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EVENTUALLY REGULAR SEMIGROUPS THAT ARE GROUP-BOUND

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Necessary and sufficient conditions are given for certain classes of eventually regular semigroups to be group-bound or even periodic.

1. Introduction and Preliminaries.

Wherever possible the notation and conventions of Clifford and Preston [1,2] will be used.

An element of a semigroup S is called *group-bound* if it has some power that is in a subgroup of S and is called *eventually regular* if it has some power that is regular. A semigroup S is *group-bound* [*eventually regular*] if all of its elements are group-bound [eventually regular]. Thus the class of eventually regular semigroups includes all regular semigroups and all group-bound semigroups. For more properties of eventually regular semigroups see [3].

It is of interest to know whether a semigroup under consideration is group-bound. For example it is well known that $\mathcal{D} = \mathcal{J}$ for a group-bound semigroup. This follows for instance from the conjuction of [6, Theorem 1.2 (vi)] and [6, Remark 1.7]. There exist regular semigroups for which $\mathcal{D} \neq \mathcal{J}$. In Section 2 necessary and sufficient conditions for certain classes of eventually regular semigroups to be group-bound or

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even periodic are given whence $\mathcal{D} = \mathcal{J}$ for these semigroups.

Recall from [3] or [4] the idempotent-separating congruence on a semigroup S, $\mu = \mu(S)$ defined by,

$\mu = \{(a,b) \in S \times S \mid$ if $x \in S$ is regular then each of $x \mathcal{R} xa$, $x \mathcal{R} xb$ implies $xa \mathcal{H} xb$ and each of $x \mathcal{L} ax$, $x \mathcal{L} bx$ implies $ax \mathcal{H} bx\}$.

The congruence μ is the maximum idempotent-separating congruence on any eventually regular semigroup [3, Theorem 11]. In this paper we use the generalized meaning of fundamental given in [4]; thus a semigroup S is called *fundamental* if the only idempotent-separating congruence on S is 1_S. Alternatively, by a result of D. Easdown, we may view a semigroup S as fundamental if and only if $\mu(S) = 1_S$. Since μ is the identity relation on S/μ [4, Theorem 8] it follows that S/μ is always fundamental.

2. Eventually Regular Semigroups that are Group-Bound.

Theorems 3, 4 and 6 indicate when certain classes of eventually regular semigroups are group-bound or even periodic. Theorem 5 [Theorem 7] indicates when certain classes of eventually regular semigroups have the property that for each member S the semigroup S/μ is periodic [finite]; thus any member of such a class if fundamental is periodic [finite].

LEMMA 1. *If the* $\mathcal{H}$*-class of* a^k, H *say, is a group then* $a^n \in H$ *for all* $n \geq k$.

Proof. Suppose the identity element of H is e and let y be the group inverse of a^k in H. It follows easily that $ea = ae$ and that $ay = aey = eay = ya^k ay = yaa^k y = yae = yea = ya$. Let r be an integer such that $kr > n$ and let $v = a^{kr-n}y^r$. Then since $ay = ya$, $va^n = a^n v = a^n a^{kr-n} y^r = a^{kr} y^r = (a^k y)^r = e^r = e$. Because $n \geq k$ it is clear that $ea^n = a^n e = a^n$. Thus $a^n \in H_e = H$ as required.

COROLLARY 2. *If* $a^k \mathcal{H} a^{k+r}$ *with* $r > 0$, *then the* $\mathcal{H}$*-class of* a^k *is a group.*

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Proof. Choose $p \geqq 0$ such that r divides $k+p$ with say $rq = k+p$. Then since $a^k \,H\, a^{k+r}$ it follows easily that $a^k \,H\, a^{k+ri}$ for all $i \geqq 1$. In particular when $i = q$, $a^k \,H\, a^{k+rq}$ and so also $a^{k+p} \,H\, a^{k+rq+p}$, whence $a^{rq} \,H\, a^{2rq}$. It follows from Green's theorem that the H-class of a^{rq} is a group. From Lemma 1, $a^n \,H\, a^{rq}$ for all $n \geqq rq$ and so $a^k \,H\, a^{k+rq} \,H\, a^{rq}$ whence the H-class of a^k is a group.

THEOREM 3. *Let S be an eventually regular semigroup such that each regular $\mathcal{D}$-class of S contains at most m L-classes of S , for some integer m . Take any $a \in S$ and let k be the least integer greater than or equal to m such that a^k is regular. Then there exists a subgroup H of S such that $a^n \in H$ for all $n \geqq k$. In particular S is group-bound.*

Proof. Let S , m , and k be as stated in the theorem. The integer k exists since for example $(a^m)^p$ is regular for some $p \geqq 1$. Since a^k is regular there is an idempotent e such that $e \,R\, a^k$. For all $0 \leqq i \leqq k$, $ea^i(a^{k-i}) = a^k$ and $a^k u a^i = ea^i$ where u is such that $a^k u = e$, whence $ea^i \,R\, a^k$. By definition of m , and because $k \geqq m$, there exist $0 \leqq s < t \leqq k$ such that $ea^s = ea^t$. Since L is a right congruence, it follows that $ea^k = a^k \,L\, ea^{k+t-s}$ and that $ea^{k-t+s} \,L\, ea^k = a^k$. Because $0 < t-s \leqq k$, $ea^{k-t+s} \,R\, a^k$ follows from above. Since $ea^{k-t+s} \,R\, a^k$ and a^k is equal to the right translation of ea^{k-t+s} by a^{t-s} and $a^k \in L_{ea^{k-t+s}}$, Green's lemma yields that $a^k \,R\, a^k a^{t-s}$ whence $a^k \,R\, a^{k+t-s}$. It is now clear that $a^k \,H\, a^{k+t-s}$. It follows that the H-class of a^k is a group by Corollary 2 and so $a^n \,H\, a^k$ for all $n \geqq k$ by Lemma 1.

When applied to regular semigroups Theorem 3 reduces to Theorem 15 of Hall [5]. The next result gives a condition that is necessary and

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sufficient for a certain class of eventually regular semigroups to be group-bound.

THEOREM 4. *Let S be an eventually regular semigroup with only finitely many regular $\mathcal{D}$-classes. Then S is group-bound if and only if S contains no copy of the bicyclic semigroup.*

Proof. Necessity is clear since the bicyclic semigroup is not group-bound and any eventually regular subsemigroup of a group-bound semigroup is group-bound. Suppose that S contains no copy of the bicyclic semigroup. Thus no pair of distinct comparable idempotents are $\mathcal{D}$-related. Take $a \in S$. Since there are only finitely many regular $\mathcal{D}$-classes and infinitely many powers of a are regular it follows that there exists distinct positive integers m and n such that $a^m \mathcal{D} a^n$ with a^m regular. Since L_{a^m} and L_{a^n} are regular $\mathcal{L}$-classes there exist idempotents $e \in L_{a^m}$ and $f \in L_{a^n}$. Without loss of generality $n < m$ and so $L_{a^m} \leq L_{a^n}$, whence $ef = e$. Putting $g = fe$ yields $g = g^2$, $g \leq f$ and $g \mathcal{L} e \mathcal{D} f$, whence $g = f$ and so $L_{a^n} = L_{a^m}$. Dually, $R_{a^n} = R_{a^m}$ and so $a^n \mathcal{H} a^m$, whence from Corollary 2, S is group-bound.

THEOREM 5. *Let S be an eventually regular semigroup and let m be an integer such that each regular $\mathcal{D}$-class of S contains at most m idempotents. Then S/μ is periodic. Thus if S is fundamental then S is periodic.*

Proof. The following representation of an arbitrary semigroup T will be used. Let X be the set of regular $\mathcal{L}$-classes of T and let Y be the set of regular $\mathcal{R}$-classes of T. Define $\phi : T \to PT_X \times PT_Y^*$ by $s\phi = (\rho_s, \lambda_s)$, where $\rho_s : L_x \to L_{xs}$ if x is regular and $x\mathcal{R}xs$ and is undefined otherwise and $\lambda_s : R_x \to R_{sx}$ if x is regular and $x\mathcal{L}sx$ and is undefined otherwise. Here PT_Y^* denotes the dual of the semigroup PT_Y. Then ϕ is a representation of T and $\mu(T) = \ker \phi$ (see [3, page 30]). Now put $T = S$, $Z = \{1, 2, \ldots, m\}$ and let n be an integer such that b^n is idempotent for each b in PT_Z. Take $c \in S$, choose $k \geq 1$ such that $(c^m)^k$ is regular and put $a = c^k$. It will suffice to show that $a\mu$ is periodic in S/μ. Note that a^m is regular.

Let D be any regular $\mathcal{D}$-class of S and let D/L denote the (finite) set of L-classes contained in D. Using the method of Hall [5, Theorem 15] it can be shown that ρ_a^m maps $(D/L) \cap$ range ρ_a^m one to one onto itself. It follows that $\rho_a^m \, H \, \rho_a^n \, H \, \rho_a^{nm}$ in PT_X. Put $Z_D = (D/L) \cap$ range ρ_a^m. Since $|Z_D| \leq m$, $(\rho_a^m)^n$ maps Z_D identically to itself, by the choice of n. Thus ρ_a^{mn} maps $\cup\{Z_D | D \in \mathcal{D}\}$ = range ρ_a^{mn} identically to itself, whence ρ_a^{mn} is an idempotent of PT_X. Dually, λ_a^{mn} is an idempotent of PT_Y^*, whence $(a^{mn}, a^{2mn}) \in \ker \phi = \mu$ and so $a^{mn}\mu$ is idempotent in S/μ.

Remark 1. [5, page 19] In general a fundamental regular semigroup in which each $\mathcal{D}$-class contains one L-class is not necessarily periodic. As an example let G be a non-periodic group, let $G' = \{g' | g \in G\}$ be a disjoint copy of G and put $S = G \cup G'$. Extend the multiplication from G to S be defining, for all $g, g_1, g_2 \in G$, $g_1'g = g_1'g_2' = g_1'$ and $gg_1' = (gg_1)'$. Then S is a fundamental semigroup, $\mathcal{D} = L$ and S is of course not periodic.

THEOREM 6. *Let S be an eventually regular semigroup with only finitely many regular elements. Then S is periodic.*

Proof. Take any $a \in S$. Then a^{k_1} is regular for some $k_1 > 1$ as for example k_1 could be $2k_0$ where k_0 is an integer such that $(a^2)^{k_0}$ is regular. Similarly there exists $k_2 > 1$ such that $(a^{k_1})^{k_2}$ is regular and $k_3 > 1$ such that $(a^{k_1k_2})^{k_3}$ is regular. Now

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$k_1 < k_1k_2 < k_1k_2k_3$ and by continuing this process it is clear that there exist distinct integers n and m such that $a^n = a^m$. Thus a is periodic, whence S is periodic.

As a final specialization we have the following theorem:

THEOREM 7. [3, Theorem 15]. *Let* S *be any semigroup with only finitely many idempotents. Then* S/μ *is finite. Thus if* S *is fundamental then* S *is finite.*

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