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THE STRUCTURE OF THE ALGEBRA OF HANKEL TRANSFORMS AND THE ALGEBRA OF HANKEL-STIELTJES TRANSFORMS

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1. Introduction. Let M be the space of all bounded regular complex-valued Borel measures defined on $I = [0, \infty)$. M is a Banach space with $||\mu|| = \int d|\mu|(x)$ $(\mu \in M)$. (Integrals in this paper extend over all of I unless otherwise specified.) Let ν be a fixed real number no smaller than $-\frac{1}{2}$ and let $\mathscr{J}_\nu(z) = (c_\nu z^\nu)^{-1} J_\nu(z)$ if $z \neq 0$ and $\mathscr{J}_\nu(0) = 1$, where J_ν is the Bessel function of the first kind of order ν and $c_\nu = [2^\nu \Gamma(\nu + 1)]^{-1}$; $\mathscr{J}_\nu$ is an entire function, as can be seen from the power series definition of

$$J_\nu(z) = z^\nu \sum_{n=0}^{\infty} (-1)^n [2^{\nu+2n} n! \Gamma(n + \nu + 1)]^{-1} z^{2n}.$$

The Hankel-Stieltjes transform of order ν is given by $\mathscr{H}_\nu \mu(y) = \int \mathscr{J}_\nu(xy)\, d\mu(x)$ $(\mu \in M)$. The integral converges absolutely because of the familiar relations $J_\nu(x) = O(x^\nu)$ as $x \to 0$ and $J_\nu(x) = O(1/\sqrt{x})$ as $x \to \infty$.

Usually ν will be held fixed, so when there is no danger of ambiguity we write $\hat{\mu}$ in place of $\mathscr{H}_\nu \mu$; $\mathscr{H}_{-\frac{1}{2}}$ is the cosine transform.

If $X \subseteq M$, let $X^\wedge = \{\hat{\mu} | \mu \in X\}$ and let m_ν be the measure on $[0, \infty)$ defined by $dm_\nu(x) = c_\nu x^{2\nu+1}\, dx$. Let A_ν consist of all measurable functions f on $[0, \infty)$ for which $\mu_f \in M$ where $d\mu_f(x) = f(x)\, dm_\nu(x)$. We define $||f|| = ||\mu_f||$ and $\hat{f} = (\mu_f)^\wedge$. Of course two functions which differ only on a set of Lebesgue measure zero will be identified, as will f and μ_f. A_ν can be considered to be a subspace of M.

These transforms behave much like the Fourier and Fourier-Stieltjes transforms. The following lemma contains some of these analogies ($\mathscr{C}$ is the space of infinitely differentiable functions with compact support in I).

1.1. LEMMA. (a) *If $f \in A_\nu$ and $\hat{f} \in A_\nu$, then f can be redefined on a null set so that $(\hat{f})^\wedge = f$.*

(b) $\mathscr{C} \subset (A_\nu)^\wedge$.

(c) $\int \hat{\mu}(x)\, d\lambda(x) = \int \hat{\lambda}(y)\, d\mu(y)$ $(\mu \in M, \lambda \in M)$.

(d) *If $\mu \in M$ and $\hat{\mu}(y) = 0$ $(y \in I)$, then $\mu = 0$.*

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Proof. (a) can be proved using many of the methods that are used in proving the analogous fact for Fourier transforms. To prove (b) let $f \in \mathscr{C}$; then $f \in A_\nu$ and repeated integrations by parts show that $\hat{f} \in A_\nu$. By (a), $f = (\hat{f})^\wedge$, and so $f \in (A_\nu)^\wedge$. (c) is a direct consequence of Fubini's theorem. To prove (d), assume that $\hat{\mu} = 0$; then by (c), $\int \hat{f}\, d\mu = 0$ for every $f \in A_\nu$, and by (b) and (a), $\int g\, d\mu = 0$ for every $g \in \mathscr{C}$, whence $\mu = 0$.

A_ν has a well-known convolution which is readily extended to M.

For $\nu > -\frac{1}{2}$, let

$$\Phi_\nu(x, y, z) = \begin{cases} \dfrac{2^{3\nu-1}\Gamma(\nu+1)^2\Delta(x, y, z)^{2\nu-1}}{\Gamma(\nu+\frac{1}{2})\pi^{\frac{1}{2}}(xyz)^{2\nu}}, \\ 0. \end{cases}$$

The first value being assumed only if there is a triangle of sides x, y, and z with area $\Delta(x, y, z)$. We take M' to be the subspace of M consisting of those measures concentrated on $(0, \infty)$. Then if $\mu \in M'$ and $\lambda \in M'$, define

$$\mu *_\nu \lambda(E) = \int \int \left\{ \int_E \Phi_\nu(x, y, z)\, dm_\nu(x) \right\} d\mu(y)\, d\lambda(z).$$

If δ denotes the unit mass concentrated at 0, we have the unique decomposition of each $\mu \in M$, $\mu = \mu' + a\delta$ ($\mu' \in M'$, and a is a complex number). The convolution is extended to all of M by treating δ as a multiplicative identity.

From the definition of Φ we see that

$$\Phi_\nu(x, y, z) \geqq 0 \qquad (0 < x, y, z < \infty), \tag{1.1}$$

and from [**9**, p. 367],

$$\int \mathscr{J}_\nu(xu)\Phi_\nu(x, y, z)\, dm_\nu(x) = \mathscr{J}_\nu(yu)\mathscr{J}_\nu(zu). \tag{1.2}$$

Setting $u = 0$ in (1.2) yields

$$\int \Phi_\nu(x, y, z)\, dm_\nu(x) = 1. \tag{1.3}$$

When there is no danger of ambiguity, "$*$" will be written in place of "$*_\nu$". The convolution has all the usual properties. It is rather elementary to show that if μ and λ are in M, then so is $\mu * \lambda$. From (1.1) and (1.3), it follows that $||\mu * \lambda|| \leqq ||\mu|| \cdot ||\lambda||$ and from (1.2) that $(\mu * \lambda)^\wedge = \hat{\mu}\hat{\lambda}$. This last fact together with part (d) of the lemma show that $*$ is commutative and associative. Moreover, if f and g are in A_ν, then $\mu_f * \mu_g = \mu_{f*g}$, where

$$f * g(x) = \int \int \Phi_\nu(x, y, z) f(y) g(z)\, dm_\nu(y)\, dm_\nu(z).$$

M together with the convolution $*_\nu$ will be denoted by M_ν.

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2. Statement of results. In this paper we will study the structure of the algebras M_ν and A_ν. We will show that if $-\frac{1}{2} \leqq \nu < \eta$, then $(M_\eta)^\wedge$ and $(A_\eta)^\wedge$ can be embedded in $(M_\nu)^\wedge$ and $(A_\nu)^\wedge$, respectively. This embedding together with the knowledge that if 2ν is an integer then M_ν and A_ν can be identified with the spaces of rotation invariant measures and radial integrable functions on R^n for $n = 2\nu + 2$ will give us a simple proof of the well-known fact that the maximal ideal space of A_ν is I and of the fact that the maximal ideal space of M_ν is $I^* = [0, \infty]$, the one-point compactification of $[0, \infty)$. Finally we will investigate the factorization of members of A_ν and M_ν.

3. The inclusions $(M_\eta)^\wedge \subset (M_\nu)^\wedge$ and $(A_\eta)^\wedge \subset (A_\nu)^\wedge$ for $\eta > \nu$. One way in which the theory of Hankel transforms arises is in the study of functions and measures on the Euclidean spaces R^n for $n = 1, 2, 3, \ldots$ which possess certain symmetries; e.g., a function defined on R^n is *radial* if there is a function φ defined on I for which $f(x) = \varphi(|x|)$ for almost every x in R^n.

For a fixed positive integer n let $\nu = \frac{1}{2}(n - 2)$ and let $L_r(R^n)$ denote the class of radial functions in the convolution Banach algebra $L(R^n)$ of functions integrable on R^n; $L_r(R^n)$ is, in fact, a closed subalgebra of $L(R^n)$. If f and g are in $L(R^n)$, let $f \circ g$ be their convolution and let $\tilde{f}$ be the Fourier transform of f; then

$$\tilde{f}(y) = \int_{R^n} f(x) e^{-ix\cdot y}\, dx$$
$$= (2\pi)^{\frac{1}{2}n} \int \varphi(r) \frac{J_\nu(|y|r)}{(|y|r)^\nu} r^{n-1}\, dr$$

(see [**1**, pp. 69–79]). Thus, if 2ν is an integer and $n = 2\nu + 2$, a linear transformation $\mathscr{S}$ can be established from A_ν to $L_r(R^n)$ satisfying $||\mathscr{S}f|| = ||f||$ and $(\mathscr{S}f)^\sim(y) = \hat{f}(|y|)$ for f in A_ν and y in R^n. Indeed, $\mathscr{S}$ is an isometric algebraic isomorphism between A_ν and $L_r(R^n)$ since

$$[\mathscr{S}(f * g)]^\sim = \hat{f}\hat{g} = (\mathscr{S}f \circ \mathscr{S}g)^\sim \qquad (f, g \in A_\nu).$$

Let $M_r(R^n)$ consist of the rotation invariant Borel measures on R^n for $n = 1, 2, 3, \ldots$; μ is rotation invariant means that $\mu(TE) = \mu(E)$ for every orthogonal transformation T of R^n and every Borel subset E of R^n. Then, $\mathscr{S}$ is easily extended to an algebraic isometry between M_ν and $M_r(R^n)$ for $\nu = \frac{1}{2}(n - 2)$.

It is sometimes the case that a theorem can be easily proved for M_ν or A_ν when 2ν is an integer by using $\mathscr{S}$ to identify these spaces with $M_r(R^{2\nu+2})$ and $L_r(R^{2\nu+2})$.

If m and n are positive integers, there is a natural algebraic homomorphism of $L(R^{n+m}) \rightarrow L(R^n)$ given by $\mathscr{T}f(x_1) = \int f(x_1, x_2)\, dx_2$, where $f \in L(R^{n+m})$, $x_1 \in R^n$, $x_2 \in R^m$, and the integral extends over all of R^m. It is easy to check that $||\mathscr{T}f|| \leqq ||f||$ and $(\mathscr{T}f)^\sim(y_1) = \tilde{f}(y_1, 0)$ for $f \in L(R^{n+m})$, $y_1 \in R^n$,

$0 = (0, 0, \ldots, 0) \in R^m$. The following lemma generalizes these facts to M_ν and A_ν.

3.1. LEMMA. *If* $-\frac{1}{2} \leqq \nu \leqq \eta < \infty$, *then there exists an operator*

$$\mathscr{T}_{\eta\nu} = \mathscr{T}: M_\eta \to M_\nu$$

such that
(a) $||\mathscr{T}|| = 1$,
(b) $\mathscr{H}_\nu \mathscr{T} \mu = \mathscr{H}_\eta \mu \ (\mu \in M_\nu)$, *and*
(c) $\mathscr{T}: A_\eta \to A_\nu$.

The proof of the lemma will follow the corollaries below.

3.2 COROLLARY. *If* $-\frac{1}{2} \leqq \nu \leqq \eta < \infty$, *then*

$$(M_\eta)^\wedge \subset (M_\nu)^\wedge \quad \textit{and} \quad (A_\eta)^\wedge \subset (A_\nu)^\wedge.$$

Proof. This is a direct application of (b) and (c) of Lemma 3.1.

3.3. COROLLARY. *If* $-\frac{1}{2} \leqq \nu \leqq \eta \leqq \xi < \infty$, *then* $\mathscr{T}_{\eta\nu}\mathscr{T}_{\xi\eta} = \mathscr{T}_{\xi\nu}$.

Proof. This follows since $\mathscr{H}_\nu \mathscr{T}_{\eta\nu}\mathscr{T}_{\xi\eta} = \mathscr{H}_\eta \mathscr{T}_{\xi\eta} = \mathscr{H}_\xi = \mathscr{H}_\nu \mathscr{T}_{\xi\nu}$.

In order to prove Lemma 3.1, we need the following formula:

$$(3.1) \qquad \mathscr{J}_\eta(x) = \frac{c_{\eta-\nu-1}}{c_\eta} \int_0^1 \mathscr{J}_\nu(xz)(1 - z^2)^{\eta-\nu-1} \, dm_\nu(z) \qquad (\eta > \nu).$$

(3.1) is obtained from Sonine's first integral formula (see [**9**, p. 373]):

$$J_{\mu+\nu+1}(x) = \frac{x^{\mu+1}}{2^\mu \Gamma(\mu + 1)} \int_0^{\frac{1}{2}\pi} J_\nu(x \sin \theta) \sin^{\nu+1}\theta \cos^{2\mu+1}\theta \, d\theta$$
$$(\operatorname{Re} \nu > -1, \operatorname{Re} \mu > -1)$$

by making the change of variable $z = \sin \theta$, setting $\eta = \mu + \nu + 1$, and multiplying both sides by $(c_\eta x^\eta)^{-1}$.

We can now prove Lemma 3.1. Suppose that $-\frac{1}{2} \leqq \nu \leqq \eta < \infty$ and that $\mu \in M_\eta$. We will construct $\lambda \in M$ such that $\mathscr{H}_\nu \lambda = \mathscr{H}_\eta \mu$ and $||\lambda|| \leqq ||\mu||$.

Let $\beta = c_{\eta-\nu-1}/c_\eta$; associated with each $\mu \in M_\eta$ is a linear functional $T(\mu)$ defined on C (the continuous functions defined in I which vanish at infinity) by:

$$(3.2) \qquad T(\mu)f = \beta \int_0^1 (1 - z^2)^{\eta-\nu-1} \left\{ \int f(zx) \, d\mu(x) \right\} dm_\nu(z),$$

so that

$$|T(\mu)f| \leqq \beta \int_0^1 (1 - z^2)^{\eta-\nu-1} \, dm_\nu(z) \, ||\mu|| \cdot ||f||_\infty;$$

thus $T(\mu)$ is bounded. Hence by the Riesz representation theorem, there is a measure $\lambda \in M$ such that

$$(3.3) \qquad T(\mu)f = \int f(x) \, d\lambda(x).$$

If we replace $f(x)$ by $\mathscr{J}_\nu(yx)$ in (3.2) and (3.3) and use Fubini's theorem we see that

$$\mathscr{H}_\nu\lambda(y) = \int d\mu(x)\beta \int_0^1 (1 - z^2)^{\eta-\nu-1}\mathscr{J}_\nu(xyz)\, dm_\nu(z)$$
$$= \int \mathscr{J}_\eta(yz)\, d\mu(x) = \mathscr{H}_\eta\mu(y)$$

by (3.1).

Let $\mathscr{T}\mu = \lambda$; then $\mathscr{H}_\nu(\mathscr{T}\mu) = \mathscr{H}_\eta(\mu)$. From (3.2) and (3.3) we have

$$||\mathscr{T}\mu|| \leqq \left\{\beta \int_0^1 (1 - z^2)^{\eta-\nu-1} dm_\nu(z)\right\}||\mu|| = \mathscr{J}_\eta(0)||\mu|| = ||\mu||$$

so that $||\mathscr{T}|| \leqq 1$. To see that $||\mathscr{T}|| = 1$, suppose that $\mu \in M$ is a positive measure, then so is $\mathscr{T}\mu$, and we have $||\mu|| = \mathscr{H}_\eta\mu(0) = \mathscr{H}_\nu\mathscr{T}\mu(0) = ||\mathscr{T}\mu||$.

To show that $\mathscr{T}A_\eta \subseteq A_\nu$, suppose that $\mu \in A_\eta$ and let E be a set of zero Lebesgue measure. Then if $0 \leqq z \leqq 1$, zE has zero Lebesgue measure, thus if f is the characteristic function of E, then $\int f(zx)\, d\mu(x) = 0$ $(0 \leqq z \leqq 1)$, and so $\mathscr{T}\mu(E) = T(\mu)f = 0$.

4. The maximal ideal spaces of M_ν and A_ν. We are now in a position to describe the maximal ideal spaces of M_ν and A_ν. That of A_ν is well known but we could not find a proof in the literature, and so we give a simple one using Lemma 3.1.

4.1. THEOREM. *Suppose that $\nu \geqq -\frac{1}{2}$; then to each homomorphism H of A_ν onto the complex numbers corresponds a unique point $y_H \in I$ such that*

$$H(f) = \hat{f}(y_H) \qquad (f \in A_\nu).$$

Moreover, given the weak topology, the space of homomorphisms is homeomorphic to I with the usual topology.

Proof. Reiter proved in [**4**, pp. 473–474] that the maximal ideal space of $L_r(R^n)$ is I with the usual topology, and so the theorem is proved when 2ν is an integer.

Now assume that $-\frac{1}{2} < \nu < \infty$ and let H be a non-zero complex homomorphism of A_ν. Let $\hat{H}(\hat{f}) = H(f)$ and $|||\hat{f}||| = ||f||$ $(f \in A_\nu)$. Then $(A_\nu)^\wedge$ is a commutative Banach algebra, and so $\hat{H}$ is continuous on $(A_\nu)^\wedge$ [**3**, p. 69]. If η is an integer exceeding ν, we see by Lemma 1.1 and Corollary 3.2 that

$$\mathscr{C}^\wedge \subset (A_\eta)^\wedge \subset (A_\nu)^\wedge$$

and the second inclusion is dense in the norm $|||\cdot|||$ because $\mathscr{C}$ is dense in A_ν. Thus $\hat{H}$ defines a non-zero complex homomorphism on $(A_\eta)^\wedge$ which must be given by $\hat{H}(\hat{f}) = f(y_H)$ for $f \in (A_\eta)^\wedge$ and some fixed $y_H \in I$. Thus by the continuity of $\hat{H}$, it follows that $H(f) = \hat{f}(y_H)$ for $f \in A_\nu$.

The converse statement follows from the fact that $(A_\nu)^\wedge$ separates points of I, and the statement about the topology of the space of homomorphisms follows from the fact that $(A_\nu)^\wedge$ consists of continuous functions which vanish at infinity.

Because of the special nature of the convolution in M_ν we can relate the maximal ideal space of M_ν to that of A_ν. The following lemmas are the keys to this relation.

4.2. LEMMA. *Let* $\nu > -\frac{1}{2}$, *then if* $\mu, \lambda \in M'$, *we have* $\mu * \lambda \in A_\nu$.

Proof. Let E be a set of zero Lebesgue measure. Then from the definition of convolution,

$$\mu * \lambda(E) = \int \int \left\{ \int_E \Phi_\nu(x, y, z)\, dm_\nu(x) \right\} d\mu(y)\, d\lambda(z).$$

But the innermost integral is zero for every x and y, and so $\mu * \lambda(E) = 0$, which completes the proof.

4.3. LEMMA. *If* $\mu \in M$, *then*

$$\hat{\mu}(\infty) = \lim_{y\to\infty} \hat{\mu}(y)$$

exists and satisfies

(4.1) $\qquad \hat{\mu}(\infty) = \mu(\{0\})$, *and so* $\mu \in M'$ *if and only if* $\mu(\infty) = 0$.

Proof. In general, for f continuous at 0, $\int f(x)\, d\delta(x) = f(0)$. Since $\mathscr{J}_\nu$ is analytic and $\mathscr{J}_\nu(0) = 1$,

$$\hat{\delta}(y) = \int \mathscr{J}_\nu(xy)\, d\delta(x) = \mathscr{J}_\nu(0) = 1 \quad \text{for } y \in I,$$

and so (4.1) holds if $\mu = \delta$. $J_\nu(x) = O(1/\sqrt{x})$ as $x \to \infty$, and so

$$\mathscr{J}_\nu(xy) = O((xy)^{-(\nu+\frac{1}{2})});$$

thus for each $x > 0$, $\mathscr{J}_\nu(xy) \to 0$ as $y \to \infty$. Moreover, Poisson's integral for $J_\nu(z)$ (see [**9**, p. 47, formula (1)]) yields

$$\mathscr{J}_\nu(z) = \Gamma(\nu+1)\left[2^\nu \Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\right]^{-1} \int_0^\pi \cos(z\cos\theta)\sin^{2\nu}\theta\, d\theta$$

which is uniformly bounded by $\mathscr{J}_\nu(0) = 1$ for real z, and so $\mathscr{J}_\nu(xy) \to 0$ as $y \to \infty$ boundedly for $x > 0$. Hence, by Lebesgue's dominated convergence theorem, if $\mu \in M'$, then $\hat{\mu}(y) = \int \mathscr{J}_\nu(xy)\, d\mu(x) \to 0$ as $y \to \infty$; therefore (4.1) holds for $\mu \in M'$.

Finally, if $\mu \in M$, μ has a unique decomposition $\mu = \mu' + a\delta$ for some $\mu' \in M'$ and some complex number a; thus $\mu(\{0\}) = \mu'(\{0\}) + a\delta(\{0\}) = a$, and

$$\lim_{y\to\infty} \hat{\mu}(y) = \lim_{y\to\infty} \hat{\mu}'(y) + a \lim_{y\to\infty} \hat{\delta}(y) = a,$$

and so $\hat{\mu}(\infty) = a = \mu(\{0\})$.

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We can now consider $(M_\nu)^\wedge$ to be an algebra of continuous functions defined on $I^* = [0,\infty]$. The following theorem describes the maximal ideal space of M_ν.

4.4. THEOREM. *Suppose that $\nu > -\frac{1}{2}$; then to each homomorphism H of M_ν onto the complex numbers corresponds a unique point $y_H \in I^*$ such that*

$$H(\mu) = \hat{\mu}(y_H) \qquad (\mu \in M_\nu).$$

Moreover, given the weak topology, the space of homomorphisms is homeomorphic to I^*.

Proof. We consider two cases.

Case (i). $H(\mu) \neq 0$ for some $\mu \in M'$. Then H restricted to A_ν is a non-zero homomorphism because $\mu * \mu \in A_\nu$ and $H(\mu * \mu) = H(\mu)^2 \neq 0$. Thus there is $y_H \in I$ such that $H(f) = \hat{f}(y_H)$ $(f \in A_\nu)$.

Now $[H(\mu)]^2 = H(\mu * \mu) = [\hat{\mu}(y_H)]^2$ and $[H(\mu)]^3 = H(\mu * \mu * \mu) = [\hat{\mu}(y_H)]^3$, and so $H(\mu) = \hat{\mu}(y_H)$. Finally, $H(\delta) = 1 = \hat{\delta}(y_H)$, and the theorem follows because if $\lambda \in M$, then $\lambda = \mu + a\delta$ for some $\mu \in M'$ and complex number a.

Case (ii). $H(\mu) = 0$ for every $\mu \in M'$. Since H is a non-zero homomorphism, we have $H(\delta) = 1 = \hat{\delta}(\infty)$; thus if $\mu \in M$, $\mu = \mu' + a\delta$ for some unique $\mu' \in M'$ and complex a and we have $H(\mu) = H(\mu' + a\delta) = H(\mu') + aH(\delta) = a = \mu(\{0\}) = \mu(\infty)$ by Lemma 4.3.

The balance of the proof is the same as that of the preceding theorem.

The following theorem exhibits a major difference between the structures of A_ν and $L(R^n)$.

If I is a closed ideal of A_ν, let $Z(I) = \{y | \hat{f}(y) = 0 \text{ for every } f \in I\}$. $Z(I)$ is called the zero set of I.

4.5. THEOREM. *If $\nu \leqq \frac{1}{2}$ and $y_0 > 0$, then $\{y_0\}$ is the zero set of at least two distinct ideals.*

Proof. In [7] we showed that the functions of $(A_\nu)^\wedge$ have p continuous derivatives on $(0,\infty)$, where p is the greatest integer not exceeding $\nu + \frac{1}{2}$. The kth derivative is given by

$$\hat{f}^{(k)}(y) = \int x^k \mathscr{J}_\nu^{(k)}(xy) f(x)\, dm_\nu(x) \qquad (0 \leqq k \leqq p, f \in A_\nu, y > 0).$$

In the course of the proof, we show that $x^k \mathscr{J}_\nu^{(k)}(xy)$ is bounded in x, and so in fact the functional on A_ν, given by

$$D_k f = \hat{f}^{(k)}(y_0) \qquad (0 \leqq k \leqq p, y_0 > 0),$$

is continuous.

Let $I_k = \{f | f \in A_\nu, D_0 f = D_1 f = \ldots = D_k f = 0\}$. The Leibniz differentiation formula shows that I_k is an ideal and the continuity of the functionals D_k shows that I_k is closed. Since $\mathscr{C}$ is contained in $(A_\nu)^\wedge$, it follows that $I_0, I_1, \ldots, I_p$ are all distinct.

We remark that Reiter [4] extended an example of Schwartz [8] to prove this in the case when 2ν is an integer by showing that $I_0 \neq I_1$.

5. Representation of functions and measures by convolutions. Rudin has shown [5] that if $f \in L(R^n)$, there are functions g and h in $L(R^n)$ such that

$$f = g \circ h \tag{5.1}$$

or, equivalently,

$$\tilde{f} = \tilde{g}\tilde{h}. \tag{5.2}$$

If f is a radial function, Rudin's proof yields radial functions g and h satisfying (5.1). In fact, in his construction he never uses the structure of R^n and it easily generalizes to the following result.

5.1. THEOREM. *If $f \in A_\nu$, then there are functions $g \in A_\nu$ and $h \in A_\nu$ such that $f = g * h$.*

We wish to investigate the generalization of this factorization to M_ν.

Because of the following lemma, there are only certain factorizations which should interest us.

5.2. LEMMA. *If K is a compact subset of I^* such that $\hat{\mu}(y) \neq 0$ $(y \in K)$, then there is a $\lambda \in M_\nu$ such that $\hat{\mu}(y)\hat{\lambda}(y) = 1$ $(y \in K)$. If $K = I^*$, then $\mu * \lambda = \delta$.*

A proof of this is given in [2, p. 124] in the context of locally compact Abelian groups, but the proof can be adapted to apply here.

A measure satisfying the hypothesis of the lemma with $K = I^*$ will be called a *unit*. Every measure η in M_ν has a trivial factorization, for if μ is a unit and λ is such that $\mu * \lambda = \delta$ (λ exists because of Lemma 5.2), then η can be factored: $\eta = (\eta * \mu) * \lambda$.

We now state the following definitions.

Suppose that $\mu \in M_\nu$; then μ is *reducible* if we can write $\mu = \lambda * \eta$, where λ and η are in M_ν and neither is a unit. μ will be called *irreducible* if it is not reducible.

The question at hand, then, is: Which measures are reducible? We give a partial answer in terms of the zero set:

$$\mathscr{Z}(\mu) = \{y \mid 0 \leqq y \leqq \infty, \hat{\mu}(y) = 0\}.$$

5.3. THEOREM. (a) *If $\mu \in M_\nu$ and $\mathscr{Z}(\mu)$ is empty, then μ is irreducible.*

(b) *If $\nu > -\frac{1}{2}$, then there are reducible and irreducible measures μ in M_ν such that $\mathscr{Z}(\mu)$ contains exactly one positive real number.*

(c) *If $\mathscr{Z}(\mu) = \{\infty\}$, then μ is reducible if and only if μ is absolutely continuous with respect to Lebesgue measure.*

(d) *If $\mathscr{Z}(\mu)$ contains at least two points, then μ is reducible.*

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Proof. (a) From Lemma 5.2 we see that a measure is a unit if and only if its zero set is empty, thus the only possible factorization of a unit is into the convolution of units since the zero set of a convolution is the union of the zero sets of the factors.

(b) Let y_0 be a positive real number and let

$$I_0 = \{\mu | \mu \in M_\nu, \mathscr{Z}(\mu) = \{y_0\}\}.$$

We wish to show that I_0 contains both reducible and irreducible measures. We show that I_0 is not empty by constructing a particular measure in I_0. We will use this measure for both parts of the proof.

Choose $\varphi \in \mathscr{C}$ such that $\varphi(y_0) = -1$, $\varphi'(y_0) = 1$, and $\varphi(y) \neq -1$ if $y \neq y_0$. Then $f = \hat{\varphi}$ is in A_ν. Let $\lambda = \delta + f$. Then λ is in I_0 and $\hat{\lambda}'(y_0) = 1$.

I_0 contains reducible measures since $\lambda * \lambda$ is in I_0.

We will now show that I_0 contains irreducible measures. Let us assume by way of contradiction that every measure in I_0 is reducible. Suppose that μ is a measure such that $\mathscr{Z}(\mu) = \{y_0\}$; then $\mu = \mu_1 * \mu_2$, where μ_1 and μ_2 are measures such that neither $\mathscr{Z}(\mu_1)$ nor $\mathscr{Z}(\mu_2)$ is empty. But for $j = 1$ or $j = 2$ we have $\mathscr{Z}(\mu_j) \subseteq \mathscr{Z}(\mu) = \{y_0\}$, therefore $\mathscr{Z}(\mu_1) = \mathscr{Z}(\mu_2) = \{y_0\}$. A simple induction argument can be used to show that if λ is the measure constructed above with the property that $\mathscr{Z}(\lambda) = \{y_0\}$ and $\lambda'(y_0) = 1$, and if N is a positive integer, then there are measures $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $\mathscr{Z}(\lambda_1) = \mathscr{Z}(\lambda_2) = \ldots = \mathscr{Z}(\lambda_N) = \{y_0\}$ and

$$\lambda = \lambda_1 * \lambda_2 * \ldots * \lambda_N. \tag{5.3}$$

We proved in [**7**] that if f is in A_ν, then $\hat{f}(y_0 + h) - \hat{f}(y_0) = O(h^a)$, where $a = \min(\nu + \frac{1}{2}, 1) > 0$. The same result holds with almost the same proof for measures.

Thus from (5.3) and the fact that $\hat{\mu}(y_0) = 0$ for $\mu \in I_0$, we have

$$\hat{\mu}(y_0 + h) - \hat{\mu}(y_0) = \hat{\mu}(y_0 + h) \qquad (\mu \in I_0),$$

so that $\hat{\lambda}(y_0 + h) = \hat{\lambda}_1(y_0 + h) \ldots \hat{\lambda}_N(y_0 + h) = [O(h^a)]^N = O(h^{Na})$ $(N = 1, 2, 3, \ldots)$. Thus $\hat{\lambda}'(y_0) = 0$ which contradicts our construction of λ so that $\hat{\lambda}'(y_0) = 1$.

(c) Suppose that $\mu \in M$ and that $\mathscr{Z}(\mu) = \{\infty\}$. If μ is absolutely continuous, it is reducible by Theorem 5.1. We now show that if μ has a non-zero singular part, then μ is irreducible.

Assume that we can find λ and η in M' and numbers a and b such that $\mu = (\lambda + a\delta) * (\eta + b\delta) = \lambda * \eta + a\eta + b\lambda + ab\delta$. Since $\mathscr{Z}(\mu) = \{\infty\}$, Lemma 4.3 tells us that $\mu \in M'$ so that $ab = 0$. Assume that $b = 0$. We then have

$$\mu = \lambda * \eta + a\eta.$$

Now $\lambda * \eta$ is absolutely continuous by Lemma 4.2, and so if μ is to have a non-singular part we must have $a \neq 0$. But since $\hat{\mu}$ has no finite zeros, $\hat{\lambda}(y) + a$

has no finite zeros and since $\hat{\lambda}(y) + a \to a \neq 0$ as $y \to \infty$, we see that $\mathscr{Z}(\lambda + a\delta) = \emptyset$, so that $\lambda + a\delta$ is a unit. Thus the only factorization of μ is trivial and so μ is irreducible.

(d) Suppose that $\mathscr{Z}(\mu)$ contains at least two points of I^*. We consider two cases.

Case (i). $\mathscr{Z}(\mu)$ contains an interval $[a, b]$. Let $\epsilon < \frac{1}{3}(b - a)$ and choose $\varphi_1, \varphi_2 \in \mathscr{C}$ such that

$$\begin{aligned} \varphi_1(y) &= 1 && (0 \leqq y \leqq a), \\ 0 < \varphi_1(y) &< 1 && (a < y < a + \epsilon), \\ \varphi_1(y) &= 0 && (y \geqq a + \epsilon), \end{aligned}$$

and

$$\begin{aligned} \varphi_2(y) &= 1 && (0 \leqq y \leqq b - \epsilon), \\ 0 < \varphi_2(y) &< 1 && (b - \epsilon < y < b), \\ \varphi_2(y) &= 0 && (y \geqq b). \end{aligned}$$

Since φ_1 and φ_2 are in $\mathscr{C}$, there are functions f_1 and f_2 in A_ν such that

$$\hat{f}_i = \varphi_i \qquad (i = 1, 2).$$

Let

$$\lambda = f_1 * \mu + f_2 - \delta \qquad \text{and} \qquad \eta = f_1 + (f_2 - \delta) * \mu.$$

It is easy to check that $\hat{\lambda}(y)\hat{\eta}(y) = \hat{\mu}(y)$ for all y so that $\lambda * \eta = \mu$. Finally, λ and η are not units since $\hat{\lambda}(a) = \hat{\eta}(b) = 0$.

Case (ii). Suppose that $\mathscr{Z}(\mu)$ contains no interval, and assume that y_1 and y_2 are points of $\mathscr{Z}(\mu)$ (take $y_1 < y_2$). Since the interval $[y_1, y_2]$ is not contained in $\mathscr{Z}(\mu)$, there must be a point $y_0 \in [y_1, y_2]$ such that $\hat{\mu}(y_0) \neq 0$. We may assume without loss of generality that $\operatorname{Re} \hat{\mu}(y_0) > 0$. Thus we can find numbers a and b such that $y_1 < a < b < y_2$ and such that

$$\operatorname{Re} \hat{\mu}(y) > 0 \qquad (a \leqq y \leqq b). \tag{5.4}$$

We will construct measures η and λ such that

$$\hat{\eta}(y) = \begin{cases} \hat{\mu}(y) & (0 \leqq y \leqq a), \\ 1 & (b \leqq y < \infty), \end{cases} \tag{5.5}$$

$$\hat{\lambda}(y) = \begin{cases} 1 & (0 \leqq y \leqq a), \\ \hat{\mu}(y) & (b \leqq y \leqq \infty), \end{cases} \tag{5.6}$$

and $\eta * \lambda = \mu \cdot \eta$ and λ will not be units since

$$\hat{\eta}(y_1) = \hat{\mu}(y_1) = 0 \qquad \text{and} \qquad \hat{\lambda}(y_2) = \hat{\mu}(y_2) = 0.$$

To perform the construction, let ϵ, φ_1, φ_2, f_1, and f_2 be as in Case (i). Choose $\varphi_3 \in \mathscr{C}$ such that

$$\begin{aligned} \varphi_3(y) &= 0 && (0 \leqq y \leqq a), \\ 0 < \varphi_3(y) &< 1 && (a < y < b), \\ \varphi_3(y) &= 0 && (y \geqq b). \end{aligned}$$

Then there is a function $f_3 \in A_\nu$ such that $(f_3)^\wedge = \varphi_3$. Let

$$\eta = \mu * f_1 + \delta - f_2 + f_3;$$

then $\hat{\eta}(y) \neq 0$ if $a \leqq y \leqq b$ because of (5.4). By Lemma 5.2, there is a measure η_1 in M_ν such that

$$\hat{\eta}(y)\hat{\eta}_1(y) = 1 \qquad (a \leqq y \leqq b).$$

Define λ by

$$\lambda = \mu * \eta_1 * [\delta - f_1 * f_1 - (\delta - f_2) * (\delta - f_2)] \\ - f_3 * \eta_1 * [f_1 + \mu * (\delta - f_2)] + \mu * (\delta - f_2) + f_1.$$

It is an easy matter to check that $\hat{\lambda}(y)\hat{\eta}(y) = \hat{\mu}(y)$ for all y and that (5.5) and (5.6) hold.

The question still remains open for the case $\mathscr{Z}(\mu) = \{0\}$. It is easy to construct reducible measures satisfying this condition but we do not know whether there are irreducible ones.

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