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# CYCLES AND HARMONIC FORMS ON LOCALLY SYMMETRIC SPACES

BY

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ABSTRACT. Two constructions of cohomology classes for congruence subgroups of unit groups of quadratic forms over totally real number fields are given and shown to coincide. One is geometric, using cycles, and the other is analytic, using the oscillator (Weil) representation. Considerable background material on this representation is given.

0. **Introduction**. This paper is a considerably expanded version of the Coxeter-James Lecture which I delivered at the Canadian Mathematical Congress in Victoria, British Columbia on December 12, 1981. The main theme of the paper is the equality between the harmonic forms arising as the Poincaré duals of totally geodesic cycles for certain locally symmetric spaces and the harmonic forms on these same spaces constructed via the oscillator (or Weil) representation. This relationship is precisely stated (in the orthogonal case) in the Main Theorem – Chapter III, Section 2. I also tried to present foundational material concerning the oscillator representation and theta functions and particularly the splitting of the metaplectic extension in a more invariant way than usual.

There are three chapters in the paper. The first chapter presents the cycles – generalizations of a closed geodesic for the Poincaré metric on a Riemann surface of genus greater than 1.

The second chapter deals with the oscillator representation and the functional equation of the theta function. I prove that if $Q$ is a non-degenerate quadratic form over $\mathbb{Z}/2$, $\Gamma_Q$ is the group of integral symplectic matrices which when reduced modulo 2 give isometries of $Q$ and $\tilde{\Gamma}_Q$ is the metaplectic extension then there is a theta multiplier $\epsilon : \tilde{\Gamma}_Q \rightarrow Q/\mathbb{Z}$. By this I mean that for each $Q$ there exists a tempered distribution $\Theta$ on $R^n$ so that for all $\gamma \in \tilde{\Gamma}_Q$: $\omega(\gamma)\,\Theta = \epsilon(\gamma)^{-1}\Theta$ where $\omega$ is the oscillator representation. $\Theta$ and $\epsilon$ depend on $Q$. In fact, Dennis Johnson and I have proved that the commutator quotient of $\tilde{\Gamma}_Q$ is $\mathbb{Z}/8$ (assuming that $Q$ is a form in 6 or more variables) and consequently $\epsilon$ takes values in the 8th roots of unity. This result combined with the remarks at the end of Chapter II, Section 3, gives an explanation for the presence of 8th roots

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of unity in the transformation law for the classical theta function (considered as a function of the period matrix). Indeed, in case $Q$ is the split form, $\epsilon$ is the usual theta multiplier. In this case I show that $\epsilon$ becomes trivial over the principal congruence subgroup of level 4, Theorem II.3.2, by using the usual Steinberg relations. In case $Q$ is the anisotropic form in 2 variables, then $\Gamma_Q = SL_2(Z)$ and $\epsilon : \tilde{S}L_2(Z) \to Z/24$ is the multiplier for the Dedekind eta function.

The third chapter is an exposition of joint work with S. Kudla. In this chapter, the cohomology classes associated by Poincaré duality to the special cycles of the first chapter are related to harmonic forms constructed from the theory of the oscillator representation developed in the second chapter.

In the first section of Chapter III, I construct a theta correspondence or lifting from Siegel modular forms to harmonic forms on locally symmetric spaces of orthogonal groups. This corrrespondence is realized by an integral transform. The kernel of this transform is constructed from a system of Schwartz functions obtained by applying an operator, generalizing the exterior derivative, to the Gaussian. This operator, called the Howe operator in [9] (its formula was suggested by Roger Howe), has many remarkable properties and merits further study.

In the second section of Chapter III, I prove the equality of the two spaces of forms referred to in the paragraph above. The key ingredient in the proof is a formula relating periods of forms on locally symmetric spaces of orthogonal groups to Fourier coefficients of the Siegel modular forms obtained by applying to them the adjoint of the lifting constructed above. This formula, denoted $(S)$ in this paper, is the analogue of the main formulas of Hirzebruch-Zagier [4] and Shintani [18]. It is proved in the last section of this paper.

My results with Kudla together with the recent injectivity results of Rallis [17] hold out the possibility of computing the dimensions of many cohomology groups of orthogonal, unitary and quaternion unitary locally symmetric spaces in terms of spaces of "classical" modular forms in certain tube domains. What remains to be done is to prove surjectivity of our lifting in some cases. This seems to be a very difficult problem but it is perhaps within reach of the powerful new techniques of representation theory.

To conclude, it would appear that the results and methods of this paper generalize to *finite volume* quotients of orthogonal symmetric spaces. Since this is rather surprising I will now sketch how this goes.

The lifting kernel $\theta_\varphi$ is a closed $nq$ form in the orthogonal variable so the lift maps into *closed* harmonic $nq$ forms and accordingly into the absolute cohomology of degree $nq$. The statement and proof of the Main Theorem must now be modified to use Poincare duality on a non-compact manifold. We now use the pairing, again denoted $[\ ,\ ]$, between $nq$ forms with arbitrary support and compactly supported $(p - n)q$ forms given by ($M$ denotes the locally symmetric space):

$$[\eta, \omega] = \int_M \eta \wedge \omega$$

Now define the Siegel modular form $\theta_\varphi(\eta)$ for $\eta$ compactly-supported of degree

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$(p - n)q$ as $[\eta, \theta_\varphi]$. One finds easily that:

$$(f, \theta_\varphi(\eta)) = [\theta_\varphi(f), \eta]$$

where the inner product on the left is the Petersson inner product on $S_{m/2}(\Gamma')$. Now consider the following two subspaces of the cohomology of degree $(p - n)q$ with compact supports. The first, $H_\theta^\perp$, is the space of all classes of closed compactly supported $(p - n)q$ forms which are orthogonal under $[\ ,\ ]$ to the image of $S_{m/2}(\Gamma')$. The second, $H_{\text{cycle}}^\perp$, is the space of all classes of closed compactly supported $(p - n)q$ forms with period zero over all the special cycles $C_\beta$ of positive type (see the discussion preceding the Main Theorem). Then it follows immediately from a suitable generalization of the formula $(S)$ that:

$$H_\theta^\perp = H_{\text{cycle}}^\perp$$

Then, by Poincaré duality, the subspace of the absolute cohomology of degree $nq$ spanned by the Poincaré duals of special cycles of positive type coincides with the space of lifts.

I would like to dedicate this paper, a summary of a number of years work, to my parents. I would like to express my gratitude to my collaborator, Steve Kudla, for a considerable amount of help with Chapter III. I would like to thank Dennis Johnson and Bill Dwyer for helpful conversations concerning Chapter II. They made me aware of the significance of quadratic forms over $Z/2$ and the groups $\Gamma_Q$. Finally, I have profited greatly from the ideas of Roger Howe.

## I. Special cycles in locally symmetric spaces

1. **Locally symmetric spaces**. A Riemannian manifold $D$ is a *symmetric space* if for each $x$ in $D$ there is an isometry $\theta$ of $D$ having $x$ as an isolated fixed-point. From this definition it follows that $D$ is homogeneous, so $D = G/K$ with $G$ the group of isometries of $D$. We assume that $G$ is a non-compact semi-simple Lie group (and $K$ a maximal compact subgroup). Then $D$ is contractible and has infinite volume.

A Riemannian manifold $M$ is a *locally symmetric space* if its universal cover is a symmetric space. From this definition, it follows that $M$ is the quotient of $D$ by a discrete subgroup $\Gamma \subset G$ acting properly discontinuously on $D$. Thus we may represent $M$ as a double coset space $M = \Gamma \backslash G/K$. The manifold $M$ is a space of type $K(\Gamma, 1)$ and consequently one has:

(I.1.1) $$H^*(M;R) = H^*(\Gamma; \mathbb{R})$$

(I.1.2) $$H^*(M; \tilde{V}) = H^*(\Gamma; V)$$

where $V$ is an $\mathbb{R}(\Gamma)$-module, $\tilde{V}$ is the associated local coefficient system and the right-hand side of (I.1.1) and (I.1.2) are abstract group cohomology groups.

To give some examples of the type of symmetric and locally symmetric spaces we are considering, we note that they include the hyperbolic plane and its quotients, the Riemann surfaces of genus greater than 1 equipped with the Poincaré metric. Hyperbolic space and its quotients give one generalization of these fundamental examples. Another generalization is the Siegel space and its quotient by $Sp_n(\mathbb{Z})$ which we now describe because it will play an important role in what follows.

We define the symplectic group $Sp_n(\mathbb{R})$ to be the isometry group of the standard skew-symmetric form on $\mathbb{R}^{2n}$; that is, the form $\langle\ ,\ \rangle$ satisfying $\langle e_i, f_j\rangle = \delta_{ij}$ where $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is the standard basis for $\mathbb{R}^{2n}$. We define an almost complex structure to be positive definite if the bilinear form $B$ on $\mathbb{R}^{2n}$ defined by $B(x,y) = \langle x, Jy\rangle$ is symmetric and positive definite. Then the space $D$ of positive definite complex structures on $\mathbb{R}^{2n}$ is the symmetric space for $Sp_n(\mathbb{R})$. Note that $B$ is symmetric if and only if $J \in Sp_n(\mathbb{R})$. From this it is easy to prove that $Sp_n(\mathbb{R})$ acts transitively on $D$. Moreover the isotropy group of the complex structure $J_0$ defined by $J_0(e_j) = f_j$, $J_0(f_j) = -e_j$ is $Sp_n(\mathbb{R}) \cap 0(2n)$, a maximal compact subgroup of $Sp_n(\mathbb{R})$ isomorphic to $U(n)$. The quotient of $D$ by $Sp_n(\mathbb{Z})$, the group of integral matrices in $Sp_n(\mathbb{R})$, is a very important space $-$ the moduli space for principally polarized abelian varieties. Another model for the previous symmetric space is the Siegel space $\mathfrak{H}_n$ of $n$ by $n$ complex symmetric matrices $\tau = u + iv$ with imaginary part $v$ positive definite. The complex structure $J_0$ corresponds to the matrix $iI_n$ where $I_n$ is the $n$ by $n$ identity matrix.

Other generalizations of the hyperbolic plane come from the symmetric spaces of the orthogonal groups. Let $V$ be an $n$ dimensional real vector space with a quadratic form $(\ ,\ )$ of signature $(p,q)$. Let $D$ be the open submanifold of the Grassmannian $G_q(\mathbb{R})$ consisting of the negative $q$-planes; that is, $q$-planes $Z$ such that the restriction of $(\ ,\ )$ to $Z$ is negative definite. Then $D$ is a symmetric space and $0(p,q)$, the orthogonal group of $(\ ,\ )$ acts transitively on $D$. To each negative $q$-plane $Z$ we may associate the involution $r_Z$ which is $-1$ on $Z$ and $+1$ on $Z^\perp$ the positive definite quadratic form $(\ ,\ )_Z$ on $V$ defined by:

$$(u,v)_Z = (r_Z u, v).$$

The form $(\ ,\ )_Z$ is said to be a Hermite majorant for $(\ ,\ )$. The subspace $m$ of all positive definite forms on $V$ defined by:

$$m = \{(\ ,\ )_Z : Z \in D\}$$

is called the space of Hermite majorants of $(\ ,\ )$. A positive definite form $R$ is in $m$ if and only if it majorizes $(\ ,\ )$; that is it satisfies $R(x,x) \geq |(x,x)|$ for all $x$ and it is minimal among all majorants. The identification of $D$ and $m$ is at the heart of Siegel's work on theta functions of indefinite quadratic forms.

We now describe in some detail the most general examples considered in [9]. Our later construction of harmonic forms via the Weil representation will apply to these examples. Let $\mathbb{F}$ represent either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the Hamilton quaternions $\mathbb{H}$. Let $\iota$ be the involution of $\mathbb{F}$ which is respectively the identity, complex conjugation or quaternionic conjugation. Let $(\ ,\ )$ be a hermitian (relative to $\iota$) form on $\mathbb{F}^m$ of signature $(p,q)$; hence $(\ ,\ )$ maps $\mathbb{F}^m \times \mathbb{F}^m$ to $\mathbb{F}$ and satisfies for $x$, $y$, $z \in \mathbb{F}^m$ and $c \in \mathbb{F}$:

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(i) $$(x + y, z) = (x, z) + (y, z)$$

(ii) $$(x, yc) = (x, y)c$$

(iii) $$(xc, y) = \iota(c)(x, y)$$

(iv) $$(y, x) = \iota((x, y)).$$

As in the previous example, let $D$ be the open submanifold of the Grassmannian $G_q(\mathbb{F})$ consisting of all negative $q$-planes; that is, $q$-planes $Z$ such that the restriction of ( , ) to $Z$ is negative definite. Then $D$ is a symmetric space and the group $G$ of isometries of ( , ) acts transitively on $D$. To obtain compact quotients of $D$ one must repeat the above construction taking $F$ to be respectively a real quadratic field $\varkappa = \mathbb{Q}(\sqrt{n})$, a totally imaginary extension of $\varkappa$. We require ( , ) to be $\varkappa$-valued and to be of signature $(p, q)$ at one completion of $\varkappa$ and positive definite at the other. For example one might take the following forms in the three cases:

(i) $$x_1^2 + \ldots + x_p^2 - \sqrt{n}(x_{p+1}^2 + \ldots + x_m^2)$$

(ii) $$|z_1|^2 + \ldots + |z_p|^2 + \sqrt{n}(|z_{p+1}|^2 + \ldots + |z_m|^2)$$

(iii) $$|q_1|^2 + \ldots + |q_p|^2 - \sqrt{n}(|q_{p+1}|^2 + \ldots + |q_m|^2)$$

Then we choose for $\Gamma$ a subgroup of finite index in the (algebraic) integral isometries of ( , ). Then $\Gamma \subset G$, $M = \Gamma \backslash D$ will always be compact and if $\Gamma$ contains no elements of finite order $M$ will be a manifold. For more details see Borel [1]. We will call these examples, the basic examples.

2. **Differential forms on a symmetric space**. In this section we give convenient representations of differential forms, the exterior derivative and the Hodge Laplacian on a locally symmetric space. Choose a maximal compact subgroup of $G$ (or a point on $D$). Let $g$ be the Lie algebra of $G$, $l$ the Lie algebra of $K$ and $p$ the orthogonal complement of $l$ for ( , ) the Killing form of $g$. Then the left invariant distribution on $G$ defined by $p$ is the horizontal distribution for the Riemannian connection on $M$. To be quite precise we should say that the frame bundle of $M$ reduces as a bundle with connection to the bundle $p : \Gamma \backslash G \to \Gamma \backslash G / K$ with the connection described above. Let $\{\omega_i\}$ be a basis for the left-invariant horizontal forms (elements of $p^*$, the dual of $p$). If $\eta$ is a $k$-form on $\Gamma \backslash G / K$ then $p^*\eta$ may be written in terms of $\omega_{i_1} \wedge \omega_{i_2} \wedge \ldots \wedge \omega_{i_k}$ for $i_1 < i_2 < \ldots < i_k$ according to $p^*\eta = \sum_I f_I \omega_I$ (here $I$ is a multi-index). Then the system of functions $\{f_I\}$ gives an element $\tilde{\eta} \in (C^\infty(\Gamma \backslash G) \oplus \Lambda^p p^*)^K$. That is, $\tilde{\eta}$ satisfies:

(I.2.1) $$\tilde{\eta}(\gamma g) = \tilde{\eta}(g)$$

(I.2.2) $$\tilde{\eta}(gk) = \sigma(k^{-1})\tilde{\eta}(g)$$

where $\sigma$ is the natural action of $K$ on $\Lambda^p p^*$. Conversely, every element $\{f_I\} \in (C^\infty(\Gamma \backslash G) \oplus \Lambda^p p^*)^K$ gives rise to a $p$-form $\eta$ on $M$ satisfying $p^*\eta = \Sigma f_I \omega_I$. An identical situation holds for sections of any homogeneous bundle $E_\sigma$ defined by a representation $\sigma$ of $K$.

We now record how the exterior derivative $d$ and the Hodge Laplacian $\Delta$ carry over to this representation of a differential form. Let $\{E_i\}$ be a basis for the left invariant vector fields which is dual to $\{\omega_i\}$. Then:

$$\tilde{d}\eta = \Sigma E_i \otimes A(\omega_i)\cdot\tilde{\eta}$$

where $A(\omega_i):\Lambda^p p^* \to \Lambda^{p+1}p^*$ is the operator of exterior multiplication by $\omega_i$.

Let $\{X_i : i = 1, \ldots, n\}$ be a basis for $l$ satisfying $(x_i, x_j) = -\delta_{ij}$ and $\{Y_j : j = 1, \ldots, m\}$ an orthogonal basis for $p$. Then $C = -\sum_{i=1}^{m} Y_j^2 + \sum_{i=1}^{n} X_i^2$ is a second order differential operator on $G$ which commutes with the left and right actions of $G$. Hence $C$ induces an operator on $(C^\infty(\Gamma\backslash G) \otimes \Lambda^p p^*)^K$ and we have:

(I.2.3) $$\Delta\tilde{\eta} = (C \otimes I)\tilde{\eta} \quad \text{(Kuga's Lemma).}$$

Finally we will need a formula in terms of $\tilde{\eta}$ for the period of a differential $p$-form $\eta$ over a $p$-dimensional sub-symmetric space $\Gamma_1\backslash G_1/K_1 \subset \Gamma\backslash G/K$. Let $p_1 \subset p$ be the complement to $l_1$ in $g_1$. Choose an orthonormal basis for $p^*$ so that the first $p$ basis vectors $\omega_1, \omega_2, \ldots, \omega_p$ form an oriented orthonormal basis for $p_1^*$. Then $p^*\eta = \Sigma\, \eta_1\omega_I$. Let $I_0$ be the multi-index $\{1, 2, \ldots, p\}$. Then we have:

(I.2.4) $$\int_{\Gamma_1\backslash G_1/K_1} \eta = \frac{1}{\text{vol } K_1}\int_{\Gamma_1\backslash G_1} \eta_{I_0}.$$

3. **The cohomology of locally symmetric spaces**. We begin our study of cohomology by noting that every form on $D$ which is invariant under $G$ is harmonic by I.2.3. Thus any non-zero invariant form represents a non-zero cohomology class in $M = \Gamma\backslash D$. Matsushima, Osaka Journal, 1962, proved that the cohomology of "low degree" relative the dimension of $M$ consists entirely of such classes. Thanks to recent work of Borel-Wallach [2] and Zuckerman [21] we now know that $H^i(M, \mathbb{R})$ consists entirely of such classes provided $i$ is less than an important geometric invariant $r$ of $D$ known as the rank – the dimension of a maximal isometrically embedded totally geodesic flat torus. For many (but not all) $D$ there are compact quotients $M = \Gamma\backslash D$ and classes in $H^r(\Gamma\backslash D; \mathbb{R})$ which are not represented by invariant forms. The construction of some such classes is the subject of the next section.

4. **Special cycles in locally symmetric spaces**. Let $\pi: D \to \Gamma\backslash D$ be the projection map in what follows. Let $\sigma_1$ be an isometry of order 2 so that $\sigma_1\Gamma\sigma_1 = \Gamma$ and let $D_1$ be the fixed-point set of $\sigma_1$ in $D$. Let $M_1 = \pi(D_1)$. Then we say $M_1$ is a *special cycle* in $\Gamma\backslash D$. We first note a very useful lemma which makes special cycles pleasant to deal with. Let $\Gamma_1 = \{\gamma \in \Gamma : \sigma_1\gamma\sigma_1 = \gamma\}$.

LEMMA I.4.1. (Harris Jaffee). *If $\Gamma$ is torsion free than $\pi(D_1) = \Gamma_1\backslash D_1$.*

PROOF. Suppose $x$ and $y$ in $D_1$ have the same image under $\pi$. Then there exists $\gamma \in$

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$\Gamma$ so that $\gamma x = y$. Then $\tau = \gamma^{-1}\sigma_1\gamma\sigma_1$ is an element of $\Gamma$ fixing $x$. Consequently $\tau$ is the identity and $\gamma \in \Gamma_1$. With this the Jaffee lemma is proved.

COROLLARY. *$M_1$ is itself a locally symmetric space $M_1 = \Gamma_1 \backslash G_1 / K_1$ where $G_1$ (resp. $K_1$) is the centralizer of $\sigma_1$ in $G$ (resp. $K$).*

We now make a conjecture which Raghunathan and I proved in many cases.

CONJECTURE. For $\sigma_1$, $\Gamma$ as above there exists a subgroup $\Gamma'$ of finite index in $\Gamma$ so that $\pi'(D_1)$ is not a boundary (here $\pi' = D \to \Gamma' \backslash D$ is the projection).

The conjecture seems a reasonable one for several reasons. First, it is consistent with all known vanishing theorems. Second, it has been proved in a large number of cases. Finally every special cycle comes with a complementary special cycle as we now explain. We assume there exist points on $D_1$, to be called rational points, so that the associated Cartan involutions $\theta$ have the property that $\theta\Gamma\theta \cap \Gamma$ has finite index in $\Gamma$. This is always the case for the basic examples.

LEMMA I.4.2. (Millson-Raghunathan [13]). *Every special cycle has a complementary special cycle.*

PROOF. Choose a rational point $x \in D_1$. Then the reflection $\theta$ at $x$ has the property that $\theta\Gamma\theta \cap \Gamma$ has finite index in $\Gamma$. We consider the isometry $\sigma_2 = \theta\sigma_1$. We note $\sigma_2^2 = \theta\sigma_1\theta\sigma_1$ fixes $x$ and the tangent space to $D$ at $x$. Hence $\sigma_2$ is an involution and $\sigma_2\Gamma\sigma_2 \cap \Gamma$ has finite index in $\Gamma$. By replacing $\Gamma$ by a subgroup of finite index in $\Gamma$ we may assume $\theta$ and $\sigma_2$ normalize $\Gamma$. Consequently, if $D_2$ denotes the fixed-point set of $\sigma_2$ in $D$ then $M_2 = \pi(D_2)$ is compact. Clearly it has dimension complementary to that of $M_1$ and the lemma is proved.

We remark that $D_1$ and $D_2$ intersect only at $z$. However, $M_1$ and $M_2$ can have many intersections corresponding to $\gamma \in \Gamma$ so that $\gamma D_1 \cap D_2 \neq \emptyset$.

In [13], Raghunathan and I studied the intersection number of $M_1$ and $M_2$. We were able to prove in many cases that it was non-zero. We also proved that if the intersection number was non-zero then there was a finite cover of $M$ so that the Poincaré dual of any component of the inverse image of $M_1$ could not be represented by an invariant form.

5. **Linear cycles in the basic examples**. For the basic examples, the special cycles have a geometric interpretation in terms of the linear structure of the Grassmannian which makes them appear to be the analogue of certain Schubert cycles.

We choose a $\varkappa$-rational orthogonal splitting $\mathbb{F}^n = X \oplus X^\perp$ where $X$ has $\mathbb{F}$ dimension $k$ and signature $(r, s)$ at the indefinite completion of $(\ ,\ )$. Thus $X^\perp$ has $\mathbb{F}$ dimension $n - k$ and signature $(p - r, q - s)$. Let $\sigma_1$ be the involution which has $+1$ eigenspace equal to $X$ and $-1$ eigenspace equal to $X^\perp$. Then $D_1$ is the set of negative $q$-planes $Z$ which are compatible with the above splitting in the sense that

$$Z = Z \cap X + Z \cap X^\perp.$$

We will denote this subspace $D_X$. To get the complementary cycle choose a rational $q$-plane $Z_0$ in $D$. Then we have a splitting $\mathbb{F}^n = Z_0 \oplus Z_0^\perp$ and $\theta$, the reflection at $Z_0$ is represented by the linear transformation which is $+1$ on $Z_0$ and $-1$ on $Z_0^\perp$. We find that $D_2$ is the set of negative $q$-planes compatible with the splitting $\mathbb{F}^n = U \oplus U^\perp$ where $U = X \cap Z_0 + X^\perp \cap Z_0^\perp$ and $U^\perp = X \cap Z_0^\perp + X^\perp \cap Z_0$. Note that the above decomposition of $\mathbb{F}^n$ is just the eigenspace decomposition of $\mathbb{F}^n$ relative to $\sigma_2 = \theta\sigma_1$. The real dimension of $D$ is $pq$, $2pq$ or $4pq$ in three cases and the dimension of $D_1$ is $rs + (p - r)(q - s)$, $2[rs + (p - r)(q - s)]$ or $4[rs + (p - r)(q - s)]$ in the three cases.

The most important special case of the above construction is the case in which $X$ is positive definite. In this case $Z \cap X = \{0\}$ and consequently $D_1 = \{Z \in D : Z \subset X^\perp\}$. We will say cycles obtained from such $X$ have positive type. If $X$ is negative definite we say $D_1$ has negative type. Finally we will say other cycles (where $X$ is indefinite) have mixed type. We will often write $D_X$ for $D_1$ and $C_X$ for $\pi(D_X)$. In case $X$ is positive and $x$ is a spanning set for $X$, we will write $D_x$ and $C_x$ in place of $D_X$ and $C_X$.

The considerable difference between the cycles of pure type and of mixed type is illustrated in the case $G = SO(2, 2)$. Then $D$ is a product of two hyperbolic planes. The cycles of pure type give rise to algebraic curves − Hirzebruch-Zagier cycles [4], whereas those of mixed type give rise to totally real tori. They are transcendental cycles and have been used by Oda [15].

In case $\mathbb{F} = \mathbb{R}$ and $X$ is a positive 1 dimensional space then $D_1$ is the set of all negative $q$-planes perpendicular to a fixed line. If $q$ is even, then Millson-Raghunathan [13] proved that there are uniform $\Gamma \subset SO(p, q)$ so that the cycles $M_1$ and $M_2$ had non-zero intersection number. Consequently there is a class in $H^q(\Gamma\backslash D; \mathbb{R})$ which is not represented by an invariant form. Since the rank of $D$ is $q$, this implies that the Borel-Wallach, Zuckerman theorem is best possible for orthogonal groups of even rank. For subgroups of $0(n, 1)$ there is a very simple geometric argument proving that $H^1(\Gamma\backslash D; \mathbb{R}) \neq \{0\}$ − see Millson [12]. By Rallis [17], we now know that the Borel-Wallach, Zuckerman theorem is best possible for all orthogonal groups.

## II. The oscillator representation and theta functions

1. **The existence of the oscillator representation**. Let $W$ be a vector space over $\mathbb{R}$ of dimension $2n$ and let $\langle\ ,\ \rangle$ by a symplectic form on $W$. Define the Heisenberg group $H(W)$ by:

$$H(W) = W \times S^1 \text{ as a set}$$

and has group law

$$(w_1, t_1)\cdot(w_2, t_2) = (w_1 + w_2, e^{i\pi\langle w_1, w_2\rangle} t_1 t_2)$$

Thus $H(W)$ is a group extension $S^1 \to H(W) \to W$ whose characteristic class is represented relative the above product decomposition by the cocyle:

$$\alpha(w_1, w_2) = e^{i\pi\langle w_1, w_2\rangle}$$

Let $\chi$ be the identity character of $S^1$ and $\lambda = 2\pi i$ (it is convenient to retain $\lambda$ as a parameter). Then we have the famous theorem:

Stone-von Neumann theorem. *There is one and only one equivalence class of irreducible unitary representations of $H(W)$ with central character $\chi$.*

We will denote this class by $\rho_\chi$. We now give a realization of $\rho_\chi$. Let $F$ be a Lagrangian, i.e. maximal isotropic, subspace of $W$. We consider the abelian subgroup $N = F \times S^1$ of $H(W)$. We extend $\chi$ to a character of $N$ by making it trivial on $F$. Then $\operatorname{ind}_N^{H(W)}\chi$ is a unitary representation of $H(W)$. By definition the space of this representation is the space of functions $\Phi$ on $H(W)$ satisfying:

(*i*) $\Phi(xn) = \chi(n)^{-1}\Phi(x)$

(*ii*) $|\Phi|^2$ is square integrable on $H(W)/N$

We denote the space of such functions $\Phi$ by $W(F)$. Then $H(W)$ acts on $W(F)$ by left translation. The following theorem is proved in Lion-Vergne [11], page 19.

THEOREM II.1.1 *$W(F)$ is irreducible.*

REMARK. $W(F)$ is called the Schrödinger model of $\rho_\chi$. Let $E$ be a Lagrangian complement to $F$.

Clearly any function in $W(F)$ is determined by its restriction to $E$ and any square integrable function on $E$ may be extended in exactly one way to an element of $W(F)$. We may accordingly realize the Schrödinger model on $L^2(E)$ with $H(W)$ acting by twisted translations; that is, if $\varphi \in L^2(E)$ then:

$$\rho_\chi(y)\varphi(x) = e^{\lambda\langle y,x\rangle}\varphi(x) \text{ for } y \in F$$

$$\rho_\chi(x_0)\varphi(x) = \varphi(x - x_0) \text{ for } x_0 \in E$$

$$\rho_\chi(t)\varphi(x) = t\varphi(x)$$

We choose a symplectic basis $\{e_1, e_2, \ldots, e_n; f_1, f_n\}$ for $W$ compatible with the splitting $W = E \oplus F$. We define coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ on $W$ dual to the previous basis. The infinitesmal action of the Lie algebra of $H(W)$ is then given by:

$$d\rho_\chi(e_j) = -\frac{\partial}{\partial q_1}$$

$$d\rho_\chi(f_j) = -\lambda q_j$$

$$d\rho_\chi(\eta) = \lambda I$$

where we have written the Lie algebra $n$ of the Heisenberg group as $W \oplus \mathbb{R}\eta$.

Let $Sp(W)$ be the isometry group of $\langle\ ,\ \rangle$. The action $g\cdot(w,t) = (g(w),t)$ for $g \in Sp(W)$ and $(w,t) \in H(W)$ embeds $Sp(W)$ into the automorphism group of $H(W)$. We will identify $Sp(W)$ with $Sp_n(\mathbb{R})$ using the above basis.

We choose a realization $H_\chi$ for the space of $\rho_\chi$. Since $Sp(W)$ acts on $H(W)$ we obtain a new representation $\rho_\chi^g$ on $H_\chi$ for each $g \in Sp(W)$ defined by $\rho_\chi^g(h) = \rho_\chi(g(h))$. But $Sp(W)$ leaves the center of $H(W)$ fixed, consequently, the central characters of $\rho_\chi$ and $\rho_\chi^g$ are the same. Then by the Stone-von Neumann theorem there is a unitary operator $\omega_\chi(g)$ satisfying:

$$\omega_\chi(g)\rho_\chi(h)\omega_\chi(g)^{-1} = \rho_\chi(g(h)).$$

The mapping $g \to \omega_\chi(g)$ is easily seen to be a projective representation of $Sp(W)$: that is, it is a homomorphism into the unitary group modulo scalars which we denote $PU(H_\chi)$. Then we have the following diagram:

$$\begin{array}{ccc} S^1 & \xrightarrow{id} & S^1 \\ \downarrow & & \downarrow \\ \overline{Sp}(W) & \xrightarrow{\omega_\chi} & U(H_\chi) \\ \downarrow & & \downarrow \\ Sp(W) & \xrightarrow{\omega_\chi} & PU(H_\chi) \end{array}$$

where $\overline{Sp}(W)$ is the "pull-back" extension under $\omega_\chi$ of the natural extension of $PU(H_\chi)$.

Let $\tilde{Sp}(W)$ denote the commutator subgroup of $\overline{Sp}(W)$. Since $Sp(W)$ is perfect we have a surjective mapping $\pi$: $\tilde{Sp}(W) \to Sp(W)$.

LEMMA II.1.1. *The kernel of* $\pi$ *is finite.*

PROOF: Let $l$ be the Lie algebra of the circle, $sp$ (resp. $\overline{sp}$) the Lie algebra of $Sp(W)$ (resp. $\overline{Sp}(W)$). Then we have $l \to \tilde{sp} \to sp$. Since $sp$ is simple, this extension splits and since $l$ is one dimensional and $sp$ has no characters the action of $sp$ on $l$ is trivial. Thus $\overline{sp} = sp \oplus l$ and $\pi$ is an isomorphism of Lie algebras. Thus the kernel of $\pi$ is a discrete subgroup of $S^1$ and consequently is finite.

We now examine more closely the infinitesimal oscillator representation $d\omega_\chi$: $sp \to \operatorname{End} H_\chi$. The algebra $sp$ acts on $n$, the Lie algebra of $H(W)$, by differentiating the action of $Sp(W)$ on $HW$). We write this action as $[x, y]$ for $x \in sp$, $y \in n$.

LEMMA II.1.2. $d\omega_\chi$ *is the unique Lie algebra homomorphism taking* $sp \to \operatorname{End} H_\chi$ *and satisfying:*

$$d\rho_\chi([x, y]) = [d\omega_\chi(x), d\rho_\chi(y)] \text{ for all } x \in sp,\ y \in n.$$

PROOF: By the previous analysis $d\omega_\chi$ satisfies the above identity. If $T$ is any other homomorphism then $d\omega_\chi - T$ must be a scalar homorphism as $d\rho_\chi$ is irreducible. But $sp$ is equal to its commutator sublagebra and the lemma is proved.

Now let $\Delta$ be the kernel of $\pi$. Then we have:

$$\begin{array}{ccccc} \Delta & \longrightarrow & S^1 & \longrightarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Sp}(W) & \to & \overline{Sp}(W) & \to & U \\ \downarrow & & \downarrow & & \downarrow \\ Sp(W) & \to & Sp(W) & \to & PU \end{array} \qquad *$$

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Thus, $\omega_\chi$ lifts to a linear representation of $\widetilde{Sp}(W)$.

We now wish to determine $\Delta$. For any finite subgroup $\Delta \subset S^1$ we will say the extension $S^1 \to \overline{Sp}(W) \to Sp(W)$ is reducible to $\Delta$ if there is a group $\widetilde{Sp}(W)$ so that the diagram * holds. The following two lemmas allow us to study the same problem for the extension induced over any subgroup $H \subset Sp(W)$ provided $\pi_1(H)$ maps onto $\pi_1(Sp(W))$.

LEMMA II.1.3. *The extension $S^1 \to \overline{Sp}(W)$ is reducible to $\Delta$ if and only if the quotient extension $S^1/\Delta \to Sp(W)/\Delta \to Sp(W)$ is trivial.*

PROOF: We denote the two maps in the diagram * by $i: \widetilde{Sp}(W) \to \overline{Sp}(W)$ and $p: \overline{Sp}(W) \to \overline{Sp}(W)/\Delta$. If $\widetilde{Sp}(W)$ exists then $p \circ i(\widetilde{Sp}(W))$ splits the above extension and if $\sigma$ is a section then defining $\widetilde{Sp}(W) = p^{-1}(\sigma(Sp(W)))$ gives the diagram *.

LEMMA II.1.4. *Let $G$ be a subgroup satisfying $\pi_1(G) \to \pi_1(Sp(W))$ is onto. Then the restriction of $S^1 \to \overline{Sp}(W) \to Sp(W)$ to $G$ reduces to $\Delta$ if and only if the original extension $S^1 \to \overline{Sp}(W) \to Sp(W)$ reduces to $\Delta$.*

PROOF: We have a diagram:

$$\begin{array}{ccc} S^1/\Delta & \to & S^1/\Delta \\ \downarrow & & \downarrow \\ \overline{G}/\Delta & \to & \overline{Sp}(W)/\Delta \\ \downarrow & & \downarrow \\ G & \to & Sp(W) \end{array}$$

Since $\pi_1(G)$ maps onto $\pi_1(Sp(W))$, the triviality of the left-hand extension implies that of the right by Shapiro [18]. This proves the "only if" part of the lemma. The converse is obvious.

We now study $d\omega_\chi$. It will be useful to identify the space of symmetric 2-tensors on $W$, to be denoted $S^2W$ with $sp$. This is done by the mapping $S^2W \to \text{End } W$ which associates to the symmetric product $w_1 \circ w_2 = w_1 \otimes w_2 + w_2 \otimes w_1$ in $S^2W$ the endomorphism sending $v \in W$ to $\langle w_1, v\rangle w_2 + \langle w_2, v\rangle w_1$.

There is a remarkably simple way to obtain the infinitesimal oscillator representation. We define the Weyl algebra $\mathcal{W}$ to be the quotient of the tensor algebra $T(W)$ by the ideal generated by $\{w_1 \otimes w_2 - w_2 \otimes w_1 - \lambda\langle w_1, w_2\rangle 1\}$. Then $\mathcal{W}$ is isomorphic to the quotient of $\mathcal{U}(n)$, the universal enveloping algebra of $n$ by the ideal generated by $\{n - \lambda 1\}$. It is also isomorphic under $d\rho_\chi$ to the algebra of polynomial coefficient differential operators on $F$. $\mathcal{W}$ is an associative algebra and hence has a Lie algebra structure with $[x, y] = xy - yx$. The algebra $\mathcal{W}$ inherits a filtration $\{\mathcal{W}^i\}$ from the filtration on $\mathcal{U}(n)$. Then $\mathcal{W}^2$ is a sub-algebra and $\mathcal{W}^1 = n = W \oplus \mathbb{R}$ is an ideal. It is clear that $\mathcal{W}^2/\mathcal{W}^1$ is isomorphic to $S^2W$. Moreover we have a canonical linear section $j: S^2W \to \mathcal{W}^2$ given by $j(x \circ y) = \dfrac{1}{2\lambda}(x \otimes y + y \otimes x)$. The image of $j$ is stable for the bracket operation on $\mathcal{W}$ and $j$ becomes an algebra isomorphism between $sp(W)$ and $j(S^2W)$. Also if $s \in S^2W$ the action of $j(s)$ on $\mathcal{W}^1$ is the same as the usual action of

$sp(W)$ on $n = W \oplus \mathbb{R}$.

Now $d\rho_\chi$ extends to $\mathcal{W}$ and of course satisfies for $s \in S^2W$, $x \in n$:

$$d\rho_\chi([j(s), x]) = [d\rho_\chi(j(s)), d\rho_\chi(x)]$$

By Lemma II.1.5 we have $d\rho_\chi(j(s)) = d\omega_\chi(s)$. Noting that if $s = x \circ y$ we have $j(s) = \frac{1}{2\lambda}(x \otimes y + y \otimes x)$ and hence

$$d\rho_\chi(j(s)) = \frac{1}{2\lambda}(d\rho_\chi(x)d\rho_\chi(y) + d\rho_\chi(y)d\rho_\chi(x))$$

we obtain a useful formula.

Lemma II.1.5.

$$d\omega_\chi(x \circ y) = \frac{1}{2\lambda}(d\rho_\chi(x)d\rho_\chi(y) + d\rho_\chi(y)d\rho_\chi(x))$$

This formula may be loosely stated as follows: we obtain the infinitesmal oscillator representation by extending the infinitesmal Stone-von Neumann representation to the quadratic elements in the Weyl algebra so that anti-commutators are preserved.

We can now compute the finite group $\Delta$ recalling now that $\lambda = 2\pi i$. Indeed by Lemma II.1.4 it is enough to consider the metaplectic extension pulled all the way back to the circle $SO(2)$ which is the maximal compact subgroup of $Sp(P)$ where $P \subset W$ is the plane spanned by $\{e_1, f_1\}$. We identify $\mathbb{R}$ with the Lie algebra of $SO(2)$ by mapping $t$ to $(-t/2)(e_1^2 + f_1^2)$ (the element $(-1/2)(e_1^2 + f_1^2)$ in $S^2(P)$ corresponds to the element $J$ of $Sp(P)$ satisfying $J(e_1 = f_1, J(f_1) = -e_1)$. Then the exponential map from $\mathbb{R}$ to $SO(2)$ is given by $\exp(t) = e^{2\pi tJ}$. Thus, the kernel of the exponential map is generated by $(-1/2)(e_1^2 + f_1^2)$. By Lemma II.1.5, this element acts by $(-1/4\pi i)((\partial^2/\partial q_1^2) - 4\pi^2 q_1^2)$. The eigenfunctions of this operator on the Schwartz space are the Hermite functions and the eigenvalues are half-integer multiples of $i$. Thus, $e^{4\pi J}$ acts trivially and $d\omega_\chi|\mathbb{R}$ descends to a representation of the 2-fold cover of $SO(2)$. We obtain the famous theorem of Segal and Shale.

Theorem. *The projective representation $\omega_\chi$ lifts to a linear representation of the 2-fold cover of $Sp(W)$.*

Remark. We will use the symbol $\omega_\chi$ for both the projective representation of $Sp(W)$ and the linear representation of the 2-fold cover $\widetilde{Sp}(W)$. The representation $\omega_\chi$ is che oscillator (or Weil) representation. $\widetilde{Sp}(W)$ is called the metaplectic group and will usually be denoted $Mp(W)$. The extension $Mp(W) \to Sp(W)$ will be called the metaplectic extension.

Let $P$ be a symplectic plane and $G(P)$ the two element subgroup of $Sp(W)$ consisting of the identity and the element $\iota$ which is $-1$ on $P$ and $+1$ on the orthogonal complement of $P$ in $W$. Then we have the following lemma.

LEMMA II.1.6. *The metaplectic extension restricted to $G(P)$ is not trivial.*

PROOF: We have just seen that the two-fold cover of the circle $SO(2) \subset Sp(P)$ induced by the metaplectic extension is the non-trivial 2-fold cover of the circle by itself. But the inclusion $G(P) \subset SO(2)$ is just the usual inclusion $\{\pm 1\} \subset S^1$. The lemma follows.

COROLLARY. *The metaplectic extension remains non-trivial upon restriction to any subgroup which contains any $G(P)$ (for example the integral symplectic group or its congruence subgroup of level 2).*

2. **Some elements in the metaplectic group**. In the next section we will need to make computations with certain elements of $Mp(W)$ which we define in the paragraphs below. We will use the notation that if $\tilde{G}$ is a covering group of a group $G$ and $\tilde{g} \in \tilde{G}$ then the image of $\tilde{g}$ in $\tilde{G}$ will be denoted $g$. Let $P$ be the subgroup of $Sp(W)$ which is the stabilizer of $F$. Let $N$ be the subgroup of $P$ consisting of those elements acting trivially on $F$ and $M$ be the stabilizer of $E$ and $F$. Then $P = M \cdot N$. Let $P^0$ denote the identity component of $P$.

LEMMA II.2.1. *The metaplectic extension restricted to $P^0$ is trivial.*

PROOF. The lemma will follow if we can show that the mapping on fundamental groups $\pi_1(P^0) \to \pi_1(Sp(W))$ is trivial. But $P^0$ has $SO(n)$ as a deformation retract and $Sp(W)$ has $U(n)$ as a deformation retract. Thus the above inclusion is equivalent to $\pi_1(SO(n)) \to \pi_1(U(n))$. But the inclusion from $SO(n)$ into $U(n)$ factors through $SU(n)$ which is simply-connected.

Let $s$ be a continuous section of the induced extension of $P^0$. We recall that the metalinear group $ML_n(\mathbb{C})$ is defined to be the fiber product of the double cover of $\mathbb{C}^*$ and the determinant map, $\det: GL_n(\mathbb{C}) \to \mathbb{C}^*$. Hence, we have the fiber square:

$$\begin{array}{ccc} ML_n(\mathbb{C}) & \to & \mathbb{C}^* \\ \downarrow & & \downarrow \\ GL_n(\mathbb{C}) & \to & \mathbb{C}^* \end{array}$$

The upper horizontal arrow we denote $\sqrt{\det}$ for it satisfies $(\sqrt{\det \tilde{g}})^2 = \det g$. We get corresponding extensions of $U(n)$ and $GL_n(\mathbb{R})$ denoted $MU(n)$ and $ML_n(\mathbb{R})$ respectively. We note that if $m \in GL_n^+(\mathbb{R})$, that is $\det m > 0$, then $\sqrt{\det s(m)} > 0$. We denote $s(GL_n^+(\mathbb{R}))$ by $ML_n^+(\mathbb{R})$. We will identify $GL_n^+(\mathbb{R})$ and $ML_n^+(\mathbb{R})$ using $s$.

We now construct some elements in $s(P^0)$. We let $X_{ij}(\lambda)$ for $\lambda \in \mathbb{R}$ and $1 \leqq i, j \leqq n$ with $i \neq j$ denote the matrix in $P^0$ with $\lambda$ in the $i, j$ position, $-\lambda$ in the $n + j$, $n + i$ position, 1's along the diagonals and zeroes elsewhere. Let $Y_{ij}(\lambda)$ for $1 \leqq i < j \leqq n$ denote the matrix in $P^0$ with $\lambda$ in the $i$, $n + j$ position and $\lambda$ in the $j$, $n + i$ position, 1's along the diagonal and zeroes elsewhere.

Let $Y_{ii}(\lambda)$ for $1 \leqq i \leqq n$ denote the matrix in $P^0$ with $\lambda$ in the $i$, $n + i$ position, 1's along the diagonal and zeroes elsewhere. Finally let $\tilde{X}_{ij}(\lambda) = s(X_{ij}(\lambda))$, $\tilde{Y}_{ij}(\lambda) = s(Y_{ij}(\lambda))$ and $\tilde{Y}_{ii}(\lambda) = s(Y_{ii}(\lambda))$. We note, since $s$ is a homomorphism, any relation among $X_{ij}$, $Y_{ij}$ and $Y_{ii}$ gives rise to the corresponding relation among $\tilde{X}_{ij}$, $\tilde{Y}_{ij}$ and $\tilde{Y}_{ii}$. We also need the matrices in $Sp_n(\mathbb{R})$ given by: $n(u) = \begin{pmatrix} 1_n & u \\ 0 & 1_n \end{pmatrix}$ where $u$ is a symmetric $n$ by $n$ matrix, $a(v) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$ where $v$ is an invertible $n$ by $n$ matrix, and $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$.

We also let $\tilde{n}(u) = s(n(u))$, $\tilde{a}(v) = s(a(v))$, and $\tilde{J}$ be the matrix in the connected component of the inverse image of $SU(n)$ in $Mp_n(\mathbb{R})$ which covers $J$. We put ${}'\tilde{n}(-u) = \tilde{J}n(u)\tilde{J}^{-1}$. Hence ${}'\tilde{n}(u)$ covers ${}'n(u)$.

LEMMA II.2.2. *We have the following relations in* $\widetilde{Sp}_n(\mathbb{Z})$:

$$[\tilde{X}_{ij}(1), \tilde{Y}_{ij}(1)] = \tilde{Y}_{ii}(2) \qquad \text{(R1)}$$

$$[\tilde{X}_{ij}(1), \tilde{Y}_{jj}(2)] = \tilde{Y}_{ii}(2)\,\tilde{Y}_{ij}(2) \qquad \text{(R2)}$$

PROOF: The corresponding relations for $X_{ij}$, $Y_{ij}$, $Y_{ii}$ may be verified by matrix multiplication or by observing that they are Steinberg relations for $Sp_n(\mathbb{R})$. The above relations follow upon applying the section $s$.

3. **Splitting the metaplectic extension, the theta distribution and $\kappa^{1/2}$.** We now choose a lattice $L \subset W$ so that $\langle\ ,\ \rangle$ is integral on $L$ and is unimodular. We take up the question of the behaviour of the metaplectic extension upon restriction to discrete subgroups $\Gamma \subset Sp(L)$ where $Sp(L)$ denotes the subgroup of $Sp(W)$ which stabilizes $L$. We have induced extensions $\mathbb{Z}/2 \to \tilde{\Gamma} \to \Gamma$ and $S^1 \to \bar{\Gamma} \to \Gamma$ which we denote $*$. We recall our observation of the previous section that to split $*$ it is necessary and sufficient to lift $\omega_\chi|\Gamma$ to a linear representation of $\Gamma$.

We will construct a concrete model for $\rho_\chi$ and $\omega_\chi$ called the lattice model. The inclusion $L \subset W$ allows us to form an extension $S^1 \to \mathbb{R} \to L$.

LEMMA II.3.1. *$R$ is a maximal abelian subgroup of $H(W)$.*

PROOF: We have the following formula for commutators in $H(W)$:

$$[(w_1, t_1), (w_2, t_2)] = (0, e^{2\pi i\langle w_1, w_2\rangle}).$$

Thus if $w_1$, $w_2 \in L$ then $(w_1, t_1)$ and $(w_2, t_2)$ commute since $\langle\ ,\ \rangle$ is integral on $L$. Also if $(w, t)$ commutes with $(\ell, 1)$ for all $\ell \in L$ then $\langle w, \ell\rangle$ is an integer for all $\ell \in L$. Consequently $w \in L$ and the lemma is proved.

COROLLARY: *$S^1 \to R \to L$ is trivial.*

PROOF: $L$ is a free abelian group.

Now let $Q$ be a quadratic form on $L/2L$ with values in $\mathbb{Z}/2$ associated to $\langle\ ,\ \rangle$ (this means $Q$ satisfies $Q(x + y) = Q(y) + Q(y) + \langle x, y\rangle$). We define a function $\chi_Q$ from $R$ to $S^1$ by:

$$\chi_Q(\ell, t) = e^{i\pi Q(\bar{\ell})}t \text{ (here } - \text{ denotes reduction modulo 2)}$$

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LEMMA II.3.2. *$\chi_Q$ is a character.*

PROOF: The proof is evident.

We now consider the representations $\mathrm{ind}_R^{H(W)}\chi_Q$. We denote the Hilbert space of this representation by $\mathscr{L}_\chi$ and the representation (temporarily) by $\rho$. We note that we may take for $\mathscr{L}_\chi$ the space of functions $F$ on $H(W)$ satisfying:
(i) $F(xr) = \chi_Q(r)^{-1}F(x)$
(ii) $|F|^2$ is square integrable on $H(W)/R$. Then $H(W)$ acts by left translations.

Clearly any element of $\mathscr{L}_\chi$ is determined by its restriction to $W$ (identified with $(w, 1) \in H(W)$). We find that the image of $\mathscr{L}_\chi$ under this restriction mapping is the space of functions for $W$ saitsfying for $\ell \in L$ and $w \in W$:
(i) $f(w + \ell) = e^{i\pi\langle \ell, w\rangle}e^{-i\pi Q(\bar{\ell})}f(w)$
(ii) $\int_{W/L}|f(w)|^2dw$ is finite
The action of $W \subset H(W)$ on this space is now by twisted translations:

$$\rho(w)\cdot f(w') = e^{-i\pi\langle w, w'\rangle}f(w' - w)$$

The functions $f$ are (non-holomorphic) analogues of theta functions.

LEMMA II.3.3. *$\rho$ is irreducible.*

PROOF: We show that the only bounded operators commuting with $\rho$ are the scalar multiples of the identity. We note:

$$\rho(r)F(x) = F(r^{-1}x) = F(xr^{-1}[r^{-1}, x]) = \chi_Q(r)\chi_Q([x, r^{-1}])F(x)$$

Thus $r \in R$ acts by the multiplication operators $\chi_Q(r)\chi_Q([x, r^{-1}])$. We claim such functions separate the points of $H(W)/R$. Indeed if $x_1 = (w_1, t_1)$ and $x_2 = (w_2, t_2)$ then $\chi_Q(r)\chi_Q([x_1, r^{-1}]) = \chi_Q(r)\chi_Q([x_2, r^{-1}])$ for all $r \in R$ implies $[x_1, r^{-1}] = [x_2, r^{-1}]$ for all $r \in R$. But by the commutator formula of Lemma II.2.1 this implies $e^{2\pi i\langle w_1, \ell\rangle} = e^{2\pi i\langle w_2, \ell\rangle}$ for all $\ell \in L$ which implies $w_1 - w_2 \in L$ and $x_1R = x_2R$.

We find then that any bounded operator $M$ on $\mathscr{L}_\chi$ commuting with all $\{\rho(r): r \in R\}$ commutes with all multiplication operators by continuous functions on $H(W)/R$ hence is given (in the second model above) by $Mf(x) = m(x)f(x)$ where $m(x)$ is a bounded function on $W/L$. From the identity $\rho(w) \circ M \circ \rho(w)^{-1} = M$ we find $m(x + w) = m(x)$ for all $w$. Consequently $m$ is constant and the lemma is proved.

Thus $\rho$ is a realization of the Stone-von Neumann representation of $H(W)$. It is called the lattice model. We should emphasize that the space $\mathscr{L}_\chi$ depends on the choice of quadratic form $Q$. Of course we have a realization of the oscillator representation $\omega_\chi$ on $\mathscr{L}_\chi$. Let $O(Q)$ denote the orthogonal group of $Q$ so $O(Q) \subset Sp(L/2L)$. Let $\Gamma_Q$ denote the inverse image in $Sp(L)$ of $O(Q)$ under the canonical mapping $Sp(L) \to Sp(L/2L)$. We have the following theorem.

THEOREM II.3.1. *The projective representation $\omega_\chi: Sp(W) \to PU(\mathscr{L}_\chi)$ becomes a linear representation when restricted to $\Gamma_Q$ for $Q$ any non-singular quadratic form over $\mathbb{Z}/2$ associated to $\langle\ ,\ \rangle$.*

PROOF. We first note that if we form the lattice model $\rho_\chi$, with the character $\chi$ corresponding via Lemma II.2.2 to the quadratic form $Q$, then $\Gamma_Q$ preserves the inducing data used to define $\rho_\chi$. Thus $\Gamma_Q$ acts on $\mathscr{L}_\chi$ by $A:\Gamma_Q \to U(\mathscr{L}_\chi)$ given by:

$$A(\gamma)f(x) = f(\gamma^{-1}(x)).$$

It is immediate that:

$$A(\gamma)\rho_\chi(n)A(\gamma^{-1}) = \rho_\chi(\gamma(n)).$$

Hence $A(\gamma)$ represents $\omega_\chi(\gamma)$ and the theorem is proved.

COROLLARY. *The $S^1$ extension associated to the metaplectic extension of $\Gamma_Q$ is trivial.*

We now apply the previous theorem to obtain a formula for the characteristic class $\sigma$ of the extension $\mu_2 \to \tilde{\Gamma}_Q \to \Gamma_Q$. We consider the long exact cohomology sequence attached to the coefficient sequence $\mu_2 \to S^1 \to S^1$, where the second map is squaring. The part we are interested in is:

$$H^1(\Gamma_Q, S^1) \xrightarrow{\delta} H^2(\Gamma_Q, \mu_2) \xrightarrow{j} H^2(\Gamma_Q, S^1)$$

($\mu_n$ will be used to denote the $n$th roots of unity).

We have seen that $j(\sigma)$ is trivial, consequently, by exactness, there exists a character $\mu$ of $\Gamma_Q$ so that $\delta\mu = \sigma$. Let us examine this more closely in terms of representing chains. Let $c$ by any $\mu_2$ valued cocycle representing $\sigma$. Then there exists an $S^1$ valued 1-cochain $\alpha$ on $\Gamma_Q$ so that $c(g_1, g_2) = \alpha(g_1g_2)\alpha(g_2)^{-1}\alpha(g_1)^{-1}$. Since $c$ takes values in $\mu_2$ we see that $\alpha^2$ is a character of $\Gamma_Q$ and $\delta\alpha^2 = -c$.

We now give a formula for $\mu$ in terms of the oscillator representation. The oscillator representation is a linear representation $\omega_\chi: \tilde{\Gamma}_Q \to U(\mathscr{L}_\chi)$. Thus there exists a character $\epsilon: \tilde{\Gamma}_Q \to S^1$ so that:

$$\omega_\chi(\gamma) = \epsilon(\gamma)A(\gamma)$$

Let $\Theta$ denote the linear functional on the smooth functions of $\mathscr{L}_\chi$ which is the Dirac delta at the identity of $H(W)$. Then since $A(\gamma)$ leaves $\Theta$ invariant we have:

$$\omega_\chi(\gamma)\Theta = \epsilon(\gamma)^{-1}\Theta$$

We observe that $\epsilon^2$ annihilates the center of $\tilde{\Gamma}_Q$ and consequently induces a character, to be denoted $\lambda$, of $\Gamma_Q$. Then $\lambda(\gamma) = \epsilon^2(s(\gamma))$ where $s(\gamma)$ is any element of $\tilde{\Gamma}_Q$ lying over $\gamma$.

Now let $s:\Gamma_Q \to \tilde{\Gamma}_Q$ be a set-theoretic cross-section chosen so that $c(g_1, g_2) = s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1}$ is a $\mu_2$ valued cocycle representing the class §. Applying $\omega_\chi$ and operating on the theta distribution we find $c(g_1, g_2) = \epsilon(s(g_1g_2))\epsilon(s(g_2))^{-1}\epsilon(s(g_1))^{-1}$. Applying the previous discussion with $\alpha(g) = \epsilon(s(g))$ and noting $\alpha^2(g) = \epsilon^2(s(g)) = \lambda(g)$ we find $\delta\lambda$ represents §.

We obtain the following formula for a representative cocycle $c$ for the metaplectic extension of $\Gamma_Q$ in terms of the theta multiplier $\epsilon$. For each $\gamma \in \Gamma_Q$, let $\sqrt{\lambda}(\gamma)$ be one of the two square roots of $\lambda(\gamma)$. Then, we have the following lemma.

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LEMMA II.3.4. *The metaplectic class restricted to* $\Gamma_Q$ *is represented by:*

$$c(\gamma_1, \gamma_2) = \delta\lambda(\gamma_1, \gamma_2) = \frac{\sqrt{\lambda(\gamma_1\gamma_2)}}{\sqrt{\lambda(\gamma_1)}\sqrt{\lambda(\gamma_2)}}$$

COROLLARY: *If* $\Lambda'$ *is any subgroup of* $\tilde{\Gamma}_Q$ *which leaves* $\Theta$ *fixed and* $\Lambda$ *is the image of* $\Lambda'$ *in* $\Gamma_Q$ *then the restriction of the metaplectic extension to* $\Lambda$ *is trivial.*

REMARKS: An equivalent statement of the previous lemma is that the extension $\tilde{\Gamma}_Q \to \Gamma_Q$ is the pull-back via $\lambda$ of the 2-fold extension of the circle. We note that the corollary is obvious directly because $\Lambda'$ cannot contain the center of $\tilde{\Gamma}_Q$ (which acts by $-1$ on $\Theta$).

We now construct $\Lambda'$. We assume $Q$ is the form given by:

$$Q(x_1, x_2, \ldots, x_{2n}) = \sum_{i=1}^{n} x_i x_{n+i}$$

In this case $\Gamma_Q$ consists of integral symplectic matrices such that ${}^t ac$ and ${}^t bd$ have even diagonal entries and is known as the *theta group.*

LEMMA II.3.5. *For* $n > 1$, *the character* $\epsilon$ *is trivial on the subgroup of* $\tilde{\Gamma}_Q$ *generated by* $\tilde{n}(2u)$ *and* ${}^t\tilde{n}(2u)$ *where u is an n by n symmetric integral matrix. (See the end of the previous section for the meaning of the notation in this lemma).*

PROOF. We claim that the matrix $\tilde{n}(2u)$ is a product of commutators in $\tilde{\Gamma}_Q$. Since $\tilde{n}(u_1)\tilde{n}(u_2) = \tilde{n}(u_1 + u_2)$, it is enough to verify the claim for $\tilde{Y}_{ij}(2)$ and $\tilde{Y}_{ii}(2)$. But the relation (R1) of Lemma II.2.2 implies $\tilde{Y}_{ii}(2)$ is a commutator, then by applying the relation (R2) we see that $\tilde{Y}_{ij}(2)$ is a product of two commutators. Since ${}^t\tilde{n}(2u)$ is conjugate to $\tilde{n}(2u)$ (note $J \in \Gamma_Q$) the lemma follows.

It is proved in Mumford [14], Proposition A1, that the elements $Y_{ij}(2)$ and $Y_{ii}(2)$ (for $1 \leqq i, j \leqq n$) generate the subgroup of $Sp_n(Z)$ of matrices satisfying the congruence conditions; $a \equiv d \equiv 1_n \bmod 4$, $b \equiv c \equiv 0 \bmod 2$. Noting that this group contains $\Gamma(4)$, the principal congruence subgroup of level 4, we obtain the following theorem.

THEOREM II.3.2. *Let* $\Lambda \subset Sp_n(\mathbb{Z})$ *be a subgroup of* $\Gamma(4)$. *Then the restriction of the metaplectic extension to* $\Lambda$ *is trivial. Moreover, there is a section* $s: \Lambda \to \tilde{\Lambda}$ *so that* $\omega_\chi(s(\gamma)) = A(\gamma)$ *for all* $\gamma \in \Lambda$. *In particular* $\omega_\chi(s(\gamma))$ *leaves* $\Theta$ *invariant for all* $\gamma \in \Lambda$ *and* $\epsilon|s(\Lambda) \equiv 1$.

REMARK. We have not proved the above theorem in the case $n = 1$. The theorem remains true since we may embed a two dimensional symplectic space (with the split quadratic form) in a larger one (the double for example). Then the case where $n = 1$ follows from the case of larger $n$ together with the functoriality of $\omega_\chi$ and $A$ under restrictions.

For our applications of the oscillator representation to the theory of automorphic forms we will need to construct a certain line bundle over $\Lambda\backslash\mathfrak{H}_n$ which is the square-root of the locally homogeneous bundle associated to the determinant representation of the maximal compact subgroup $U$ of $Sp(W)$. Let $MU$ denote the 2-fold cover of $U$. We will call $MU$ the meta-unitary group. There is a unique homomorphism $\sqrt{\det}$ making the following diagram commutative:

$$\begin{array}{ccc} MU & \xrightarrow{\sqrt{\det}} & S^1 \\ \downarrow & det & \downarrow \\ U & \longrightarrow & S^1 \end{array} \quad \text{square}$$

We recall that a manifold $M$ of dimension $n$ is said to have an almost complex structure if its tangent frame bundle can be reduced to a $U(n)$ principal bundle $P$. We will say that an almost complex manifold has a meta-unitary structure if there is an $MU(n)$ principal bundle $\bar{P}$ and a double covering $\bar{P} \to P$ which restricts to the cover $MU(n) \to U(n)$ on each fiber.

LEMMA II.3.6. $\Lambda\backslash\mathfrak{H}_n$ *has a meta-unitary structue.*

PROOF: The tangent bundle of $\Lambda\backslash\mathfrak{H}_n$ may be reduced to the bundle:

$$U \to \Lambda\backslash Sp(W) \to \Lambda\backslash Sp(W)/U.$$

Recall we have a section $s:\Lambda \to \tilde{\Lambda}$ and $\Lambda' = s(\Lambda)$. Put $\bar{P} = \Lambda'\backslash\widetilde{Sp}(W)$. Since $P$ is also equal to $\tilde{\Lambda}\backslash\widetilde{Sp}(W)$ we find that $\bar{P}$ is the required double cover of $P$.

We now define the line bundle $L$ over $\Lambda\backslash\mathfrak{H}_n$ to be the line bundle associated to the principal bundle $\bar{P}$ and the representation $\sqrt{\det}$ of $MU(n)$. $L$ is also the line bundle associated to the automorphy factor $j$ on $Mp(W) \times \mathfrak{H}_n$ defined by $j(g,\tau) = \sqrt{\det}$ $(c\tau + d)$ where $\sqrt{\ }$ is chosen so that $j(k, iI_n) = \sqrt{\det k}$ for $k \in MU(n)$. $L$ is also a square root of determinant $\kappa$ of the Hodge bundle over the space of principally polarized abelian varieties. We now make some remarks concerning the transformation law for the classical theta function.

There is a unique (up to scalar multiples) element $\varphi_0$ (the Gaussian) in the Schwartz space of $E$ which satisfies:

$$\omega_\chi(k)\varphi_0 = \sqrt{\det(k)}^{-1}\varphi_0 \text{ for } k \in MU(n)$$

Then the function $\theta(g) = \Theta(\omega_\chi(g)\varphi_0)$ transforms according to:

(i) $\theta(\gamma g) = \epsilon(\gamma)\theta(g) \qquad \gamma \in \tilde{\Gamma}_Q$

(ii) $\theta(gk) = \sqrt{\det(k)}\sqrt{-1}\,\theta g \qquad k \in MU(n)$

Defining $\theta(\tau) = j(g_\tau, iI_n)\theta(g_\tau)$ where $g_\tau(iI_n) = \tau$, we find $\theta(\tau)$ satisfies for $\gamma \in \tilde{\Gamma}_Q$:

$$\theta(\gamma\tau) = \epsilon(\gamma)\sqrt{\det}(c\tau + d)\theta(\tau)$$

REMARKS. $\epsilon(\gamma)$ is called the *theta multiplier*. It is a character of $\tilde{\Gamma}_Q$. To deal with *compact* orthogonal locally symmetric spaces we need to prove the invariance of $\Theta$ under $\Lambda \subset Sp_n(\mathcal{O})$ for the integers $\mathcal{O}$ in a totally real field. Here $\Lambda$ acts by a tensor product of oscillator representations. See Borel-Wallach [2], VIII.7.

4. **Dual reductive pairs and liftings**. Now let $V$ be a real vector space of dimension

$m$ equipped with a quadratic form $(\ ,\ )$ of signature $(p,q)$. We assume there exists a lattice $L$ in $V$ so that $(\ ,\ )$ takes integral values on $L$. We let $Sp_n(\mathbb{R})$ denote the isometry group of the standard skew-symmetric form $\langle\ ,\ \rangle$ on $\mathbb{R}^{2n}$ and $Mp_n(\mathbb{R})$ denote the metaplectic group. Then $(\ ,\ ) \otimes \langle\ ,\ \rangle$ is a symplectic form on $W = V \otimes \mathbb{R}^{2n}$. We perform the construction of Section 1 to obtain the oscillator representation $\omega$ of $Mp(W)$. We let $E$ denote the Lagrangian subspace of $\mathbb{R}^{2n}$ spanned by $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ the first $n$ basis vectors in the standard basis for $\mathbb{R}^{2n}$. We then realize the Schrödinger model for this representation on $L^2(V \otimes E)$. We identify $V \otimes E$ with $V^n$, the $n$-fold direct sum of $V$ with itself by means of the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. We let $F$ denote the subspace of $\mathbb{R}^{2n}$ spanned by the last $n$ standard basis vectors. Then $\mathbb{R}^{2n} = E \oplus F$ is a Lagrangian splitting of $\mathbb{R}^{2n}$.

It is easily proved that the restriction of $\omega_\chi$ to $Mp_n(\mathbb{R}) \subset Mp(W)$ is the $m$-fold tensor product $f$ the oscillator representation of $Mp_n(\mathbb{R})$. We describe this representation on the elements $s(n(u))$ and $s(a(v))$ of section 2 (assuming det $v > 0$). Then the operators of $\omega$ on $f \in L^2(V^n)$ are given by:

(i) $\omega(s(n(u)))f(x) = e(1/2 \text{ tr } ({}^t xxu)f(x)$

(ii) $\omega(s(a(v)))f(x) = (\det v)^{m/2} f(xv)$

Here ${}^t xy$ is the $n \times n$ matrix of inner products $(x_i, y_j)$. We have an embedding of $SO(p,q)$ into $Sp(W)$ which lifts to an embedding into $Mp(W)$ – by lemma II.2.1 (choose $P$ to be the stabilizer of $V \otimes E$).

The operators $\omega_\chi(g)$ for $g \in SO(p,q)$ are given by $\omega_\chi(g)\cdot f(x) = f(g^{-1}x)$ for $f \in L^2(V^n)$. The operators $\omega_\chi(g')$ for $g' \in Mp_n(\mathbb{R})$ and $\omega_\chi(g)$ for $g \in SO(p,q)$ commute with each other. Indeed $\tilde{O}(p,q)$ and $Mp_n(\mathbb{R})$ centralize each other in $Mp(W)$. In fact each is the full centralizer of the other. Such pairs of groups have been named dual reductive pairs by Roger Howe. There are many others besides the previous pair. In fact to treat the complex and quaternionic basic examples we would need the dual reductive pairs $U(n,n) \times U(p,q)$ and $SO^*(4n) \times Sp(p,q)$. In this paper we will restrict ourselves to the orthogonal case. We will rename $\omega$ to denote the representation of $Mp_n(\mathbb{R}) \times \tilde{O}(p,q)$ obtained by restricting $\omega_\chi$ and will abbreviate $\omega(g', 1)$ and $\omega(1, g)$ to $\omega(g')$ and $\omega(g)$ for $g' \in Mp_n(\mathbb{R})$ and $g \in \tilde{O}(p,q)$.

There are two critical properties of dual reductive pairs which we now explain. The first involves a remarkable relation between the action of the invariant differential operators from $Sp_n(\mathbb{R})$ and $O(p,q)$. Let $\mathcal{U}(g)$ be the universal envelopping algebra of the Lie algebra $g$ of $O(p,q)$ and $\mathcal{U}(g')$ be the universal envelopping algebra of the Lie algebra $g'$ of $Sp_n$. Then we have the following theorem of Roger Howe [5].

THEOREM II.4.1. *Let $z$ and $z'$ be the centers of $\mathcal{U}(g)$ and $\mathcal{U}(g')$. Then $d\omega(z)$ and $d\omega(z')$ coincide as algebras of operators on the smooth vectors in $L^2(V^n)$.*

This theorem motivates the following lemma which may be proved by a direct calculation.

LEMMA II.4.1. *There is a scalar $\lambda$ so that $\omega(C') - \omega(C) = \lambda$ where $C'$ is the Casimir operator of $g'$ and $C$ is the Casimir operator of $g$.*

We have already met the second property – it is Theorem II.3.2. We intersect the subgroup $\Lambda' \subset \widetilde{Sp}(\mathbb{Z} \otimes L)$ with $Mp_n(\mathbb{R}) \times \tilde{O}(p,q)$ and we find a subgroup of finite index in the product of $\widetilde{Sp}_n(\mathbb{Z})$ and the integral points of $SO(p,q)$ which fixes $\Theta$. We call this group $\Gamma' \times \Gamma$.

If $\varphi \in S(V^n)$ and $\Gamma' \times \Gamma$ is a suitable subgroup of the group of integral matrices in $Mp_n(\mathbb{R}) \times SO(p,q)$ then the function $\theta_\varphi(g',g)$ on $Mp_n(\mathbb{R}) \times \tilde{O}(p,q)$ given by:

$$\theta_\varphi(g',g) = \Theta(\omega(g',g)\varphi) = \sum_{x \in \mathbb{Z}^{mn}} \omega(g',g)\varphi(x)$$

satisfies

$$\theta_\varphi(\gamma' g', \gamma g) = \theta_\varphi(g',g).$$

Now suppose we can choose $\varphi$ so that $\varphi$ transforms according to a finite dimensional representation $\sigma' \otimes \sigma$ under $MU(n) \times (O(p) \times O(q))$. Then as described in Section 1, $\theta_\varphi$ can be identified with a section of the exterior tensor product $E_{\sigma'} \otimes E_\sigma$ which we denote $\theta_\varphi(x',x)$. In the cases we will consider $E_{\sigma'}$ and $E_\sigma$ have natural metrics and $\theta_\varphi$ can be taken as the kernel of an integral operator mapping sections of $E_{\sigma'}$ to sections of $E_\sigma$ and vice versa. We will use the notation of Howe, Piatetski-Shapiro [6] for this correspondence – if $f$ is a section of $E_{\sigma'}$ then $\theta_\varphi(f)$ will be the section of $E_\sigma$ obtained by taking the inner product of $f$ and $\theta_\varphi$. Similarly if $\eta$ is a section of $E_\sigma$ then $\theta_\varphi(\eta)$ will be the resulting section of $E_{\sigma'}$. In summary, we have:

$$\theta_\varphi(f) = (f, \theta_\varphi)$$

$$\theta_\varphi(\eta) = (\theta_\varphi, \eta)$$

where ( , ) are the Hilbert space inner products on section of $E_{\sigma'}$ and $E_\sigma$. We use the convention that the inner products are anti-linear in the first variable. $\theta_\varphi(\eta)$ will be redefined later.

REMARK: Again we should treat the general totally real case and again we refer to Borel-Wallach [2], VIII.7.

## III. The construction of harmonic dual forms by a theta correspondence

1. **The Howe operator and the Schwartz form**. Our goal in this section is to construct a canonical element in $(\mathbf{A}^{qn}(D) \otimes \mathbf{S}(V \otimes X))^G$. Here $X$ is a rational positive subspace of $V$, $\mathbf{A}^{qn}(D)$ denotes the space of smooth $qn$ forms on $D$ and the superscript $G$ means the subspace of $G$ invariants for the diagonal action. We think of an element $F$ of the above space as a form on $D$ with values in the Schwartz functions on $V \otimes X$. Given any linear functional $T$ on $\mathbf{S}(V \otimes X)$ then $T(F(z))$ will be an ordinary differential form on $D$. If $T$ is invariant under a discrete subgroup $\Gamma$ of $G$ then $T(F(z))$ will induce an element of $\mathbf{A}^{qn}(\Gamma \backslash D)$. In particular, if we choose $T = \Theta$ then we obtain a map:

$$(\mathbf{A}^{qn}(D) \otimes \mathbf{S}(V \otimes X))^G \to \mathbf{A}^{qn}(\Gamma \backslash D).$$

For this reason it is important to construct forms of the above type.

We first construct an element of $(\mathbf{A}^0(D) \otimes \mathbf{S}(V \otimes X))^G$. We choose a majorant $(\ ,\ )_{z_0}$ for $(\ ,\ )$ as in I-1. Since $(\ ,\ )$ restricted to $X$ is positive definite we obtain a positive definite form $((\ ,\ )) = (\ ,\ )_{z_0} \otimes (\ ,\ )$ on the tensor product. We choose an orthonormal basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ for $X$ and identify $X$ with $R^n$ using this basis. Thus $V \otimes X$ is identified with $V^n$ and we find that for $x, y \epsilon V^n$ the inner product $((x, y))$ corresponds to the trace of the matrix with $i,j$-th entry $(x_i, y_j)_{z_0}$. Since we have identified $\mathcal{S}(V \otimes X)$ with $\mathbf{S}(V^n)$ we have the operators of the oscillator representation of $Mp_n(R)$ acting on $\mathbf{A}^*(D) \otimes \mathbf{S}(V \otimes X)$ via the action on the second factor. This action commutes with the action of $G$ on either factor, in particular it commutes with the diagonal action. We define the Gaussian $\varphi_0 \epsilon \mathbf{S}(V^n)$ by the formula:

$$\varphi_0(x) = e^{-\pi((x,x))} = \prod_{i=1}^{n} e^{-\pi(x_i, x_i)_{z_0}}.$$

Then $\varphi_0$ transforms by a character under $K'$, the maximal compact subgroup $MU(n)$ of $Mp_n(\mathbb{R})$ which fixes $J_0$. This is easily proved by an infinitesimal computation using Lemma II.1.5. We define:

$$\bar{\varphi}_0(g, x) = \varphi_0(g^{-1}x).$$

Then $\bar{\varphi}_0 \epsilon \mathbf{A}^*(D)\mathbf{A}^0(D) \otimes \mathbf{S}(V^n))^G$.

We now look for a $G$ invariant, $K'$ semi-invariant, operator $\nabla$ such that:

$$\nabla : (\mathbf{A}^*(D) \otimes S(V^n))^G \to (\mathbf{A}^{*+k}(D) \otimes S(V^n))^G$$

We give the construction of such an operator in the case $k = qn$ and $G = SO_0(p, q)$, the connected component of the identity of $O(p, q)$. In this case $K = SO(p) \times SO(q)$.

We first observe the isomorphism given by the restriction map:

$$(\mathbf{A}^*(D) \otimes S(V^n))^G \to (\Lambda^* T^*_{z_0}(D) \otimes S(V^n))^K.$$

But $T^*_{z_0}(D)$ is canonically isomorphic to $z_0^\perp \otimes z_0$. We obtain an isomorphism of the above space of $G$-invariants with $(\Lambda^*(z_0^\perp \otimes z_0) \otimes S(V^n))^K$. We are now in the framework studied by Howe [5]. We now write down an operator $\nabla'$ which is $K$ invariant, $K'$ semi-invariant and satisfies:

$$\nabla' : (\Lambda^*(z_0^\perp \otimes z_0) \otimes S(V^n))^K \to (\Lambda^{*+nq}(z_0^\perp \otimes z_0) \otimes S(V^n))^K.$$

Such an operator will give rise to the desired operator $\nabla$.

We give a formula in coordinates for $\nabla'$. We choose a basis $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_m\}$ compatible with the splitting $V = z_0^\perp \oplus z_0$. Then we let $\{x_{ij} : i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\}$ denote coordinates relative the basis $\{e_j \otimes \epsilon_j\}$ for $\otimes$. We use the index convention that $\alpha$, $\beta$ will stand for indices between 1 and $p$ and $\mu$, $\nu$ for those between $p + 1$ and $m$. We normalize the Riemannian metric on $D$ to coincide on $T_{z_0}(D)$ with the negative of the tensor product $(\ ,\ ) \otimes (\ ,\ )$ restricted to $z_0^\perp \otimes z_0$. For this metric $\{e_\alpha \otimes e_\mu : 1 \leqq \alpha \leqq p, p + 1 \leqq \mu \leqq p + q\}$ is on orthonormal basis. Using the metric, $e_\alpha \otimes e_\mu$ gives rise to an element $(e_\alpha \otimes e_\mu)^\#$ in $T^*_{z_0}(D)$ which we

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identify with the Maurer-Cartan form $\omega_{\alpha\mu}$ in $p^*$. This is a $K$-equivariant identification.

We have operators $\partial/\partial x_{ij}$, $M(x_{ij})$ on $S(V^n)$ where $M(x_{ij})$ denotes multiplication by $x_{ij}$. We also have operators $A(\omega_{ij})$ on $\Lambda^*(z_0^\perp \otimes z_0)$ where $A(\omega_{ij})$ denotes exterior multiplication by $\omega_{ij}$. Then we define the Howe operator by:

$$\nabla' = \frac{(-1)^{nq}}{2^{nq/2}} \prod_{i=1}^{n} \prod_{\mu=p+1}^{m} \sum_{\alpha=1}^{p} \left( \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} - M(x_{\alpha i}) \right) \otimes A(\omega_{\alpha\mu}).$$

Finally, we define:

$$\varphi = \nabla' \varphi_0 \in (\Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n))^K.$$

In the next section we will need a formula for the element $\bar{\varphi} \in (\mathbf{A}^{nq}(D) \otimes S(V^n))^G$ restricting to $\varphi$. We have identified $\Lambda^{nq}(z_0 \otimes z_0^\perp)$ with $\Lambda^{nq}p^*$. We look for a representation (for each $x$) of $\bar{\varphi}$ as an element of $(\Lambda^{nq}p^* \otimes C^\infty(G))^K$. To do this we just use the map in the beginning of this section. We define

$$\bar{\varphi}(g,x) \in \Lambda^{nq}p^* \otimes [C^\infty(G) \otimes S(V^n)]^G \text{ by } \bar{\varphi}(g,x) = \varphi(g^{-1}x).$$

The $K$-invariance of $\varphi_0$ guarantees that for each $x$ we have $\bar{\varphi}(g,x) \in (\Lambda^{nq}p^* \otimes C^\infty(G))^K$ (we will sometimes write $\bar{\varphi}(z,x)$ for $z \in D$ since $\bar{\varphi}$ depends only on $g$ mod $K$).

To make the previous correspondence completely clear we note that we may extend $\varphi$ in terms of monomials $\omega_I$ in the $\omega_{\alpha\mu}$'s according to:

$$\varphi(x) = \sum_I f_I(x)\omega_I \qquad \text{(here } I \text{ is a multi-index)}$$

Then the $f_I$'s satisfy:

$$f_I(kx) = \sum_J \sigma_{IJ}(k) f_J(x)$$

where $\sigma_{IJ}$ is the $IJ$-th matrix element of $K$ acting on $\Lambda^{nq}p^*$. We extend the $\omega_I$'s to left-invariant on $G$ and we extend each $f_I(x)$ to $\bar{f}_I(g,x) = f_I(g^{-1}x)$. Then

$$\bar{\varphi}(g,x) = \sum_I \bar{f}(g,x)\omega_I$$

We next note that for $n = 1$, the function $\varphi(x)$ is a linear combination of monomials of the form $\omega_{\alpha_1 p+1} \wedge \omega_{\alpha_2 p+2} \wedge \ldots \wedge \omega_{\alpha_q p+q}$ where $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ is an arbitrary subset of $\{1, 2, \ldots, p\}$. We will be especially interested in the coefficient $\eta_{I_0}(x)$ of $\omega_{1p+1} \wedge \ldots \wedge \omega_{1p+q}$. We see that this coefficient is given by:

$$\eta_{I_0}(x) = \frac{1}{2^{q/2}(2\pi)^{q/2}} H_q(\sqrt{2\pi}(e_1, x))\varphi_0(x)$$

where $H_q(t)$ is the $q$-th Hermite polynomial given by:

$$H_q(t) = (-1)^q e^{t^2} \frac{d^q}{dt^q} e^{-t^2}$$

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We observe that the form $\bar{\varphi}$ for $n = 1$ determines the form $\bar{\varphi}$ for general $n$. Indeed, we have an isomorphism $\mathcal{S}(V)^{\otimes n}$ to $\mathcal{S}(V^n)$ sending $f_1 \otimes f_2 \otimes \ldots \otimes f_n$ to $\Pi_{i=1}^n f_i$. We also have the $n$-th exterior power map. $\Lambda_q(z_0^\perp \otimes z_0) \to \Lambda^{nq}(z_0^\perp \otimes z_0)$. Clearly both of these maps are $K$-homomorphisms. Combining these two mappings we obtain a $K$-homomorphism (of degree $n$):

$$\Lambda^q(z_0^\perp \otimes z_0) \otimes S(V) \to \Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n)$$

and consequently a map of $K$-invariants to be denoted $\wedge$:

$$\bigotimes_1^n (\Lambda^q(z_0^\perp \otimes z_0) \otimes S(V))^K \to (\Lambda^{nq}(z_0^\perp \otimes z_0) \otimes S(V^n))^K.$$

If $g \in G$ and $z = gz_0$ we let $(\ ,\ )_z$ denote the majorant of $(\ ,\ )$ associated to $z$. Then we have:

$$(x,y)_z = (g^{-1}x, g^{-1}y)_{z_0}.$$

We note the transformed Gaussian satisfies

$$\varphi_0(g^{-1}x) = \prod_{i=1}^n e^{-\pi(x_i, x_i)_z}.$$

We then have the following lemma whose proof is left to the reader.

LEMMA III.1.1.

$$\bar{\varphi}(z,x) = \bar{\varphi}_1(z,x_1) \wedge \bar{\varphi}_1(z,x_2) \wedge \ldots \wedge \bar{\varphi}_1(z,x_n).$$

NOTATION. $\bar{\varphi}_1(z,x_j)$ is the $q$-form obtaining by applying the partial Howe operator

$$\nabla_j = \frac{(-1)^q}{2^{q/2}} \prod_{\mu=p+1}^{m} \sum_{\alpha=1}^{p} \left( \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha j}} - M(x_{\alpha j}) \right) \otimes A(\omega_{\alpha\mu})$$

to the Gaussian in the variable $x_j$.

We will later need a naturality property of the form $\bar{\varphi}_1(z,x)$ under restriction.

Let $y$ be a vector in $V$ of positive length and $x$ another vector of positive length so that $x = x' + x''$ with $x'$ a multiple of $y$ and $(x'', y) = 0$. Let $V_y$ denote the orthogonal complement of $y$ in $V$, $G_y$ be the subgroup of $G$ which fixes $y$, $D_y$ the set of negative $q$-planes contained in $V_y$ *and* $i_y : D_y \to D$ the inclusion. We may consider the dual pair $Mp_n(\mathbb{R}) \times G_y \subset Mp(\mathbb{R}^{2n} \otimes V_y)$. The theory of the previous section produces an element $\bar{\varphi}' \in (\mathbf{A}^q(D_y) \otimes \mathbf{S}(V_y))^{G_y}$. We then have the following lemma.

LEMMA III.1.2.

$$i_y^* \bar{\varphi}(z,x) = \bar{\varphi}_0(x') \bar{\varphi}'(z,x'')$$

We now summarize the key tensorial properties of $\bar{\varphi}$ in the following theorem.

THEOREM III.1.1. (i) $\bar{\varphi}(z,x)$ *is a closed nq-form on D for every x in* $V^n$.
(ii) $\bar{\varphi}(z,x)$ *transforms under* $MU(n)$ *according to the representation* $(\sqrt{\det})^m$.

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REMARK. We will not prove $\bar{\varphi}$ is closed. As the reader will observe in the section 3 this is essential. The proof may be found in Kudla-Millson [9].

2. **The theta correspondence**. In the last section we constructed a canonical element $\bar{\varphi} \in (\mathbf{G}^{nq}(D) \otimes \mathbf{S}(V^n))^G$. We now consider the element $\theta_\varphi \in \mathbf{A}^{nq}(\Gamma\backslash D) \otimes C^\infty(Mp_n(\mathbb{R}))$ defined by:

$$\theta_\varphi(g', z) = \Theta(\omega(g')\bar{\varphi}) = \sum_{x \in L^n}' \omega(g')\bar{\varphi}(z, x).$$

By $\sum'$, we mean the sum over only those $x \in L^n$ which are congruent to some fixed non-degenerate (i.e. rank $n$) $n$-tuple $x_0 \in L^n$ modulo some integer $N$. Elements of $V^n$ will be called $n$-frames and elements of $L^n$ will be called integral $n$-frames. We assume $\gamma \in \Gamma'$ implies $\gamma \equiv 1 \bmod N$.

Clearly, $\theta_\varphi$ defines a closed differential $nq$ form on $M = \Gamma\backslash D$ for a suitable congruence subgroup (again denoted $\Gamma$) of the integral points of $O(p, q)$. The transformation law in $g'$ is very subtle but is now clear (after the considerable work of Chapter II). Since $\Theta$ is invariant under $\Gamma'$ we have

(i) $\theta_\varphi(\gamma' g', z) = \theta_\varphi(g', z)$

and since $\varphi$ transforms under $K'$ like $(\sqrt{\det})^m$ we have:

(ii) $\theta_\varphi(g'k', z') = [\sqrt{\det}(k')]^m \, \theta_\varphi(g', z)$.

The formulas (i) and (ii) together imply that $\theta_\varphi$ is a section of the line bundle $L^m$ over $M' = \Gamma'\backslash\mathfrak{H}_n$ (recall $L = \kappa^{1/2}$). We use $\tau$ to denote the coordinate in $\mathfrak{H}_n$. Then $\tau = u + iv$ with $u$ and $v$ real $n$ by $n$ symmetric matrices and $v$ positive definite. We define an element $g'_\tau \in Mp_n(\mathbb{R})$, satisfying $g'_\tau(iI_n) = \tau$, by the following formula:

$$g'_\tau = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1}u \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$$

Then we define $\theta_\varphi(\tau, z)$ by the formula:

$$\theta_\varphi(\tau, z) = j(g'_\tau, iI_n)^{m/2}\theta_\varphi(g'_\tau, z) = (\det v)^{-m/4}\theta_\varphi(g'_\tau, z)$$

We may use $\theta_\varphi$ as a kernel of an integral transform as in Chapter II, section 4 and we obtain an integral transform:

$$\mathcal{L} : C_0^\infty(M', L^m) \to \mathbf{A}^{nq}(M).$$

Since $\theta_\varphi$ is closed we also obtain a map:

$$C_0^\infty(M', L) \to H^{nq}(M, \mathbb{R}).$$

Now $L$ is a holomorphic line bundle. Holomorphic sections of $L$ are classical Siegel modular forms; that is, holomorphic functions on $\mathfrak{H}_n$ satisfying the transformation law:

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^{m/2} f(\tau) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subset Sp_n(\mathbb{Z}).$$

We denote the holomorphic cusp-forms satisfying the above transformation law by

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$S_{m/2}(\Gamma')$. Clearly we can integrate a holomorphic cusp form against $\theta_\varphi$ and obtain a lifting $\mathcal{L}: S_{m/2}(\Gamma') \to \mathbf{A}^{nq}(M)$. A computation of Casimir values and Lemma II.4.1 yield the following theorem of Kudla-Millson [9].

THEOREM III.2.1. *The lift of a holomorphic cusp form is a closed harmonic nq form on M.*

We have constructed a mapping from spaces of classical Siegel modular forms to harmonic forms on locally symmetric spaces of orthogonal groups. We want to relate the image of this map to the dual classes of special cycles. Let $\mathbf{H}^{nq}$ denote the space of harmonic $nq$ forms in $M$. We assume henceforth that $M$ is compact.

We now define some linear combinations of special cycles. We will say an $n$-frame $x = \{x_1, x_2, \ldots, x_n\}$ has length equal to an $n$ by $n$ matrix $\beta$ if $\beta$ is equal to the matrix $((x_i, x_j))$. If $\beta > 0$ we write ${}^t xx = \beta$.

Let $C'_\beta$ denote a set of $\Gamma$ orbit representatives for the set of $n$-frames in $L^n$ of length $2\beta$ saitsfying the previous congruence condition. Then $\mathcal{C}'_\beta$ is finite and we define:

$$C_\beta = \sum_{x \in \mathcal{C}'_\beta} C_x.$$

The notation $C_x$ is explained in I.1.5.

We now define two subspaces of $\mathbf{H}^{nq}$. We let $\mathbf{H}$ denote the image of $S_{m/2}(\Gamma')$ under $\mathcal{L}$. We let $\mathbf{H}_{\text{cycle}}$ denote the span of the duals of the cycles $C_\beta$ as above subject to the condition that $\beta$ is positive definite. We will now outline a proof of the following theorem of Kudla-Millson. Let $m = p + q$.

MAIN THEOREM: *If* $n < m/4$ *then* $\mathbf{H}_\theta = \mathbf{H}_{\text{cycle}}$.

The theorem follows easily from a formula for certain Fourier coefficient of $\theta_\varphi(\eta)$ where $\eta$ is a harmonic $(p - n)q$-form. We consider the non-singular pairing [ , ] between harmonic $(p - n)q$ forms $\eta$ and harmonic $nq$ forms $\omega$ given by: $[\eta, \omega] = \int_M \eta \wedge \omega$. We define $\theta_\varphi(\eta)$ for $\eta$ a harmonic $(p - n)q$ form by: $\theta_\varphi(\eta) = [\eta, \theta_\varphi]$. By the transformation law for $\theta$ we see that $\theta_\varphi$ is periodic with respect to the lattice $L' = \Gamma' \cap N$. Consequently $\theta_\varphi(\eta)$ has a Fourier expansion with respect to the characters of $L'$. Let $a_\beta(\theta_\varphi(\eta))$ denote the $\beta^{\text{th}}$ Fourier coefficient for $\beta$ an element of the dual lattice $(L')^*$ of $L'$ ($\beta$ will be a symmetric $n$ by $n$ matrix with rational entries). $a_\beta$ is a function of $v$ where $\tau = u + iv$. Then, for $\beta$ positive definite, we have the following formula — to be proved in the next section:

$$a_\beta(\theta_\varphi(\eta))(v) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta \qquad (S)$$

We now show $(S)$ implies the theorem. Clearly, it is enough to show $\mathbf{H}_\theta^\perp = \mathbf{H}_{\text{cycle}}^\perp$. This later equality we establish by proving two inclusions.

We first establish $\mathbf{H}_{\text{cycle}}^\perp \subset \mathbf{H}_\theta^\perp$. Accordingly, we assume $\eta$ is orthogonal to the dual forms of the cycles $C_\beta$. Hence $\int_{C_\beta} \eta = 0$ for all cycles $C_\beta$ with $\beta$ positive definite and accordingly $a_\beta(\theta_\varphi(\eta)) = 0$ for all such $\beta$. But then any $f$ in $S_{m/2}(\Gamma')$ has Fourier coefficients disjoint from $\theta_\varphi(\eta)$ and consequently we have

$$[\theta_\varphi(f), \eta] = (f, \theta_\varphi(\eta)) = 0$$

We now establish $\mathbf{H}_\theta^\perp \subset \mathbf{H}_{\text{cycle}}^\perp$. We assume that $\theta_\varphi(\eta)$ is orthogonal to all holomorphic cusp forms. We introduce the Poincaré series (convergent provided $n < (p + q)/4$):

$$p_\beta(\tau) = c \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} p\tau}}{j(\gamma, \tau)^m}$$

Here $\Gamma'_\infty = \Gamma' \cap N$ and $c$ is a constant chosen so that $(p_\beta, f) = a_\beta(f)$ for $f \in S_{m/2}(\Gamma')$ and $\Gamma'_\infty$ is the intersection of $\Gamma'$ and $N$. We recall that $p_\beta(\tau)$ is a holomorphic cusp form.

We will also need the series:

$$p_\beta(\tau, s) = c(s) \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} \beta\tau}}{j(\gamma, \tau)^m} \det v(\gamma\tau)^s$$

Here $c(s)$ is chosen so that $(p_\beta(\tau, s), f(\tau)) = a_\beta(f)$ for $f$ a holomorphic cusp form. Then assuming $n < (p + q)/4$ we have $p_\beta(\tau, s)$ is holomorphic in $s$ in a vertical half-plane containing 0 and $p_\beta(\tau, 0) = p_\beta(\tau)$.

Since $p_\beta(\tau)$ is a holomorphic cusp form we have:

$$(p_\beta(\tau, s), \theta_\varphi(\eta))|_{s=0} = (p_\beta(\tau), \theta_\varphi(\eta)) = 0$$

We now compute the first inner product directly. By the usual unfolding argument (valid for Re $s$ sufficiently large) we obtain:

$$\begin{aligned}(p_\beta(\tau, s), \theta_\varphi(\eta)) &= c(s) \int_{\mathcal{F}_\infty} e^{-2\pi i \operatorname{tr} \beta\tau} (\det v)^{m/2+s} \theta_\varphi(\eta) \frac{du\,dv}{(\det v)^{n+1}} \\ &= c(s) \int_{\mathcal{P}_n} e^{-2\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-n+1/2} a_\beta(\theta_\varphi(\eta)) \frac{dv}{(\det v)^{n+1/2}} \\ &= c(s) \left(\int_{C_\beta} {}^*\eta\right) \int_{\mathcal{P}_n} e^{-4\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-n+1/2} \frac{dv}{(\det v)^{n+1/2}}\end{aligned}$$

Here $\mathcal{F}_\infty$ is a fundamental domain for $\Gamma'_\infty$ in $\mathfrak{H}_n$ and $\mathcal{P}_n$ is the space of positive definite symmetric $n$ by $n$ matrices.

The above integral formula coincides with $(p_\beta(\tau, s), \theta_\varphi(\eta))$ a priori only for Re $s$ large, but, by the principle of unique analytic continuation, it must coincide with $(p_\beta(\tau, s), \theta_\varphi(\eta))$ in any region where they are both defined. The second integral has been computed in Siegel [18], Hilfsatz 37, and is convergent and non-zero provided Re $s > n - m/2$. This region includes zero under our assumption on $n$ and $m$ and consequently both the inner product and the integral are regular at $s = 0$. Evaluating the integral at $s = 0$ we obtain a non-zero constant and find:

$$cc' \int_{C_\beta} \eta = (p_\beta(\tau), \theta_\beta(\eta)) = 0$$

Hence the period of $\eta$ over $C_\beta$ is zero and the theorem is proved.

3. **The positive-definite Fourier coefficients of $\theta_\varphi(\eta)$**. The purpose of section 3 is to prove the formula:

$$a_\beta(\theta_\varphi(\eta))(v) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta$$

In this subsection we introduce some notation and some ideas concerning the cohomology of the total space of an oriented vector bundle.

If $x \in V^n$ then $G_x$ will denote the stabilizer in $G$ of the span of $x$ and $D_x$ will be the sub-symmetric space of $D$ associated to $G_x$. We let $\Gamma_x = \Gamma \cap G_x$. We assume now that the span of $x$ is a positive definite subspace of $V$.

The critical topological observation for what follows is that the space $E = \Gamma_x \backslash D$ is in a natural way a vector bundle over $C_x = \Gamma_x \backslash D_x$. There is a fibering $\pi : E \to C_x$ whose fibers are obtained by exponentiating the normal bundle of $C_x$ in $E$. We choose some $x \in C'_\beta$ and study the cohomology of $E = \Gamma_x \backslash D$.

We review briefly some facts concerning the cohomology with compact supports of an oriented vector bundle over a compact manifold. These facts are proved in Chapter I, section 6 of Bott-Tu [3]. If $E$ is an oriented vector bundle with fiber dimension $nq$ then $H_c^{nq}(E, \mathbb{R})$ is isomorphic to $\mathbb{R}$ (here the subscript $c$ denotes cohomology with compact support). Moreover, if $\varphi$ is a closed, compactly-supported $nq$ form then its image under the above isomorphism is the integral over any fiber (by Stokes theorem all fiber integrals are the same). If $\psi$ is a closed compactly supported $nq$-form with period 1 along a fiber then $\psi$ is called a Thom form and its class in $H_c^{nq}(E, \mathbb{R})$ is said to be the Thom class. It is a basic result in topology (proved in Bott-Tu) that the Thom class is Poincaré (or Lefschetz) dual to the zero section $Z$ of $E$; that is, if $\eta$ is any closed form on $E$ we have:

$$\int_E \eta \wedge \psi = \int_Z \eta \qquad (D)$$

We now apply these considerations to $\pi : E \to C_x$ (so $C_x$ is the zero section of our bundle). For any $g \in Sp_n(\mathbb{R})$ we find that $\omega(g')\bar{\varphi}$ is $\Gamma_x$ invariant and hence we may project it to $E$. Let us abbreviate $\omega(g')\bar{\varphi}$ by $\bar{\varphi}$ until the end of Lemma III.3.4. This will cause no difficulty for the only additional properties of $\bar{\varphi}$ we will use are that $\bar{\varphi}$ is closed and rapidly decreasing (see below) both of which are true for all $\omega(g')\bar{\varphi}$. We would like to multiply $\bar{\varphi}$ by an appropriate factor so that it becomes a Thom form. Since $\bar{\varphi}$ is not compactly-supported we must make some modifications in the standard theory.

We note $E$ inherits a Riemannian metric from $D$. We say a form $\tau$ on $E$ is rapidly decreasing if we have for some constants $C$ and any positive integer $n$: $\|\tau(z)\| \leq Ca(z)^{-n}$ where $a(z) = e^{r(z)}$ and $r(z)$ is the geodesic distance from the point $z \in D$ from the submanifold $D_x$. It is easy to see $\bar{\varphi}$ is rapidly decreasing. We recall also that $\bar{\varphi}$ is closed. The next lemma will allow us to replace $\bar{\varphi}$ by a compactly supported form.

LEMMA III.3.1. *$\bar{\varphi}$ is cohomologous to a compactly supported form via a rapidly decreasing primitive.*

PROOF. Let $a_\lambda$ denote the operation of dilation by the positive number $\lambda$ acting fiberwise on the vector bundle $E$. Then $r(\partial/\partial r)$ is the infinitesimal generator for the flow $a_\lambda$. Put $\tau = i_{r(\partial/\partial r)} \int_1^\infty a_\lambda^* \bar{\varphi}(d\lambda/\lambda)$. Then $\tau$ is defined in the complement of zero section of $E$ and satisfies $d\tau = \bar{\varphi}$. We multiply $\tau$ by a smooth radial function $\sigma$ vanishing at the zero section and identically 1 in the complement of a small neighbourhood fo the zero section. Then $\bar{\varphi}_c = \bar{\varphi} - d(\sigma\tau)$ is compactly supported. With this the lemma is proved.

We now treat the problem of whether $(D)$ holds for $\bar{\varphi}$. Note that since $\bar{\varphi}$ is not compactly supported some restriction must be placed on $\eta$ in order that the left-hand integral of $(D)$ coverge. We allow $\eta$ to be slowly increasing ($\|\eta\|$ bounded by a power of $a(z)$).

Let $\kappa$ be the period of $\bar{\varphi}$ over any fiber of $\pi$ and let $\eta$ be any slowly increasing closed form on $E$. Then we have the following lemma.

LEMMA III.3.2.

$$\int_E \eta \wedge \bar{\varphi} = \kappa \int_{C_x} \eta$$

PROOF. We may write $\varphi = \psi + d\nu$ where $\psi$ is compactly supported and $\nu$ is rapidly decreasing. Let $S(r)$ be the bundle of normal spheres of radius $r$ around $C_x$ in $E$. Then the volume of $S(r)$ is bounded by $a(z)^k$ for some fixed $k$ (since the curvature of $D$ is bounded below) and consequently:

$$\lim_{r\to\infty} \int_{S(r)\cap \text{fiber}} \nu = 0 \text{ and } \lim_{r\to\infty} \int_{S(r)} \eta \wedge \nu = 0$$

We conclude by Stokes Theorem that

$$\int_{\text{fiber}} \psi = \int_{\text{fiber}} \bar{\varphi} = \kappa \text{ and } \int_E \eta \wedge \bar{\varphi} = \int_E \eta \wedge \psi$$

Hence $1/\kappa\ \psi$ is a Thom form and

$$\int_E \eta \wedge \frac{1}{\kappa} \varphi = \int_E \eta \wedge \frac{1}{\kappa} \psi = \int_{C_x} \eta$$

Since $\kappa$ is constant (in $z$) the lemma is proved.

REMARK. $\kappa$ is a function of $g'$ and $x$; that is, $\kappa = \kappa(g', x)$. Although it is initially defined only for rational $x$ we may extend it to *all real* $x$ with positive definite span by defining $\kappa(g', x)$ to be the integral of $\omega(g')\bar{\varphi}$ over some normal fiber of $D_x \subset D$.

Finally since $\bar{\varphi}$ is rapidly decreasing the sum $\Omega = 1/\kappa \sum_{\Gamma_x\backslash\Gamma} \gamma^* \bar{\varphi}$ converges, projects to $M$ and defines the dual form to the cycle $C_x$ in $M$ by Lemma 2.1 of Kudla-Millson [8]. Rewriting slightly the condition that $\Omega$ is dual to $C_x$ we obtain:

$$\int_M \eta \wedge \sum_{\Gamma_x\backslash\Gamma} \omega(g')\gamma^* \bar{\varphi} = \kappa(g', x) \int_{C_x} \eta$$

We may summarize the results obtained so far in this section by the following lemma.

LEMMA III.3.3. *Let $\varphi$ be a rapidly decreasing closed nq form on $E = \Gamma_x \backslash D$ and $\eta$ be a $\Gamma$-invariant form on E. Then the following are equivalent:*

(i) $\int_{\text{fiber}} \varphi = \kappa$

(ii) $\int_E \eta \wedge \varphi = \kappa \int_{C_x} \eta$

(iii) $\int_M \eta \wedge \sum_{\Gamma_x \backslash \Gamma} \gamma^* \varphi = \kappa \int_{C_x} \eta$

The previous lemma completes the topological preliminaries and we are now ready to study the integral for the $\beta^{\text{th}}$ Fourier coefficient of $\theta_\varphi(\eta)$ namely:

$$a_\beta(\theta_\varphi(\eta))(v) = \frac{1}{\operatorname{vol} \mathcal{D}(v)} \int_{\mathcal{D}(v)} \theta_\varphi(\eta)(u + iv) e^{-2\pi i \operatorname{tr} \beta u} \, du.$$

Here $\mathcal{D}(v)$ is a fundamental domain for $\Gamma'_\infty$ acting on the subset of $\mathfrak{H}_n$ defined by $\operatorname{Im} \tau = v$.

Let $\theta_\varphi(\tau, z, \beta)$ be the function defined by:

$$\theta_\varphi(\tau, z, \beta) = (\det v)^{-m/4} \sum_{x \in L^n}{}' \{\omega(g'_\tau)\bar{\varphi}(z,x) \colon (x,x) = 2\beta\}$$

Then an argument identical to that of Kudla-Millson [8], page 254, yields the following lemma.

LEMMA III.3.4.

$$a_\beta(\theta_\varphi(\eta))(v) = \int_M \eta \wedge \theta_\varphi(iv, z, \beta)$$

We note that $\theta_\varphi(\tau, z, \beta)$ is $\Gamma$ invariant but is no longer $\Gamma'$ invariant. We now rewrite $\theta_\varphi(\tau, z, \beta)$ as follows. Recall that we have chosen a set of representatives $C'_\beta$ for the $\Gamma$-orbits of frames in $L^n$ of length $2\beta$. We define for $x \in C'_\beta$:

$$\theta_\varphi(\tau, z, x) = (\det v)^{-m/4} \sum_{\gamma \in \Gamma_x \backslash \Gamma} \omega(g'_\tau)\gamma^* \bar{\varphi}(z,x)$$

Recall in the previous section we defined:

$$\kappa(g', x) = \int_{\text{fiber}} \omega(g')\bar{\varphi}(z,x)$$

We define $\kappa'(\tau, x)$ by the formula:

$$\kappa'(\tau, x) = (\det v)^{-m/4} \kappa(g'_\tau, x)$$

We find then:

$$a_\beta(\theta_\varphi(\eta)(v) = \sum_{x \in C'_\beta} \int_M \eta \wedge (\det v)^{-m/4} \sum_{\Gamma_x \backslash \Gamma} \omega(g'_{iv})\gamma^* \bar{\varphi}(z,x)$$

hence, by Lemma III.3.3:

$$a_\beta(\theta_\varphi(\eta))(v) = \sum_{x \in C'_\beta} \kappa'(iv, x) \int_{C_x} \eta \qquad **$$

Thus, to prove $(S)$ we must compute $\kappa'(iv, x)$. In fact it is equally easy to compute $\kappa'(iv, x)$, $\kappa'(\tau, x)$ or $\kappa(g', x)$. We note $\kappa'(iv, x) = (\det v)^{-m/4}\kappa(\sqrt{v}, x)$.

The function $\kappa(g', x)$ has a large number of symmetries which we now describe. We first claim that $\kappa(g', x)$ depends (in the second variable) only upon $2\beta = (x, x)$. To see this it is sufficient to prove $\kappa(g', gx) = \kappa(g', x)$. But we may compute $\kappa(g', gx)$ by integrating $\omega(g')\bar{\varphi}(z, gx)$ along the transform by $g$ of the fiber we used to calculate $\kappa(g', x)$ note $gz_0 \in D_{gx}$. But $\omega(g')\bar{\varphi}(z, gx) = \omega(g')\bar{\varphi}(g^{-1}z, x)$ and the result follows by the change of variable theorem.

Thus, there exists a smooth function $\kappa''$ on $G' \times \mathcal{P}_n$ with $\kappa(g', x) = \kappa''(g', \beta)$ for $\beta = {}^t xx$.

As a function of $g'$, $\kappa$ is determined by its restriction to $ML_n^+(\mathbb{R}) \subset Mp_n(\mathbb{R})$ since it transforms by characters under the actions of $N$ and $K'$. Let $u$, $v$ denote elements in $ML_n^+(\mathbb{R})$. Then we have:

$$\begin{aligned} \kappa(vu, x) &= (\det v)^{m/2}\kappa(u, xv) \\ \kappa''(vu, \beta) &= (\det v)^{m/2}\, \kappa''(u, {}^t v\beta v) \end{aligned} \qquad (A)$$

where $v$ acts from the right on the frame $x$.

From the previous two paragraphs we see that it is sufficient to compute $\kappa(u, e)$ where $e = \{e_1, e_2, .\ \ , e_n\}$ and $u \in ML_n^+(\mathbb{R})$. Let $A'$ be the diagonal subgroup of $ML_n^+(\mathbb{R})$. We now claim that the function $\kappa(u, e)$ is determined by its restriction to $A'$.

LEMMA III.3.5. *$\kappa(u, e)$ is a spherical function on $ML_n^+(\mathbb{R})$.*

PROOF. By the Cartan decomposition for $ML_n^+(\mathbb{R})$we may write $u = k_1 a' k_2$ with $k_1 \in s(SO(n))$, $k_2 \in s(SO(n))$ and $a' \in A'$. Clearly $\kappa(k_1 a' k_2, e) = \kappa(k_1 a', e)$ since $\bar{\varphi}$ is invariant under $s(SO(n))$. Also by $(A)$ we have:

$$\kappa(k_1 a', e) = \kappa(a', ek_1) = \kappa(a', e)$$

because $(ek_1, ek_1) = (e, e)$. With this the lemma is proved.

We now compute $\kappa'(\tau, x)$ in the case $n = 1$, this is the key calculation of the chapter. Let $a'_\mu$ be the diagonal matrix with diagonal entries $(\mu, \mu^{-1})$.

LEMMA III.3.6. *In case $n = 1$ we have:*

(i) $\kappa(a'_\mu, e_1) = \mu^{m/2} e^{-\pi\mu^2}$

(ii) $\kappa'(\tau, x) = e^{i\pi 2\beta\tau}$ *where* $(x, x) = 2\beta$

PROOF. We must compute the period of the form $\omega(g')\bar{\varphi}(z, x)$ along a fiber of the tube $E = \Gamma_x \backslash D$. We have reduced this to a study of $f(\mu) = \kappa(a'_\mu, e_1)$. In this case $D_x$ is the set of negative $q$-planes contained in the orthogonal complement of $e_1$. We choose $z_0 \in D_x$ to be the $q$ plane spanned by $e_{p+1} \wedge e_{p+2} \wedge \ldots \wedge e_{p+q}$. Then the fiber through $z_0$ is the set of negative $q$-planes pependicular to $\{e_2, e_3, \ldots, e_p\}$. This is a sub-

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symmetric space $G_1/K_1$ where $G_1$ is the subgroup of $G$ which fixes $\{e_2, e_3, \ldots, e_p\}$. We find that $p_1^*$ consists of the Maurer-Cartan forms $\omega_{1p+1}, \omega_{1p+2}, \ldots, \omega_{1p+q}$. Applying the formula I.2.4 we find that the period $f(\mu)$ of $\omega(a'_\mu)\varphi(g^{-1}e_1))$ is given by:

$$f(\mu) = \frac{1}{\text{vol } K_1} \int_{G_1} \eta_{l_0}(a'_\mu, g^{-1}e_1)\, dg$$

where $\eta_{l_0}(a'_\mu, g^{-1}e_1)$ is the coefficient of $\omega_{1p+1} \wedge \omega_{1p+2} \wedge \ldots \wedge \omega_{1p+q}$ in the expression for $\omega(a'_\mu)\varphi(g^{-1}e_1)$ in terms of left-invariant $q$-forms on $G$. We have seen in the last section that:

$$\eta_{l_0}(a'_\mu, g^{-1}e_1) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \mu^{m/2} H_q((\sqrt{2\pi}\,\mu g e_1, e_1)) e^{-\pi\mu^2\|g^{-1}e_1\|^2}.$$

Noting that the stabilizer of $e_1$ in $G_1$ is a maximal compact subgroup $K_1$ in $G_1$ we may compute the above integral by using the Cartan decomposition, $G_1 = K_1A_1K_1$. If $a_r$ denotes the generic element of $A_1$ (with an upper 2 by 2 block of chr and shr) we obtain

$$\eta_{l_0}(a'_{\mu, a_r^{-1}}e_1) = \frac{1}{(2\pi)^{q/2}} \mu^{m/2} H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-\pi\mu^2(\text{ch}^2\text{r} + \text{sh}^2\text{r})}.$$

and:

$$f(\mu) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \mu^{m/2} \text{ vol } (S^{q-1}) \int_0^\infty H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-\pi\mu^2(\text{ch}^2\text{r} + \text{sh}^2\text{r})} \text{ sh}^{q-1}\text{r}\, dr.$$

Hence

$$f(\mu) = \frac{1}{2^{q/2}(2\pi)^{q/2}} \frac{q\pi^{q/2}}{\Gamma((q/2) + 1)} \mu^{m/2} e^{\pi\mu^2} \times \int_0^\infty H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-2\pi\mu^2\text{ch}^2\text{r}} \text{ sh}^{q-1}\text{r}\, dr.$$

We now compute

$$I(\mu) = \int_0^\infty H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-2\pi\mu^2\text{ch}^2\text{r}} \text{ sh}^{q-1}\text{r}\, dr$$

by first computing its Mellin transform $\mathcal{M}I(s)$. We have:

$$\begin{aligned}
\mathcal{M}I(s) &= \int_0^\infty \int_0^\infty H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-2\pi\mu^2\text{ch}^2\text{r}} \text{ sh}^{q-1}\text{r}\, dr\ \mu^s \frac{d\mu}{\mu} \\
&= \int_0^\infty \int_0^\infty H_q(\sqrt{2\pi}\ \mu \text{ chr}) e^{-2\pi\mu^2\text{ch}^2\text{r}} \mu^s \frac{d\mu}{\mu} \text{ sh}^{q-1}\text{r}\, dr \\
&= \frac{1}{(2\pi)^{s/2}} \int_0^\infty \int_0^\infty H_q(\mu) e^{-\mu^2} \mu^s \frac{d\mu}{\mu} \frac{\text{sh}^{q-1}\text{r}}{(\text{chr})^s}\, dr \\
&= \frac{1}{(2\pi)^{s/2}} \int_0^\infty H_q(\mu) e^{-\mu^2} \mu^s \frac{d\mu}{\mu} \int_0^\infty \frac{\text{sh}^{q-1}\text{r}}{(\text{chr})^s}\, dr.
\end{aligned}$$

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To obtain the first integral we have only to note:

$$\int_0^\infty H_q(\mu)e^{-\mu^2}\mu^s \frac{d\mu}{\mu} = (-1)^q \int_0^\infty \frac{d^q}{d\mu^q} e^{-\mu^2}\mu^s \frac{d\mu}{\mu}$$

$$= (s-1)(s-2)\dots(s-q)\frac{1}{2}\Gamma\left(\frac{s-q}{2}\right).$$

As for the second integral, we leave to the reader the verification of the following formula (substitute $t = \mathrm{ch}^2 r$):

$$\int_0^\infty (\mathrm{ch} r)^a(\mathrm{sh} r)^b\, dr = \frac{\Gamma\left(-\frac{a+b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{2\Gamma\left(-\frac{a}{2}+\frac{1}{2}\right)}.$$

We then obtain (with $a = -s$, $b = q - 1$):

$$\int_0^\infty \frac{(\mathrm{sh} r)^{q-1}}{(\mathrm{ch} r)^s}\, dr = \frac{\Gamma\left(\frac{s-q-1}{2}\right)\Gamma\left(\frac{q}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}$$

and accordingly (using the duplication formula for $\Gamma$):

$$\mathcal{M}I(s) = \frac{2^{q-2}\Gamma\left(\frac{q}{2}\right)}{(2\pi)^{s/2}}(s-1)(s-2)\dots(s-q)\frac{\Gamma(s-q)}{\Gamma(s)}\Gamma\left(\frac{s}{2}\right)$$

and finally

$$\mathcal{M}I(s) = \frac{2^{q-2}}{(2\pi)^{s/2}}\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{s}{2}\right).$$

From this we obtain:

$$I(\mu) = 2^{q-1}\Gamma\left(\frac{q}{2}\right)e^{-2\pi\mu^2}$$

and

$$\kappa(a'_\mu, e_1) = f(\mu) = \mu^{m/2}e^{-\pi\mu^2}$$

Hence if $\lambda = \sqrt{2\beta}$ we obtain:

$$\kappa''(\mu, 2\beta) = \kappa(\mu, \lambda e_1) = \frac{1}{\lambda^{m/2}}\kappa(\mu\lambda, e_1) = \mu^{m/2}e^{-\pi\lambda^2\mu^2} = \mu^{m/2}e^{-\pi 2\beta\mu^2}$$

Now letting $\mu = \sqrt{v}$ we obtain:

$$\kappa'(iv, \lambda e_1) = \kappa(\mu, \lambda e_1) = \mu^{m/2}e^{-\pi 2\beta\mu^2} = v^{m/4}e^{-\pi 2\beta v}$$

Using the transformation law under $N$ and multiplying by $v^{-m/4}$ we obtain $\kappa'(\tau, x) = e^{i\pi 2\beta\tau}$ and the lemma is proved.

It is somewhat surprising that the formula for $\kappa(\tau, x)$ for general $n$ follows from the formula for $n = 1$ by a simple formal argument. This is one reason for introducing the topological considerations of the beginning of this section.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be an $n$-tuple of positive real numbers and let $a'_\mu$ represent the matrix in $A'$ with these diagonal entries. We have:

$$\omega(a'_\mu)\bar{\varphi}(z, e) = (\mu_1\mu_2 \ldots \mu_n)^{m/2}\bar{\varphi}_1(z, \mu_1 e_1) \wedge \bar{\varphi}_1(z, \mu_2 e_2) \wedge \ldots \wedge \bar{\varphi}_1(z, \mu_n e_n).$$

MAIN LEMMA.

$$\kappa(a'_\mu, e) = \kappa(a'_{\mu_1}, e_1)\kappa(a'_{\mu_2}, e_2) \ldots \kappa(a'_{\mu_n}, e_n) = \prod_{i=1}^{n} \mu_i^{m/2} e^{-\pi\mu_i^2}$$

PROOF. By Lemma III.3.3 it is sufficient to prove that the form $\psi_\mu$ given by:

$$\psi_\mu = (\mu_1\mu_2 \ldots \mu_n)^{-m/2} e^{\pi(\mu_1^2 + \mu_2^2 + \ldots + \mu_n^2)}\omega(a'_\mu)\bar{\varphi}(z, e)$$

satisfies for any $\Gamma$-invariant closed form $\eta$ on $\Gamma_e \backslash D_e$:

$$\int_{\Gamma_e \backslash D} \eta \wedge \psi_\mu = \int_{\Gamma_e \backslash D_e} \eta.$$

We define an $(n - 1)q$ form $\Phi_{n-1}$ on $D$ by

$$\Phi_{n-1} = (\mu_1\mu_2 \ldots \mu_{n-1})^{-m/2} e^{\pi(\mu_1^2 + \ldots + \mu_{n-1}^2)}\bar{\varphi}_1(z, \mu_1 e_1) \wedge \ldots \wedge \bar{\varphi}_1(z, \mu_{n-1} e_{n-1})$$

We consider the integral:

$$I_n = \int_{\Gamma_e \backslash D} \eta \wedge \Phi_{n-1} \wedge \mu_n^{-m/2} e^{\pi\mu_n^2}\bar{\varphi}_1(z, \mu_n e_n).$$

We define a form $\Omega_{n-1}$ on $\Gamma_{e_n} \backslash D$ by:

$$\Omega_{n-1} = \sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^* \Phi_{n-1}.$$

We claim $\|\Omega_{n-1}\|$ is a bounded function on $\Gamma_{e_n} \backslash D$. To see this we introduce one more function, a partial Gaussian $F$ defined by:

$$F(g) = e^{-\pi/2(\|g^{-1}\mu_1 e_1\|^2 + \ldots + \|g^{-1}\mu_{n-1} e_{n-1}\|^2)}.$$

Of course $F$ depends only on $g$ mod $K$. Our claim will be established if we prove that $F$ satisfies the following three properties:

(1) $\|\Phi_{n-1}\| \leqq CF$ for some constant $C$

(2) $\sum_{\Gamma_e \backslash \Gamma_{e_n}} F(\gamma g)$ converges for $g \in G_{e_n}$

(3) $F(g_1 a_r k)$ is a non-increasing function of $r$ where $g = g_1 a_r k$ is the Berger decomposition (see Rossman [17], page 169) with $g_1 \in G_{e_n}$, $k \in K$ and $a_r$ as below.

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Here $a_r$ is the element of $G$ defined by:

$$a_r e_i = e_i \text{ for } i \neq n \text{ or } i \neq p+1$$
$$a_r e_n = \operatorname{ch} r\, e_n + \operatorname{sh} r\, e_{p+1}$$
$$a_r e_{p+1} = \operatorname{sh} r\, e_n + \operatorname{ch} r\, e_{p+1}.$$

We leave (1) and (2) to the reader and prove (3). We note:

$$F(g_1 a_r k) = e^{-(\pi/2)(\|a_r^{-1} g_1 \mu_1 e_1\|^2 + \dots + \|a_r^{-1} g_1 \mu_{n-1} e_{n-1}\|^2)}$$

Property (3) then follows from the easy observation that if $x \in V$ satisfies $(x, e_n) = 0$ then $\|a_r^{-1} x\|^2 \geqq \|x\|^2$. (In the previous formula $\| \ \|$ denoted the norm for the basic majorant $(\ ,\ )_{z_0}$, ).

Now we compute $I_n$ by folding and unfolding the integral with respect to $\Gamma_{e_n}$.

$$\begin{aligned} I_n &= \int_{\Gamma_e \backslash D} \eta \wedge \Phi_{n-1} \wedge \mu_n^{-m/2} e^{\pi \mu_n^2} \bar{\varphi}_1(z, \mu_n e_n) \\ &= \int_{\Gamma_{e_n} \backslash D} \eta \wedge \Omega_{n-1} \wedge \mu_n^{-m/2} e^{\pi \mu_n^2} \bar{\varphi}_1(z, \mu_n e_n) \\ &= \int_{\Gamma_{e_n} \backslash D_{e_n}} i_n^*(\eta \wedge \Omega_{n-1}) \\ &= \int_{\Gamma_e \backslash D_{e_n}} i_n^*(\eta \wedge \Phi_{n-1}) \end{aligned}$$

The next to last inequality follows from the previous lemma and Lemma III.3.3. Here $i_n$ denotes the inclusion of $D_{e_n}$ into $D$ or any quotient thereof.

We now apply Lemma III.1.2 which allows us to conclude that $i_n^* \bar{\varphi}(z, e_i) = \bar{\varphi}'(z, e_i)$ for $i = 1, 2, \dots, n-1$. Thus the last integral above is in fact the integral $I_{n-1}$ for the smaller orthogonal group $SO(p-1, q)$. Continuing in this way we obtain the lemma.

The formula $(S)$ will follow from the next lemma.

LEMMA III.3.7.

$$\kappa'(\tau, x) = e^{i\pi \operatorname{tr} 2\beta\tau} \textit{ where } (x, x) = 2\beta$$

PROOF. We have:

$$\kappa(a'_\mu, e) = (\mu_1 \mu_2 \dots \mu_n)^{m/2} e^{-\pi(\mu_1^2 + \mu_2^2 + \dots + \mu_n^2)}$$

Hence, by Lemma III.3.5:

$$\kappa(u, e) = (\det u)^{m/2} e^{-\pi \operatorname{tr} {}^t u u} \text{ for } u \in GL_n^+(\mathbb{R})$$

Let $2\beta$ be as above and write $2\beta = {}^t u_0 u_0$ for $u_0 \in GL_n^+(\mathbb{R})$. Then, by the formula $(A)$, we have:

$$\kappa''(u, 2\beta) = \kappa(u, eu_0) = (\det u_0)^{-m/2}\kappa(u_0 u, e)$$
$$= (\det u)^{m/2} e^{-\pi \operatorname{tr} {}^t u_0 u_0 {}^t u u}$$
$$= (\det u)^{m/2} e^{-\pi \operatorname{tr} 2\beta {}^t u u}$$

Letting $v = {}^t u u$ we obtain:

$$\kappa'(iv, eu_0) = \kappa(u, eu_0) = (\det u)^{m/2} e^{-\pi \operatorname{tr} 2\beta {}^t u u} = (\det v)^{m/4} e^{-\pi \operatorname{tr} 2\beta v}$$

Using the transformation law under $N$ we obtain:

$$\kappa(g'_\tau, \beta) = (\det v)^{m/4} e^{i\pi \operatorname{tr} 2\beta\tau}$$

Hence:

$$\kappa'(\tau, x) = e^{i\pi \operatorname{tr} 2\beta\tau}$$

and

$$\kappa'(iv, x) = e^{-\pi \operatorname{tr} 2\beta v}$$

With this the lemma is proved. Substituting the value obtained for $\kappa'(iv, x)$ into ** we obtain the required formula for $a_\beta(\theta(\eta))$.

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