but not in $A_pA_{p^n}(F)$. This would contradict the fact that ϕ is an isomorphism. So $\left|A_{p^n}(G) : A_{p^n}(F)\right|$ is finite and not divisible by p. But

$$|G : F| = \left|G : A_{p^n}(G)\right| : \left|A_{p^n}(F)\right| / \left|F : A_{p^n}(F)\right| = \left|A_{p^n}(G) : A_{p^n}(F)\right| ,$$

and so $|G : F|$ is finite and not divisible by p.

We now concentrate on the variety $\underline{\underline{AA}}_2$, and shall use the following notation:

G is a projective group of rank 2 in $\underline{\underline{AA}}_2$;

$M = A_2(G)$;

$A = G/M$.

Note that conjugation in G induces an action of A on M ; that is, makes M a ZA-module. Although we write G multiplicatively, we shall feel free to use additive notation to combine elements of G, both of which lie in M. If x, y are inverse images in G of any two generators of A, we let

$$F = \langle x, y \rangle ,$$

$$N = A_2(F) .$$

By Lemma 1, F is free in $\underline{\underline{AA}}_2$, and, if $t = |G : F| = |M : N|$, t is

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finite and odd.

Since the map ϕ in the proof of Lemma 1 is an isomorphism, the inclusion $F \to G$ induces an isomorphism $F/N \tilde{\to} G/M = A$. Hence N is a ZA-submodule of M . Using the fact that $m^{gh} = m^{hg}$ for any $m \in M$, $g, h \in G$, it is easy to see that, for any subset S of A , $C_M(S) = \{m \in M : ms = m$ for all $s \in S\}$ is a ZA-submodule of M .

We shall show that it is possible to choose x and y such that $F = G$ and so G is free in $\underline{\underline{AA}}_2$. The next lemma restricts the amount of work we have to do to achieve this.

LEMMA 2. *If, for each non-trivial element* a *of* A , $C_M(a) \subseteq F$, *then* $F = G$.

Proof. $V = M/N$ is a group of odd order, on which the elementary abelian 2-group A of rank 2 acts. It follows that $V = \langle C_V(a) : a \in A, a \neq 1\rangle$. So if $m \in M$ then there exist elements $n \in N$ and $m_i \in M$ such that $m = n + \Sigma m_i$, where, for all i , $m_i a_i - m_i \in N$ for some $a_i \in A\backslash\{1\}$. Write $n_i = m_i a_i - m_i$; then $n_i + n_i a_i = 0$. We shall show below that there then exists some $n_{0i} \in N$ such that $n_{0i} a_i - n_{0i} = n_i$. Then we shall have $m_i - n_{0i} \in C_m(a_i)$. By hypothesis $C_m(a_i) \subseteq F$ and $n_{0i} \in N \subseteq F$; so $m_i \in F$. So $m = n + \Sigma m_i \in F$. Thus $M \subseteq F$, and it follows that $F = G$.

It remains to prove that, if $n_i \in N$, $a_i \in A$ are such that $n_i + n_i a_i = 0$, then there exists some $n_{0i} \in N$ such that $n_{0i} a_i - n_{0i} = n_i$. At this stage we drop the subscript i .

Let g be an inverse image of a in F , such that g is part of a free basis of F (we can choose g to be either x or y or xy). Indeed, let $\{g, h\}$ be a free basis of F . Using the Schreier-Reidemeister procedure, we find that N is a free abelian group on the basis $\{g^2, h^2, gh^2g^{-1}, ghgh^{-1}, hgh^{-1}g^{-1}\}$. And so N is equally free abelian on the basis $\{g^2, h^2, gh^2g^{-1}, ghgh^{-1}, hgh^{-1}g\}$. Writing these elements, in this order, as e_1, e_2, e_3, e_4, e_5, we see that

$e_1a = e_1^g = e_1$, $e_2a = e_3$, $e_4a = e_5$.

It follows that, if $n \in N$ is such that $n + na = 0$, then $n = \lambda(e_2-e_3) + \mu(e_4-e_5)$ for some $\lambda, \mu \in Z$. Then, if $n_0 = \lambda e_3 + \mu e_5$, $n_0^a - n_0 = n$. Now the proof of Lemma 2 is complete.

We now show how it is possible, given generators a, b of A and inverse images x, y of a, b in G , to find x', y' which are also inverse images of a, b , and such that $\langle x', y'\rangle \supseteq C_M(a) \cup C_M(b) \cup C_M(ab)$.

From the description of N at the end of Lemma 2, it is clear that $C_N(a)$ is free abelian on the basis

$$\{x^2, xy^2x^{-1}\cdot y^2, xyxy^{-1}\cdot yxy^{-1}x\} = \{x^2, xy^2x^{-1}y^2, yx^2y^{-1}x^2\} .$$

By symmetry, the second and third basis elements are centralized by b , and are therefore in $C_N(A)$. As x^2 is not centralized by b , and $C_N(A)$ is a pure subgroup of N (for, if $\lambda \in Z$, $n \in N$, and $\lambda n \in C_N(A)$, $\lambda n^a = n = 0$ for any $a \in A$; so $n^a = n = 0$ and $n \in C_N(A)$), and therefore of $C_N(a)$, $\{xy^2x^{-1}y^2, yx^2y^{-1}x^2\}$ must be a basis for $C_N(A)$.

Now, for any $m \in C_M(a)$, $tm \in C_N(a)$. So the index of $C_N(a)$ in $C_M(a)$ is certainly a factor of t^3 , and is therefore odd. $C_M(A)$, having $C_N(A)$ as a subgroup of finite index, is of rank 2 , and it is a pure subgroup of $C_M(a)$. Let e be a generator of a subgroup complementary to $C_M(A)$ in $C_M(a)$. Then, as $x^2 \in C_M(a)$, $x^2 = \alpha e + c$ for some $\alpha \in Z$, $c \in C_M(A)$. As $C_N(A) \subseteq C_M(A)$, and $\langle x^2\rangle$ is a complementary subgroup to $C_N(A)$ in $C_N(a)$, α divides the index of $C_N(a)$ in $C_M(a)$, and is therefore odd. Say $\alpha = 2\delta + 1$. Then, as e commutes with x , $(xe^{-\delta})^2 = x^2 - 2\delta e = e + c$.

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So, writing $u = e^{-\delta} \in C_M(a)$, we see that $\langle xu, y\rangle + C_M(A) \supseteq C_M(a)$. Similarly, we can choose $v \in C_M(b)$, $w \in C_M(ab)$, such that $\langle x, yv\rangle + C_M(A) \supseteq C_M(b)$, $\langle x, yw\rangle + C_M(A) \supseteq C_M(ab)$. Write now $x_0 = xu$, $y_0 = yvw$. Then, if $F_0 = \langle x_0, y_0\rangle$, F_0 is of odd index t_0 in G. If $N = A_2(F_0)$, $C_{N_0}(A) = \left\langle x_0y_0^2x_0^{-1}y_0^2, y_0x_0^2y_0^{-1}x_0^2\right\rangle$ is of odd index in $C_M(A)$.

Replacing x_0, y_0 by x_0u_0, y_0v_0, where $u_0, v_0 \in C_M(A)$, one generates a free group F_1 such that, if $N_1 = A_2(F_1)$, then $C_{N_1}(A) = \left\langle x_0y_0^2x_0^{-1}y_0^2+4v_0, y_0x_0^2y_0^{-1}x_0^2+4u_0\right\rangle$. Thus, by suitable choice of u_0 and v_0, one can change these basis elements of $C_{N_0}(A)$ each by four times any element of $C_M(A)$. It follows that, by choice of u_0 and v_0, one can change the elements of any basis of $C_{N_0}(A)$ each by four times any element of $C_M(A)$. But there is some basis of $C_{N_0}(A)$, the elements of which are odd multiples of two basis elements of $C_M(A)$. As every odd integer is congruent to $\pm 1 \pmod 4$, it is possible to choose u_0 and v_0 so as to change this basis of $C_{N_0}(A)$ to one which generates the whole of $C_M(A)$. Choose u_0 and v_0 so. Thus $\langle xuu_0, yvwv_0\rangle \supseteq C_M(A)$.

We now show that the various alterations to x and y do not interfere.

LEMMA 3. *Suppose that* $\langle g, h\rangle + C_M(A) \supseteq C_M(k)$, *where* g, h *are inverse images of* a, b *in* G, *and* $k \in A\backslash\{1\}$, *and that* $p, q \in C_M(l)$, $l \in A\backslash\{1, k\}$. *Then* $\langle gp, hq\rangle + C_M(A) \supseteq C_M(k)$.

Proof. Suppose, for example, that $k = a$. Let $m \in C_M(k)$. Then by hypothesis $m = m_1 + m_2$, where $m_1 \in C_M(A)$, $m_2 \in \langle g, h\rangle$. Then $m_2 \in \langle g, h\rangle \cap C_M(k) = \left\langle C_{N_1}(A), g^2\right\rangle$ where $N_1 = A_2(\langle g, h\rangle)$.

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Now $(gp)^2 = g^2 + p^g + p$; but $p^g + p \in C_M(l)$ as $C_M(l)$ is an A-module, and $p^g + p \in C_M(k)$ as $p^g = pa = p^k$. So

$$p^g + p \in C_M(l) \cap C_M(k) = C_M(A) ,$$

and, therefore, $m_2 \in \left\langle C_M(A), (gp)^2 \right\rangle$. Thus $m \in C_M(A) + \langle gp, hq \rangle$ as required. The proof of the lemma is similar in the cases $k = b$ and $k = ab$.

Repeated application of Lemma 3 now gives, in view of the choice of u, v, w , $\langle xuu_0, yvwv_0 \rangle + C_M(A) \supseteq C_M(a) \cup C_M(b) \cup C_M(ab)$. But u_0 and v_0 were chosen so that $xuu_0, yvwv_0 \supseteq C_M(A)$. So $\langle xuu_0, yvwv_0 \rangle \supseteq C_M(a) \cup C_M(b) \cup C_M(ab)$. By Lemma 2, this is enough to show that $G = \langle xuu_0, yvwv_0 \rangle$ and is, therefore, free of rank 2 in $\underline{\underline{AA}}_2$.

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