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A converse of Bernstein's inequality for locally compact groups

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Let G be a Hausdorff locally compact abelian group, Γ its character group. We shall prove that, if S is a translation-invariant subspace of $L^p(G)$ $(p \in [1, \infty])$,

$$\omega(a) = \sup\{\|\tau_a f - f\|_p : f \in S, \|f\|_p \le 1\}$$

for each $a \in G$ and $\lim_{a \to 0} \omega(a) = 0$, then $\bigcup_{f \in S} \Sigma(f)$ is relatively compact (where $\Sigma(f)$ denotes the spectrum of f). We also obtain a similar result when G is a Hausdorff compact (not necessarily abelian) group. These results can be considered as a converse of Bernstein's inequality for locally compact groups.

Throughout this paper we shall follow the notation of [1]. We require two technical lemmas.

LEMMA 1. *Suppose we are given* $\chi \in \Gamma$ *and* $k \in L^1(G)$ *such that* $\hat{k}(\chi) = 1$. *Then for* $\varepsilon > 0$, *we can find* $l \in L^1(G)$ *such that* $\hat{k}\hat{l} = 1$ *on a neighbourhood of* χ *and* $\|l\|_1 < 1 + \varepsilon$.

Proof. Choose $\delta \in (0, 1)$ satisfying

$$\delta(1-\delta)^{-1} < \varepsilon/2 . \tag{1}$$

Since $(\bar{\chi}k)\hat{}(0) = 1$, [7], Chapter 5, 2.3 (5), p. 114, asserts the

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existence of $\tau \in L^1(G)$ such that $\|\tau\|_1 < 1 + \varepsilon/2$, $\hat{\tau} = 1$ on a neighbourhood of zero, and

(2) $$\|(\bar{\chi}k)*\tau-\tau\|_1 < \delta .$$

Putting $\tau_\chi = \chi\tau$, (2) yields

(3) $$\|k*\tau_\chi-\tau_\chi\|_1 < \delta ,$$

and clearly, $\hat{\tau}_\chi = 1$ on a neighbourhood V_χ of χ and $\|\tau_\chi\|_1 < 1 + \varepsilon/2$.

As $\delta < 1$, it appears from (3) that the series

(4) $$\tau_\chi + \sum_{n\geq 1} (-1)^n(k*\tau_\chi-\tau_\chi)^{*n}$$

converges in $L^1(G)$ to l , say. For $\gamma \in V_\chi$, we have

$$\hat{k}(\gamma)\hat{l}(\gamma) = \hat{k}(\gamma)\left(1 + \sum_{n\geq 1} (-1)^n[\hat{k}(\gamma)-1]^n\right)$$
$$= 1 .$$

A combination of (1), (3) and (4) gives us

$$\|l\|_1 \leq \|\tau_\chi\|_1 + \sum_{n\geq 1} \delta^n$$
$$< 1 + \varepsilon/2 + \delta(1-\delta)^{-1}$$
$$< 1 + \varepsilon . \qquad //$$

LEMMA 2. *Let* $\delta \in (0, 1)$. *Suppose that* $\chi \in \Gamma$ *and* $a \in G$ *satisfy*

$$|\chi(a)-1| > 1 - \delta .$$

Then we can find p, q *in* $L^1(G)$ *such that* $\hat{p} = 1$ *on a neighbourhood of* χ *,*

$$p = \tau_a q - q$$

and

$$\|q\|_1 < (1-\delta)^{-1}(1+\delta) .$$

Proof. By [8], 2.6.1, we can find $k \in L^1(G)$ such that $\hat{k}(\chi) = 1$ and $\|k\|_1 = 1$. Since

$$\left(\overline{\chi}(a)-1\right)^{-1}\left(\tau_a k-k\right)^{\wedge}(\chi) = 1 ,$$

we can appeal to Lemma 1 to deduce the existence of $l \in L^1(G)$ such that $\|l\|_1 < 1 + \delta$ and

$$(5) \qquad \left(\overline{\chi}(a)-1\right)^{-1}\left(\tau_a k-k\right)^{\wedge}\hat{l} = 1$$

on a neighbourhood of χ. Now put

$$(6) \qquad q = \left(\overline{\chi}(a)-1\right)^{-1} k * l .$$

Then, if

$$p = \tau_a q - q ,$$

(5) shows that $\hat{p} = 1$ on a neighbourhood of χ, and from (6),

$$\begin{aligned} \|q\|_1 &\le \left|\overline{\chi}(a)-1\right|^{-1}\|k\|_1\|l\|_1 \\ &< (1-\delta)^{-1}(1+\delta) . \qquad // \end{aligned}$$

We can now prove:

THEOREM 1. *Suppose that* S *is a translation-invariant subspace of* $L^p(G)$ $(p \in [1, \infty])$ *, that*

$$(7) \qquad \omega(a) = \sup\left\{\|\tau_a f-f\|_p : f \in S,\ \|f\|_p \le 1\right\}$$

for each $a \in G$ *, and that* $\lim_{a\to 0} \omega(a) = 0$ *. Then* $D = \bigcup_{f\in S} \Sigma(f)$ *is relatively compact.*

Proof. As ω is unchanged if we replace S by S^- in (7), we can assume that S is closed.

Suppose D is not relatively compact. Then, if V is any neighbourhood of zero and $\delta > 0$ is given, we can find $a_V \in V$, $f_V \in S$ and $\chi_V \in \Sigma\left(f_V\right)$ such that

$$(8) \qquad \left|\chi_V\left(a_V\right)-1\right| > 1 - \delta$$

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(for if $|\chi(a)-1| \leq 1 - \delta$ for all $a \in V$ and all $\chi \in D$, we could appeal to (23.16) of [6] to deduce that D^- is compact, contrary to assumption). In the case $p = \infty$, it follows from (7), the assumption that $\lim_{a\to 0} \omega(a) = 0$, and the main result of [2] that f_V is equal locally almost everywhere to a uniformly continuous function. Taking $\delta = 1/4$, and recalling (8), Lemma 2 implies the existence of an open neighbourhood W_V of χ_V, and p_V, q_V in $L^1(G)$ such that $\hat{p}_V = 1$ on W_V,

$$p_V = \tau_{a_V} q_V - q_V ,$$

and $\|q_V\|_1 < 2$.

Choose any $k_V \in L^1_{W_V}(G)$ such that $\hat{k}_V(\chi_V) = 1$. Using the definitions of p_V and q_V, we have

$$(9) \qquad \begin{aligned} k_V * f_V &= p_V * k_V * f_V \\ &= \left(\tau_{a_V} q_V - q_V\right) * k_V * f_V \\ &= q_V * \left(\tau_{a_V} k_V - k_V\right) * f_V . \end{aligned}$$

Since S is assumed to be a closed translation-invariant subspace of $L^p(G)$, the proof of [7], Chapter 3, 5.8, p. 78, can be used to show that

$$(10) \qquad h * f_V \in S$$

for all $h \in L^1(G)$ (recall that when $p = \infty$, f_V is equal locally almost everywhere to a uniformly continuous function). Combining (7), (9) and (10),

$$(11) \qquad \begin{aligned} \|k_V * f_V\|_p &\leq \|q_V\|_1 \left\|\tau_{a_V} k_V * f_V - k_V * f_V\right\|_p \\ &\leq 2\omega(a_V) \|k_V * f_V\|_p . \end{aligned}$$

As $\chi_V \in \Sigma(f_V)$ and $\hat{k}_V(\chi_V) \neq 0$, we see that $k_V * f_V \neq 0$ and so, by (11),

(12) $$\omega(a_V) \geq 1/2 .$$

Now consider the net (a_V), where V ranges over the set of neighbourhoods of zero, partially ordered by

(13) $$V \prec V' \text{ if and only if } V \supset V' .$$

It is seen that (13) entails that (a_V) converges to zero; but (12) holds for all V, contradicting the assumption that $\lim_{a\to 0} \omega(a) = 0$. Hence our assumption that D is not relatively compact was false. //

REMARK. It can be shown that for the spaces $L^1(G)$ and $C(G)$, we do not require that $\lim_{a\to 0} \omega(a) = 0$ but only that there exists a compact set F of strictly positive measure such that $\omega(a) < \alpha < 1$ for all $a \in F$.

COROLLARY 1. *Let $M_b(G)$ denote the space of bounded Radon measures on G. Suppose that S is a translation-invariant subspace of $M_b(G)$, that*

(14) $$\omega(a) = \sup\{\|\tau_a\mu-\mu\|_M : \mu \in S, \|\mu\|_M \leq 1\}$$

for each $a \in G$, and that $\lim_{a\to 0} \omega(a) = 0$. Then $\bigcup_{\mu\in S} \operatorname{supp}\hat{\mu}$ is relatively compact.

Proof. It follows from (14) and [3], Corollary 3, that any $\mu \in S$ is generated by an L^1-function. Let

$$S' = \{f \in L^1(G) : f \text{ generates a measure in } S\} .$$

Then S' is a translation-invariant subspace of $L^1(G)$ satisfying the conditions of Theorem 1, from which we deduce that $\bigcup_{f\in S'} \Sigma(f)$ is relatively compact. Since $\hat{f} = \hat{\mu}_f$, where μ_f is the measure generated by f, and any $\mu \in S$ is μ_f for some $f \in S'$, we can conclude (note that for $f \in L^1(G)$, we have $\Sigma(f) = \operatorname{supp}\hat{f}$) that $\bigcup_{\mu\in S} \operatorname{supp}\hat{\mu}$ is relatively compact. //

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We shall now consider the converse when G is a Hausdorff compact group (G is not assumed to be abelian). We follow the notation used in [5]. Given a finite-dimensional continuous irreducible unitary representation $U \in \hat{G}$, with representation space H_U, $d(U)$ will denote the dimension of H_U, and I_U the identity endomorphism of H_U. The trace function on H_U will be denoted by Tr. We let $(E(G), \|\cdot\|)$ denote any of the spaces $L^p(G)$ $(p \in [1, \infty))$ or $C(G)$, each taken with its usual norm. By L_a, we will mean the left translation operator.

THEOREM 2. *Suppose that S is a left translation-invariant subspace of $E(G)$, that*

$$(15) \qquad \omega(a) = \sup\{\|L_a f - f\| : f \in S, \|f\| \le 1\}$$

for each $a \in G$, and that $\lim_{a \to 0} \omega(a) = 0$. Then $\bigcup_{f \in S} \operatorname{supp} \hat{f}$ is finite.

Proof. As ω is unchanged if we replace S by S^- in (15), we can assume that S is closed.

Consider the unit disc in S;

$$B = \{f \in S : \|f\| \le 1\} .$$

It follows immediately from the Weil criterion ([4], 4.20.1), or when $E(G) = C(G)$, from Ascoli's Theorem ([4], 0.4.11), that B is compact in $E(G)$. We can now use the Riesz Theorem ([4], p. 65) to deduce that S is finite dimensional.

Let $\{f_1, f_2, \ldots, f_n\}$ be a basis for S. Since for every $f \in S$,

$$\operatorname{supp} \hat{f} \subseteq \bigcup_{j=1}^{n} \operatorname{supp} \hat{f}_j ,$$

it will suffice to show that $\operatorname{supp} \hat{f}_j$ is finite for all $j \in \{1, 2, \ldots, n\}$.

However if this were false, there would exist $j \in \{1, 2, \ldots, n\}$ and an infinite sequence $\{U_i\}_{i=1}^{\infty}$ of distinct elements of $\hat{G}$ such that $\hat{f}_j(U_i) \ne 0$ for every $i \in \{1, 2, \ldots\}$. Define $h_i \in C(G)$ by

$$h_i(x) = d(U_i) Tr[U_i(x)^*] ,$$

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where $U_i(x)^*$ denotes the adjoint of $U_i(x)$. Since S is assumed to be a closed left translation-invariant subspace of $E(G)$, it is a left ideal (in $E(G)$); hence $h_i * f_j \in S$ for every $i \in \{1, 2, \ldots\}$. Also

(16) $$(h_i*f_j)^\wedge(U_k) = \hat{h}_i(U_k)\hat{f}_j(U_k) = \delta_{ik}\hat{f}_j(U_k) ,$$

where

$$\delta_{ik} = \begin{cases} I_{U_k} , & i = k , \\ 0 , & i \neq k . \end{cases}$$

We see that $\{h_i*f_j\}_{i=1}^{\infty}$ is linearly independent in S; for suppose there exist $\alpha_i \in \mathbb{C}$ such that

$$\sum_{i=1}^{m} \alpha_i(h_i*f_j) = 0 .$$

Then for all k,

$$\sum_{i=1}^{m} \alpha_i(h_i*f_j)^\wedge(U_k) = 0$$

and by (16),

$$\sum_{i=1}^{m} \alpha_i\delta_{ik}\hat{f}_j(U_k) = 0 ,$$

that is,

$$\alpha_k I_{U_k}\hat{f}_j(U_k) = 0 .$$

Since $\hat{f}_j(U_k) \neq 0$, it follows that $\alpha_k = 0$ for all k. Hence $\{h_i*f_j\}_{i=1}^{\infty}$ is linearly independent in S, contradicting the fact that S is finite dimensional.

Consequently $\operatorname{supp}\hat{f}_j$ is finite for all $j \in \{1, 2, \ldots, n\}$, and the theorem is proved. //

COROLLARY 2. *Suppose that S is a left translation-invariant subspace of $L^{\infty}(G)$, that*

(17) $$\omega(a) = \sup\{\|L_a f-f\|_\infty : f \in S, \|f\|_\infty \leq 1\}$$

for each $a \in G$, and that $\lim_{a\to 0} \omega(a) = 0$. Then $\bigcup_{f\in S} \operatorname{supp}\hat{f}$ is finite.

Proof. It follows from (17) and the proof of the main result of [2] that every $f \in S$ is equal almost everywhere to a uniformly continuous function. The problem is then reducible to that covered by the case $E(G) = C(G)$ of Theorem 2. //

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