SOME RESULTS ON STABLE p-HARMONIC MAPS

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1. Introduction. For each $p \in [2, \infty)$, a p-harmonic map $f: M^m \to N^n$ is a critical point of the p-energy functional

$$E = \frac{1}{p} \int_M \|df\|^p \, dv_m,$$

where M^m is a compact and N^n a complete Riemannian manifold of dimensions m and n respectively. In a recent paper [**3**], Takeuchi has proved that for a certain class of simply-connected δ-pinched N^n and certain type of hypersurface N^n in $\mathbb{R}^{n+1}$, the only stable p-harmonic maps for any compact M^m are the constant maps. Our purpose in this note is to establish the following theorem which complements Takeuchi's results.

THEOREM 1. *Let $S^{n_1} \times \ldots \times S^{n_k}$ be a product of k unit spheres of dimensions $n_1, \ldots, n_k$ with $\min\{n_1, \ldots, n_k\} > p$. Then for any compact M^m, any stable p-harmonic map from M^m to $S^{n_1} \times \ldots \times S^{n_k}$ must be a constant map.*

We note that the simple inductive proof for the case $p = 2$ in Theorem 1 as given in [**2**, p. 381] does not seem to work for the case $p > 2$. Instead we shall deduce Theorem 1 from the following more general theorem which is a generalization of the main theorem in [**2**].

Consider now N^n a complete submanifold in the Euclidean space $\mathbb{R}^{n+r}$, where the codimension r is arbitrary. Let B denote the second fundamental form given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

for any tangent vectors X, Y to N^n, where $\tilde{\nabla}$ and ∇ denote the connection on $\mathbb{R}^{n+r}$ and N^n respectively. Define the function $h: N^n \to \mathbb{R}$ by

$$h(x) = \max\{\|B(u, u)\|^2 : u \in TN_x \text{ and } \|u\| = 1\}$$

and at each $x \in N^n$, define the function $\phi: TN_x \to \mathbb{R}$ by

$$\phi(v) = \sum_{i=1}^{n} \|B(v, v_i)\|^2,$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis for TN_x. As was noted in [**2**, p. 382], the value of ϕ is independent of the choice of this orthonormal basis. At each $x \in N^n$ and for each unit vector $v \in TN_x$, we let $\operatorname{Ric}(v, v)$ denote the Ricci curvature of N^n at x in the direction v.

THEOREM 2. *If at each $x \in N^n$ and for any unit vector $v \in TN_x$, we have*

$$(p-2)h(x) + \phi(v) < \operatorname{Ric}(v, v),$$

then for any compact M^m, the only stable p-harmonic maps $f: M^m \to N^n$ are the constant maps.

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Proof of Theorem 1. For $S^{n_1}\times\ldots\times S^{n_k}\subset\mathbb{R}^{n_1+1}\times\ldots\times\mathbb{R}^{n_k+1}=\mathbb{R}^{n_1+\ldots+n_k+k}$, an easy calculation shows that at each $x\in S^{n_1}\times\ldots\times S^{n_k}$ and for any unit vector v in the tangent space to $S^{n_1}\times\ldots\times S^{n_k}$ at x, we have $h(x)=1$, $\phi(v)=1$ and $\mathrm{Ric}(v,v)\geq \min\{n_1,\ldots,n_k\}-1$. Therefore, Theorem 1 follows from Theorem 2.

2. Proof of Theorem 2. Let us first recall the second variation formula for a p-harmonic map $f:M^m\to N^n$.

Let v be a vector field on N^n and let $\phi_t:N^n\to N^n$ be the one-parameter group of transformations on N^n generated by v. Let $f_t=\phi_t\circ f$ and put

$$E_v(t)=\frac{1}{p}\int \|df_t\|^p,$$

where all integrals will be taken over M^m with respect to the volume element dv_m on M^m. Let e_a, $a=1,\ldots,m$ be a local orthonormal frame field on M^m. Then we have [**1**, p. 5]

$$E_v''(0)=(p-2)\int \|df\|^{p-4}\Big\{\sum_a \langle\nabla_{f_*e_a}v,f_*e_a\rangle\Big\}^2$$
$$+\int \|df\|^{p-2}\sum_a \{\|\nabla_{f_*e_a}v\|^2+\langle R(v,f_*e_a)v,f_*e_a\rangle$$

where $\langle\,,\,\rangle$, ∇ and $R(x,y)=\{\nabla_x,\nabla_y\}-\nabla_{[x,y]}$ are the Riemannian metric, the connection and the curvature tensor respectively on N^n.

Now for any vector v in $\mathbb{R}^{n+r}$, we let $v=v^T+v^N$, where v^T is tangent to N^n and v^N is normal to N^n.

Recall that for any vector v normal to N^n, the shape operator corresponding to v denoted by A^v is defined by

$$A^v(x)=-(\tilde{\nabla}_x v)^T \qquad \text{for all } x\in TN.$$

A^v is symmetric and satisfies

$$\langle B(x,y),v\rangle=\langle A^v(x),y\rangle \qquad \text{for all } x,y\in TN.$$

Now consider a parallel vector field v in $\mathbb{R}^{n+r}$. The second variation corresponding to v^T is given by

$$E_{v^T}''(0)=(p-2)\int \|df\|^{p-4}\Big\{\sum_a \langle\nabla_{f_*e_a}v^T,f_*e_a\rangle\Big\}^2$$
$$+\int \|df\|^{p-2}\sum_a \{\|\nabla_{f_*e_a}v^T\|^2+\langle R(v^T,f_*e_a)v^T,f_*e_a\rangle\}.$$

We have [**2**, p. 381]

$$\nabla_{f_*e_a}v^T=A^{v^N}(f_*e_a)$$

and hence

$$\Big\{\sum_a \langle\nabla_{f_*e_a}v^T,f_*e_a\rangle\Big\}^2=\Big\langle\sum_a B(f_*e_a,f_*e_a),v^N\Big\rangle^2.$$

Now we consider the quadratic form Q on $\mathbb{R}^{n+r}$ defined by

$$\begin{aligned} Q(v) &= E''_{v^T}(0) \\ &= (p-2)\int \|df\|^{p-4}\left\langle \sum_a B(f_*e_a, f_*e_a), v^N \right\rangle^2 \\ &\quad + \int \|df\|^{p-2} \sum_a \{\|A^{v^N}(f_*e_a)\|^2 + \langle R(v^T, f_*e_a)v^T, f_*e_a\rangle\}. \end{aligned}$$

We shall compute the trace of Q. At a point $y \in M^m$, we want to evaluate the trace of the integrands at the point $x = f(y) \in N^n$. Since this trace is independent of the choice of an orthonormal basis for $\mathbb{R}^{n+r}$ at the point x, we choose an orthonormal basis $\{v_i, v_q\}$, $i = 1, \ldots, n$, $q = n+1, \ldots, n+r$ such that the v_i are tangent to N^n and the v_q are normal to N^n. A direct calculation as in [**2**, p. 382] shows that

$$\begin{aligned} \operatorname{trace}(Q) &= (p-2)\int \|df\|^{p-4}\left\| \sum_a B(f_*e_a, f_*e_a) \right\|^2 \\ &\quad + \int \|df\|^{p-2} \sum_{a,i} \{\|B(f_*e_a, v_i)\|^2 + \langle R(v_i, f_*e_a)v_i, f_*e_a\rangle\}. \end{aligned}$$

Using the Schwarz inequality, we have

$$\begin{aligned} \left\| \sum_a B(f_*e_a, f_*e_a) \right\|^2 &= \sum_{a,b} \langle B(f_*e_a, f_*e_a), B(f_*e_b, f_*e_b)\rangle \\ &\leq \sum_{a,b} \|B(f_*e_a, f_*e_a)\| \, \|B(f_*e_b, f_*e_b)\| \\ &\leq \sum_{a,b} h(x) \, \|f_*e_a\|^2 \, \|f_*e_b\|^2 \\ &= h(x)\left\{ \sum_a \|f_*e_a\|^2 \right\}^2 \\ &= \|df\|^2 \left\{ \sum_a h(x) \, \|f_*e_a\|^2 \right\}. \end{aligned}$$

Now suppose f is not a constant map. Then for each a such that $f_*e_a \neq 0$ at x, we put $f_*e_a = \|f_*e_a\| \, u_a$. We have

$$\operatorname{trace}(Q) \leq \int \|df\|^{p-2} \sum_a \|f_*e_a\|^2 \{(p-2)h(x) + \phi(u_a) - \operatorname{Ric}(u_a, u_a)\},$$

where $\sum_a$ is taken over those a such that $f_*e_a \neq 0$. Therefore, by the assumption in Theorem 2,

$$\operatorname{trace}(Q) < 0$$

and so f is not stable.

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