INTEGRALS INVOLVING BESSEL AND LEGENDRE FUNCTIONS

by J. S. LOWNDES

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1. The integrals

Felsen (1) has shown that when a plane wave is incident along the axis of a rigid cone of narrow apex angle an approximate expression for the scattered wave involves an integral of the form

$$\int_{-\infty}^{\infty} \mu e^{\mu\pi/2} \psi(\mu) \operatorname{sech}^2(\mu\pi) H_{i\mu}^{(1)}(kr) P_{-\frac{1}{2}+i\mu}(\cos\theta) d\mu, \tag{1}$$

where $\psi(\mu) = \frac{1}{4}+\mu^2$, $H_{i\mu}^{(1)}(kr)$, $0<r<\infty$, is a Bessel function of the third kind, k a constant and $P_{-\frac{1}{2}+i\mu}(\cos\theta)$, $0<\theta<\pi$, is the Legendre (conical) function of the first kind.

If we set $k = i\eta$ and use the result (**2b**, p. 5)

$$H_{i\mu}^{(1)}(i\eta r) = \frac{2}{\pi i} e^{\frac{\mu\pi}{2}} K_{i\mu}(\eta r), \tag{2}$$

we find that, apart from some multiplying constant, the integral assumes the form

$$\int_0^{\infty} \mu\psi(\mu) \tanh(\mu\pi) \operatorname{sech}(\mu\pi) K_{i\mu}(\eta r) P_{-\frac{1}{2}+i\mu}(\cos\theta) d\mu, \tag{3}$$

where $K_{i\mu}(\eta r)$ is a modified Bessel function.

A number of integrals related to (3) are

$$I_1^m(\alpha, \theta) = \int_0^{\infty} f(\mu) K_{i\mu}(\alpha) P_{-\frac{1}{2}+i\mu}^m(\cos\theta) d\mu, \tag{4}$$

$$I_2^m(\alpha, \theta) = \int_0^{\infty} \psi(\mu) f(\mu) K_{i\mu}(\alpha) P_{-\frac{1}{2}+i\mu}^m(\cos\theta) d\mu, \tag{5}$$

$$I_3^1(\alpha, \theta) = \int_0^{\infty} \frac{f(\mu)}{\psi(\mu)} K_{i\mu}(\alpha) P_{-\frac{1}{2}+i\mu}^1(\cos\theta) d\mu, \tag{6}$$

where $m = 0, 1$, $0<\alpha<\infty$, $0<\theta<\pi$, $f(\mu) = \mu \operatorname{sech}(\mu\pi) \tanh(\mu\pi)$ and $P_{-\frac{1}{2}+i\mu}^m(\cos\theta)$ is the associated Legendre function of the first kind.

As $\mu\to\infty$, we have (**2a**, p. 147)

$$P_{-\frac{1}{2}+i\mu}^m(\cos\theta) \sim \frac{1}{(2\pi\sin\theta)^{\frac{1}{2}}} (i\mu)^{m-\frac{1}{2}} e^{\mu\theta + i\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)}, \quad 0<\theta<\pi, \tag{7}$$

and for a fixed α (**2b**, p. 88),

$$K_{i\mu}(\alpha) \sim \left(\frac{2\pi}{\mu}\right)^{\frac{1}{2}} e^{-\frac{\mu\pi}{2}} \sin\left[\mu \log\left(\frac{2\mu}{\alpha}\right) - \mu + \frac{\pi}{4}\right]. \tag{8}$$

Hence the integrals (4), (5) and (6) exist and converge uniformly in θ, $0 < \theta < \pi$.

The value of $I_1^0(\alpha, \theta)$ is shown in (**3**) to be

$$I_1^0(\alpha, \theta) = \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \int_1^\infty \frac{e^{-\lambda\alpha}}{(\lambda + \cos\theta)} d\lambda = -\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{\alpha \cos\theta} Ei[-\alpha(1 + \cos\theta)], \tag{9}$$

where $Ei(-x)$ is the tabulated exponential integral.

It is the purpose of this paper to show that the values of the integrals (4), (5) and (6) may be deduced from the result (9).

2. Subsidiary results

For convenience we now list some useful results. From (**2a**, pp. 148, 174) we have

$$\left.\begin{aligned} \frac{d}{d\theta} P_{-\frac{1}{2}+i\mu}(\cos\theta) &= P^1_{-\frac{1}{2}+i\mu}(\cos\theta), && \text{(a)} \\ \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{d}{d\theta} P_{-\frac{1}{2}+i\mu}(\cos\theta) &= (\tfrac{1}{4} + \mu^2) P_{-\frac{1}{2}+i\mu}(\cos\theta), && \text{(b)} \end{aligned}\right\}. \tag{10}$$

If we denote

$$J_n(\alpha, \theta) = \int_1^\infty \frac{e^{-\lambda\alpha}}{(\lambda + \cos\theta)^n} d\lambda, \quad n = 1, 2, \ldots, \tag{11}$$

then an appropriate recurrence formula is

$$\left.\begin{aligned} (n-1) J_n(\alpha, \theta) &= e^{-\alpha}(1 + \cos\theta)^{1-n} - \alpha J_{n-1}(\alpha, \theta), \quad n > 1 \\ J_1(\alpha, \theta) &= -e^{\alpha \cos\theta} Ei[-\alpha(1 + \cos\theta)], \end{aligned}\right\}. \tag{12}$$

Also,

$$\frac{d}{d\theta} J_n(\alpha, \theta) = n \sin\theta J_{n+1}(\alpha, \theta), \tag{13}$$

for a fixed α.

3. Evaluation of the integrals

From equations (4) and the result (10*a*) we have for a fixed α,

$$I_1^1(\alpha, \theta) = \frac{d}{d\theta} I_1^0(\alpha, \theta). \tag{14}$$

An interchange of the order of integration and differentiation is permissible since the integral converges uniformly in θ, $0 < \theta < \pi$. Performing the differentiation, using the results (12) and (13),

$$I_1^1(\alpha,\theta)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\sin\theta J_2(\alpha,\theta)$$

$$=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\sin\theta\{(1+\cos\theta)^{-1}e^{-\alpha}+\alpha e^{\alpha\cos\theta}Ei[-\alpha(1+\cos\theta)]\}. \tag{15}$$

From (5) and (10)

$$I_2^0(\alpha,\theta)=\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta I_1^1(\alpha,\theta)$$

$$=\frac{1}{\sin\theta}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{d}{d\theta}[\sin^2\theta J_2(\alpha,\theta)].$$

Hence, using (12) and (13) we arrive at the result,

$$I_2^0(\alpha,\theta)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\{[1+\alpha(\cos\theta-1)]e^{-\alpha}$$
$$+\alpha(2\cos\theta-\alpha\sin^2\theta)e^{\alpha\cos\theta}Ei[-\alpha(1+\cos\theta)]\}. \tag{16}$$

Also from (5) and (10*a*)

$$I_2^1(\alpha,\theta)=\frac{d}{d\theta}I_2^0(\alpha,\theta).$$

Completing the differentiation as before, we find

$$I_2^1(\alpha,\theta)=\left(\frac{\alpha^3}{2\pi}\right)^{\frac{1}{2}}\sin\theta\{(\alpha\sin^2\theta-3\cos\theta-1)(1+\cos\theta)^{-1}e^{-\alpha}$$
$$-[(\alpha\cos\theta+2)^2-(\alpha^2+2)]e^{\alpha\cos\theta}Ei[-\alpha(1+\cos\theta)]\}. \tag{17}$$

Finally, from (6) and (10) we have,

$$\frac{d}{d\theta}\sin\theta I_3^1(\alpha,\theta)=\sin\theta I_1^0(\alpha,\theta).$$

Integrating over θ between the limits 0 and θ we obtain

$$\sin\theta I_3^1(\alpha,\theta)=\int_0^\theta\sin\phi I_1^0(\alpha,\phi)d\phi,$$

since $I_3^1(\alpha,\theta)$ is finite at $\theta=0$.
Substituting for $I_1^0(\alpha,\theta)$ from (9) and interchanging the orders of integration

$$\sin\theta I_3^1(\alpha,\theta)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int_1^\infty e^{-\lambda\alpha}d\lambda\int_0^\theta\frac{\sin\phi}{\lambda+\cos\phi}d\phi$$

$$=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int_1^\infty e^{-\lambda\alpha}\log\left(\frac{\lambda+1}{\lambda+\cos\theta}\right)d\lambda.$$

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Performing the integration we arrive at the result

$$I_3^1(\alpha,\theta)=\frac{(2\pi\alpha)^{-\frac{1}{2}}}{\sin\theta}\left\{e^{\alpha\cos\theta}Ei[-\alpha(1+\cos\theta)]-e^{\alpha}Ei(-2\alpha)-e^{-\alpha}\log\left(\frac{1+\cos\theta}{2}\right)\right\}. \tag{18}$$

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Department of Mathematics
University of Strathclyde
Glasgow

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