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Rational points on cubic hypersurfaces that split off a form

T. D. Browning

With an appendix by J.-L. Colliot-Thélène

Abstract

Let X be a projective cubic hypersurface of dimension 11 or more, which is defined over $\mathbb{Q}$. We show that $X(\mathbb{Q})$ is non-empty provided that the cubic form defining X can be written as the sum of two forms that share no common variables.

Contents

1. Introduction

Let $X \subset \mathbb{P}^{n-1}$ be a cubic hypersurface, given as the zero locus of a cubic form $C \in \mathbb{Z}[x_1, \ldots, x_n]$. This paper is concerned with the problem of determining when the set of rational points $X(\mathbb{Q})$ on X is non-empty. There is a well-known conjecture that $X(\mathbb{Q}) \neq \emptyset$ as soon as $n \geqslant 10$. In fact, for non-singular cubic hypersurfaces, it is expected that the Hasse principle holds as soon as $n \geqslant 5$. This means that in order for $X(\mathbb{Q})$ to be non-empty it is necessary and sufficient that $X(\mathbb{Q}_p)$ is non-empty for every prime p. For a large class of possibly singular cubic hypersurfaces $X \subset \mathbb{P}^{n-1}$, Salberger has calculated the Brauer group $\mathrm{Br}(Y)$ associated to a projective non-singular model Y of X. A detailed proof of this calculation is provided by Colliot-Thélène in the appendix to this paper. As a consequence of this investigation one has the following prediction.

Conjecture. Let $X \subset \mathbb{P}^{n-1}$ be a cubic hypersurface defined over $\mathbb{Q}$ which is not a cone, with $n \geqslant 5$ and singular locus which is empty or of codimension at least four in X. Then the Hasse principle holds for the locus of non-singular points on X.

Let us now consider some of the progress that has been made towards this conjecture. When C is diagonal it follows from the work of Baker [Bak89] that X has $\mathbb{Q}$-rational points

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as soon as $n \geqslant 7$. At the opposite end of the spectrum, when absolutely no assumptions are made about the shape of C, a lot of work has been invested in producing a reasonable lower bound for the number of variables needed to ensure that $X(\mathbb{Q}) \neq \emptyset$. Building on work of Davenport [Dav59, Dav63], Heath-Brown [Hea07] has recently shown that $n \geqslant 14$ variables are enough to secure this property for an arbitrary cubic hypersurface defined over $\mathbb{Q}$. In the light of this body of work it is very natural to try and evince intermediate results in which the existence of rational points is guaranteed for cubic hypersurfaces in fewer than 14 variables when mild assumptions are made about the structure of the hypersurface. It is precisely this point of view that is the focus of the present investigation.

Let $\mathrm{sing}(X)$ denote the singular locus of X, as a projective subvariety of X. When $C \in \mathbb{Z}[x_1, \ldots, x_n]$ is non-singular, so that $\mathrm{sing}(X)$ is empty, it follows from work of Hooley [Hoo88] that the Hasse principle holds for X provided that $n \geqslant 9$. As is well-known, the local conditions are automatic when $n \geqslant 10$, and so $X(\mathbb{Q})$ is non-empty for non-singular X provided that $n \geqslant 10$. This fact was first proved by Heath-Brown [Hea83]. When $\mathrm{sing}(X)$ has dimension $\sigma \geqslant 0$, joint work of the author [BH09a] with Heath-Brown shows that $X(\mathbb{Q})$ is non-empty provided that $n \geqslant 11 + \sigma$.

Let $m < n$ be a positive integer. We will say that an integral cubic form C in n variables 'splits off an m-form' if there exist non-zero cubic forms C_1, C_2 with integer coefficients so that

$$C(x_1, \ldots, x_n) = C_1(x_1, \ldots, x_m) + C_2(x_{m+1}, \ldots, x_n),$$

identically in $x_1, \ldots, x_n$. We will merely say that C 'splits off a form' if C splits off an m-form for some $1 \leqslant m < n$. The following is our main result.

Theorem 1. *Let $X \subset \mathbb{P}^{n-1}$ be a hypersurface defined by a cubic form that splits off a form, with $n \geqslant 13$. Then $X(\mathbb{Q}) \neq \emptyset$.*

The essential content of Theorem 1 is that we can save one variable in the result of Heath-Brown [Hea07] when the underlying cubic form splits off a form. It should be stressed that the existence of a single $\mathbb{Q}$-rational point on X is enough to demonstrate the $\mathbb{Q}$-unirationality of X, and so the Zariski density of $X(\mathbb{Q})$ in X, when X is geometrically integral and not a cone. Variants of this result have been known for a long time (cf [CSS87, Man86, Seg44]). In the generality with which we have stated the result, it appears in the work of Colliot-Thélène and Salberger [CS89, Proposition 1.3] and in that of Kollár [Kol02].

Our work has implications for the problem of determining when an arbitrary cubic form $C \in \mathbb{Z}[x_1, \ldots, x_n]$ represents all non-zero $a \in \mathbb{Q}$, using rational values for the variables. When this property holds we say that C 'captures $\mathbb{Q}^*$'. Recall that a cubic form is said to be degenerate if the corresponding cubic hypersurface is a cone. Fowler [Fow62] has shown that any non-degenerate cubic form that represents zero automatically captures $\mathbb{Q}^*$ provided only that $n \geqslant 3$. Hence it suffices to fix attention on those forms that do not represent zero non-trivially. On multiplying through by denominators it will clearly suffice to establish that cubic forms of the shape

$$C(x_1, \ldots, x_n) - a x_{n+1}^3 \tag{1.1}$$

represent zero non-trivially, with a an arbitrary non-zero integer. But this form splits off a 1-form, and so is handled by Theorem 1. In this way we deduce the following result.

Corollary. *Let $C \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-degenerate cubic form, with $n \geqslant 12$. Then C captures $\mathbb{Q}^*$.*

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This result should be compared with the work of Heath-Brown [Hea07], which implies that $n \geqslant 13$ variables suffice. It follows from the work of Hooley [Hoo88] that this may be improved to $n \geqslant 8$ when C is non-singular. As indicated in [Hea83, Appendix 1] the latter lower bound is probably the correct one for arbitrary cubic forms, since (1.1) always has non-trivial p-adic zeros for n in this range.

Let $n \geqslant 4$. When $X \subset \mathbb{P}^{n-1}$ is a hypersurface defined by a cubic form that splits off a form, we are able to handle fewer variables when appropriate assumptions are made about one of the forms. If X is a cone then we will see in Lemma 1 that $X(\mathbb{Q}) \neq \emptyset$. If, on the other hand, X is not a cone let us suppose that the underlying cubic form splits off a non-singular m-form $C_1(x_1, \dots, x_m)$. If $m = n-1$ then X is itself non-singular and we automatically have $X(\mathbb{Q}) \neq \emptyset$ when $n \geqslant 10$. If $m \leqslant n-2$ then the residual form C_2 defines a projective cubic hypersurface of dimension $n-m-2$, and as such has singular locus of dimension at most $n-m-3$. But then it follows that X has singular locus of dimension at most $n-m-3$. Thus we may deduce from [BH09a] that $X(\mathbb{Q}) \neq \emptyset$ provided that $n \geqslant 8+n-m$. We record this observation in the following result.

Theorem 2. *Let $X \subset \mathbb{P}^{n-1}$ be a hypersurface defined by a cubic form that splits off a non-singular m-form, with $m \geqslant 8$ and $n \geqslant 10$. Then $X(\mathbb{Q}) \neq \emptyset$.*

It would be interesting to reduce the range of m needed to ensure the validity of Theorem 2. Ours is not the first attempt to better understand the arithmetic of cubic hypersurfaces that split off a form. Indeed, in Davenport's [Dav63] treatment of cubic forms in 16 variables, a fundamental ingredient in the treatment of certain bilinear equations is a separate analysis of those forms that split into two. In further work, Colliot-Thélène and Salberger [CS89] have shown that the Hasse principle holds for any cubic hypersurface in $\mathbb{P}^{n-1}$ that contains a set of three conjugate singular points, provided only that $n \geqslant 4$. Given a cubic extension K of $\mathbb{Q}$, define the corresponding norm form

$$N(x_1, x_2, x_3) := \mathrm{N}_{K/\mathbb{Q}}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3), \tag{1.2}$$

where $\{\omega_1, \omega_2, \omega_3\}$ is a basis of K as a vector space over $\mathbb{Q}$. In view of the fact that the local conditions are automatically satisfied for cubic forms in at least 10 variables, we observe the following easy consequence.

Theorem 3 (Colliot-Thélène and Salberger [CS89]). *Let $X \subset \mathbb{P}^{n-1}$ be a hypersurface defined by a cubic form that splits off a norm form, with $n \geqslant 10$. Then $X(\mathbb{Q}) \neq \emptyset$.*

It turns out that Theorem 3 will play a useful rôle in dispatching some of the cases that arise in the proof of Theorem 1. Following the strategy of Birch *et al.* [BDL62], it would however be straightforward to adapt the proof of Theorem 1 to retrieve Theorem 3.

An obvious further line of enquiry would be to investigate cubic hypersurfaces that split off two forms, by which we mean that the corresponding cubic form can be written as

$$C(x_1, \dots, x_n) = C_1(x_1, \dots, x_\ell) + C_2(x_{\ell+1}, \dots, x_m) + C_3(x_{m+1}, \dots, x_n),$$

identically in $x_1, \dots, x_n$, for appropriate $1 \leqslant \ell < m < n$. With the extra structure apparent in such hypersurfaces one would like to determine the most general conditions possible under which the conjectured value of $n \geqslant 10$ variables suffices to ensure the existence of $\mathbb{Q}$-rational points.

One of the remarkable features of our argument is the breadth of tools that it draws upon. The underlying machinery is the Hardy–Littlewood circle method, and we certainly take advantage of many of the contributions to the theory of polynomial cubic exponential sums that have been made during the last fifty years. These are detailed in § 3. A further component of our work involves a detailed analysis of the case in which one of the forms that splits off in Theorem 1 is singular and has a relatively small number of variables. To deal with this scenario it pays to reflect upon the classification of singular cubic hypersurfaces. This is a very old topic in algebraic geometry, and can be traced back to the pioneering work of Cayley [Cay69] and Schläfli [Sch64]. All of the necessary information will be collected together in § 2. The final ingredient in our work comprises good upper bounds for the number of $\mathbb{Q}$-rational points of bounded height on auxiliary cubic hypersurfaces. The estimates that we will take advantage of are presented in § 4.

When it is applicable, the Hardy–Littlewood circle method allows us to show that $X(\mathbb{Q}) \neq \emptyset$ for a given cubic hypersurface $X \subset \mathbb{P}^{n-1}$ by evaluating asymptotically the number of $\mathbb{Q}$-rational points of bounded height on X. It is a well-known but intriguing feature of the method that one can achieve such precise information by first establishing weaker upper bounds for the growth rate of $\mathbb{Q}$-rational points on appropriate auxiliary varieties. In fact, we will show in Lemma 11 that the $\mathbb{Q}$-rational points on a non-singular cubic hypersurface $X \subset \mathbb{P}^{n-1}$ satisfy the growth bound

$$\#\{x \in X(\mathbb{Q}) : H(x) \leqslant P\} \ll_{\varepsilon, X} P^{\dim X - \frac{1}{2} + \varepsilon},$$

for any $P \geqslant 1$, provided that $\dim X \geqslant 6$. Here $H : \mathbb{P}^{n-1}(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ is the usual exponential height function. This should be compared with the Manin conjecture [FMT89] which predicts that the exponent of P should be $\dim X - 1$ as soon as $\dim X \geqslant 3$.

Notation. Throughout our work, $\mathbb{N}$ will denote the set of positive integers. For any $\alpha \in \mathbb{R}$, we will follow common convention and write $e(\alpha) := e^{2\pi i \alpha}$ and $e_q(\alpha) := e^{2\pi i \alpha/q}$. The parameter ε will always denote a very small positive real number. We will use $|\mathbf{x}|$ to denote the norm $\max |x_i|$ of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, whereas $\|\mathbf{x}\|$ will be reserved for the usual Euclidean norm $\sqrt{x_1^2 + \cdots + x_n^2}$. All of the implied constants that appear in this work will be allowed to depend upon the coefficients of the cubic forms under consideration and the parameter $\varepsilon > 0$. Any further dependence will be explicitly indicated by appropriate subscripts.

2. Geometry of singular cubic hypersurfaces

The proof of Theorem 1 will depend intimately on the dimension of the hypersurfaces defined by the constituent cubic forms and the nature of their singularities. A key step will be to determine conditions on this singular locus under which the hypersurface automatically has rational points.

Let $C \in \mathbb{Z}[x_1, \ldots, x_n]$ be an arbitrary cubic form, which we assume takes the shape

$$C(\mathbf{x}) := \sum_{i,j,k} c_{ijk} x_i x_j x_k, \tag{2.1}$$

in which the coefficients $c_{ijk} \in \mathbb{Z}$ are symmetric in the indices i, j, k. Define the $n \times n$ matrix $M(\mathbf{x})$ with j, k-entry $\sum_i c_{ijk} x_i$. We will say that the cubic form C is 'good' if for any $H \geqslant 1$ and any $\varepsilon > 0$ we have the upper bound

$$\#\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| \leqslant H, \operatorname{rank} M(\mathbf{x}) = r\} \ll H^{r+\varepsilon},$$

for each integer $0 \leqslant r \leqslant n$. A crucial step in Davenport's [Dav63] treatment of general cubic forms is a proof of the fact that forms that fail to be good automatically possess non-trivial integer solutions for 'geometric reasons'. Our approach has a similar flavour, although the underlying arguments will be more obviously geometric.

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Assume throughout this section that $n \geqslant 3$ and $X \subset \mathbb{P}^{n-1}$ is a hypersurface defined by a cubic form $C \in \mathbb{Z}[x_1, \ldots, x_n]$. A lot of the facts that we will record are classical. Suppose for the moment that C is not absolutely irreducible. Then either it has a linear factor L defined over $\mathbb{Q}$, or it is a product $C = L_1 L_2 L_3$ of three linear factors that are conjugate over $\overline{\mathbb{Q}}$. By considering the equation $L = 0$ in the former case, we deduce that $X(\mathbb{Q}) \neq \emptyset$. In the latter case, we arrive at the same conclusion when $n \geqslant 4$, by considering the system of equations $L_1 = L_2 = L_3 = 0$. When $n = 3$ and C is a product of three conjugate factors we deduce that X has precisely three conjugate singular points. When $n \geqslant 3$ and X is defined by an absolutely irreducible cubic form, but is a cone, we note that the space of vertices on X must be a linear space globally defined over $\mathbb{Q}$. Thus $X(\mathbb{Q}) \neq \emptyset$ in this case too. We have therefore established the following simple result.

LEMMA 1. *Let $n \geqslant 4$. If X is not geometrically integral, or if X is a cone, then $X(\mathbb{Q}) \neq \emptyset$. When $n = 3$ the same conclusion holds unless X contains precisely three conjugate singular points.*

Recall that a cubic hypersurface X is said to be non-singular if over $\overline{\mathbb{Q}}^n$ the only solution to the system of equations $\nabla C(\mathbf{x}) = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$. Henceforth we will be predominantly interested in singular cubic forms, and then only in the cases $n = 3, 4$ and 5. Let k be a field. It has been conjectured by Cassels and Swinnerton-Dyer that any cubic hypersurface $X \subset \mathbb{P}^{n-1}$ defined over k that contains a k-rational 0-cycle of degree coprime to three, automatically has a k-rational point. The case $n = 3$ goes back to Poincaré. When the singular locus is non-empty, the case $n = 4$ can be deduced from the work of Skolem [Sko55]. A comprehensive discussion of the arithmetic of singular cubic surfaces can be found in the work of Coray and Tsfasman [CT88]. Coray [Cor76] has established the conjecture for all local fields and, in a subsequent investigation [Cor87, Proposition 3.6], has also dispatched the case in which $n = 5$ and the 0-cycle is made up of double points.

Suppose first that $n = 3$, so that $X \subset \mathbb{P}^2$ defines a cubic curve, which we assume to be geometrically integral and not a cone. When X is singular it contains exactly one singular point, which must therefore be defined over $\mathbb{Q}$. Once combined with Lemma 1 we arrive at the following result.

LEMMA 2. *Let $n = 3$ and suppose that $X(\mathbb{Q}) = \emptyset$. Then one of the following holds.*

(i) *The curve X is non-singular.*

(ii) *The curve X contains precisely three conjugate singular points.*

In case (ii) of Lemma 2 one concludes that the underlying cubic form can be written as a norm form (1.2), for appropriate $\omega_1, \omega_2, \omega_3 \in K$, where K is the cubic number field obtained by adjoining one of the singularities.

We turn now to the case $n = 4$ of cubic surfaces $X \subset \mathbb{P}^3$, which we suppose to be geometrically integral and not equal to a cone. Suppose that X is singular. The classification of such cubic surfaces can be traced back to Cayley [Cay69] and Schläfli [Sch64], but we will employ the modern treatment found in the work of Bruce and Wall [BW79]. In particular the singular locus of X is either a single line, in which case X is ruled, or else it contains $\delta \leqslant 4$ isolated singularities and these are all rational double points. It follows that $X(\mathbb{Q}) \neq \emptyset$ unless $\delta = 3$ and the three singular points are conjugate to each other over a cubic extension of $\mathbb{Q}$. In this final case, Skolem [Sko55] showed that C can be written as

$$\mathrm{N}_{K/\mathbb{Q}}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) + ax_4^2 \,\mathrm{Tr}_{K/\mathbb{Q}}(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) + bx_4^3, \tag{2.2}$$

for appropriate coefficients $\omega_1, \omega_2, \omega_3 \in K$ and $a, b \in \mathbb{Z}$, where K is the cubic number field obtained by adjoining one of the singularities to $\mathbb{Q}$. In terms of the classification over $\overline{\mathbb{Q}}$ according

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to singularity type, the only possibility here is that X has singularity type $3\mathbf{A}_i$ for $i=1$ or 2, since the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ preserves the singularity type. Bringing this all together, we have therefore established the following analogue of Lemma 2.

Lemma 3. *Let $n=4$ and suppose that $X(\mathbb{Q})=\emptyset$. Then one of the following holds.*

(i) *The surface X is non-singular.*

(ii) *The surface X contains precisely three conjugate double points.*

We now try to construct a version of Lemmas 2 and 3 for the case $n=5$. Let $Y\subset X$ denote the singular locus of $X\subset\mathbb{P}^4$, a variety of dimension at most two. As usual we assume that X is geometrically integral and not a cone. We analyse Y by considering the intersection of X with a generic hyperplane $H\in\mathbb{P}^{4*}$. In particular the hyperplane section

$$S_H=H\cap X$$

is a geometrically integral cubic surface which is not a cone (see [Har92, Proposition 18.10], for example). In taking H to be defined over $\mathbb{Q}$, we may further assume that S_H is defined over $\mathbb{Q}$. Any $\mathbb{Q}$-rational point on S_H visibly produces a $\mathbb{Q}$-rational point on X. Let $T_H\subset S_H$ denote the singular locus of S_H. Then the classification of cubic surfaces implies that T_H is either empty or it is a union of $\delta_H\leqslant 4$ points or it is a line. When T_H is non-empty it follows from Lemma 3 that either $S_H(\mathbb{Q})\neq\emptyset$ or else T_H is finite, with $\delta_H=3$ and with the three points being conjugate to each other over $\overline{\mathbb{Q}}$.

Now an application of Bertini's theorem (in the form given by Harris [Har92, Theorem 17.16], for example) shows that

$$H\cap Y=T_H.$$

When S_H is non-singular it therefore follows that $H\cap Y$ is empty for generic $H\in\mathbb{P}^{4*}$, whence Y must be finite. In the alternative case, when S_H is singular, we may conclude that $\#(H\cap Y)=3$ for generic $H\in\mathbb{P}^{4*}$, whence the maximal component of Y is a cubic curve.

Let us examine further the possibility that the singular locus Y of X has dimension one, and that it contains a cubic curve Y_0 as its component of maximal dimension. Clearly Y_0 is defined over $\mathbb{Q}$. Furthermore, we may conclude from Bézout's theorem that the line connecting any two points of Y_0 must be contained in X, since each such point is a singularity of X.

If Y_0 is reducible over $\mathbb{Q}$ then there are two basic possibilities: either it is a union of lines or it is a union of a conic and a line. In the latter case Y_0 contains a line defined over $\mathbb{Q}$ and it trivially follows that $Y_0(\mathbb{Q})\neq\emptyset$. The former case fragments into a number of subcases: either it is a union of three concurrent lines, or it contains a pair of skew lines, or it is a union of three coplanar lines, or it contains a repeated line. The second case is impossible since then the join of the two skew lines defines a 3-plane that would also be contained in X, contradicting the fact that X is geometrically irreducible. It follows from consideration of the Galois action on Y_0 that $Y_0(\mathbb{Q})\neq\emptyset$ in every case apart from the one in which Y_0 is a union of three coplanar lines.

If Y_0 is geometrically irreducible then it cannot be a twisted cubic since then the secant variety $S(Y_0)\cong\mathbb{P}^3$ would be contained in X. Our argument so far has shown that either $Y_0(\mathbb{Q})\neq\emptyset$ for trivial reasons, or else Y_0 is a cubic plane curve that is either geometrically irreducible or a union of three distinct lines. The plane P containing Y_0 is defined over $\mathbb{Q}$ and, after carrying out a linear change of variables, we may take $x_1=x_2=0$ as its defining equations. However, it then follows that the cubic form defining X can be written

$$x_1Q_1(x_1,\ldots,x_5)+x_2Q_2(x_1,\ldots,x_5),$$

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for appropriate quadratic forms Q_1, Q_2 defined over $\mathbb{Z}$. With this notation one sees that Y is the locus of solutions to the system of equations

$$x_1 = x_2 = Q_1(0, 0, x_3, x_4, x_5) = Q_2(0, 0, x_3, x_4, x_5) = 0,$$

in $\mathbb{P}^4$. It is now clear that the component Y_0 of Y of maximal dimension cannot be a cubic plane curve of the two remaining types.

It remains to deal with the case in which the singular locus Y of X is finite and globally defined over $\mathbb{Q}$. As shown by Segre [Seg86/87], we have $\delta = \#Y \leqslant 10$, the extremal case of 10 singular points being achieved by the so-called Segre threefold. Since X is assumed not to be a cone so we may assume that all of the singularities are double points. Indeed any singularity with multiplicity exceeding two must be a vertex for X. In fact, when $\delta \geqslant 6$ it is known [CLSS99, Lemma 2.2] that all the singularities are actually nodal. Appealing to Coray's partial resolution of the Cassels–Swinnerton-Dyer conjecture for threefolds, we are now ready to record our analogue of Lemmas 2 and 3.

Lemma 4. *Let $n = 5$ and suppose that $X(\mathbb{Q}) = \emptyset$. Then one of the following holds.*

(i) *The threefold X is non-singular.*

(ii) *The threefold X is a geometrically integral cubic hypersurface whose singular locus contains precisely δ double points, with $\delta \in \{3, 6, 9\}$.*

In the second case of Lemma 4 it follows from [CS89, CLSS99] that the Hasse principle holds for X when $\delta = 3$ or 6. Our investigation would be made easier if we were also in possession of this fact when $\delta = 9$. Lacking this, all that we actually require from part (ii) of Lemma 4 is that the singular locus should be finite. In his survey of open problems in Diophantine geometry, Lewis [Lew89] reports on unpublished joint work with Blass, which would appear to give Lemma 4. However, in the absence of subsequent elucidation, we have chosen to present our own proof of this result.

3. Cubic exponential sums

Let $C \in \mathbb{Z}[x_1, \dots, x_n]$ be an arbitrary cubic form, assumed to take the shape (2.1). Our work in this section centres upon various properties of the cubic exponential sums

$$S(\alpha) = S_w(\alpha; C, P) := \sum_{\mathbf{x} \in \mathbb{Z}^n} w(P^{-1}\mathbf{x}) e(\alpha C(\mathbf{x})), \tag{3.1}$$

for a suitable family of weights w on $\mathbb{R}^n$, and cubic forms that are always either good (in the sense of the previous section) or the hypersurface they define has finite (possibly empty) singular locus. Specifically, we will collect together some general upper bounds for $S(\alpha)$, some estimates for suitable moments of $S(\alpha)$ and some asymptotic formulae for $S(\alpha)$ when suitable assumptions are made about how α can be approximated by rational numbers. All of these estimates will depend on the parameter P which should be thought of as tending to infinity.

We must begin by saying a few words about the weight functions that we will be working with. Let $n_1, n_2 \geqslant 0$ such that $n_1 + n_2 = n$. When $n_i \geqslant 1$ we let $\mathbf{z}_i \in \mathbb{R}^{n_i}$ be certain vectors, which we think of as being fixed, but whose nature will be determined later. Similarly we let $\rho > 0$. All of the estimates in our work will be allowed to depend upon the choice of $\mathbf{z}_1, \mathbf{z}_2$ and ρ. Define $w_1 : \mathbb{R}^{n_1} \to \mathbb{R}_{\geqslant 0}$, via

$$w_1(\mathbf{x}_1) := \exp(-\|\mathbf{x}_1 - \mathbf{z}_1\|^2 (\log P)^4), \tag{3.2}$$

where we have written $\mathbf{x}_1 = (x_1, \dots, x_{n_1})$. Let $P_0 = P(\log P)^{-2}$. Then

$$w_1(P^{-1}\mathbf{x}_1) = \exp(-\|\mathbf{x}_1 - P\mathbf{z}_1\|^2 P_0^{-2})$$

is exactly the weight function introduced by Heath-Brown in [Hea83, § 3]. Note that

$$\nabla w_1(\mathbf{x}_1) = -2(\log P)^4 w_1(\mathbf{x}_1)(x_1 - z_1, \dots, x_{n_1} - z_{n_1}),$$

so that $\nabla w_1(\mathbf{x}_1) \ll (\log P)^4$ for any $\mathbf{x}_1 \in \mathbb{R}^{n_1}$. Next we let $w_2 : \mathbb{R}^{n_2} \to \{0, 1\}$ denote the characteristic function

$$w_2(\mathbf{x}_2) := \begin{cases} 1 & \text{if } |\mathbf{x}_2 - \mathbf{z}_2| < \rho, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $\mathbf{x}_2 = (x_{n_1+1}, \dots, x_n)$.

Each weight w appearing in our work will either be of the form $w_1(\mathbf{x})$ or $w_2(\mathbf{x})$ or $w = w_1(\mathbf{x}_1)w_2(\mathbf{x}_2)$, for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n$, depending on context. To help distinguish which estimates are valid for which choice of weight function, let us denote by $\mathcal{W}_n^{(1)}$ the set of non-negative weight functions on $\mathbb{R}^n$ that are of the shape (3.2), and let $\mathcal{W}_n^{(2)}$ denote the corresponding set of weight functions on $\mathbb{R}^n$ of the type (3.3). We let $\mathcal{W}_n$ denote the set of mixed functions $w = w_1 w_2$, with $w_i \in \mathcal{W}_{n_i}^{(i)}$ for $i = 1, 2$. In particular $\mathcal{W}_n^{(i)} \subset \mathcal{W}_n$ for $i = 1, 2$. In the definition of these sets the precise value of $\mathbf{z}_1, \mathbf{z}_2$ or ρ is immaterial, unless explicitly indicated otherwise, and the corresponding implied constants will always be allowed to depend on these quantities in any way.

We are now ready to record the upper bounds for $S(\alpha)$ that feature in our investigation.

Lemma 5. *Let $\varepsilon > 0$, let $w \in \mathcal{W}_n$ and assume that $C \in \mathbb{Z}[x_1, \dots, x_n]$ is a good cubic form. Let $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{3/2}$ and $\gcd(a, q) = 1$. Then if $\alpha = a/q + \theta$ we have*

$$S(\alpha) \ll P^{n+\varepsilon}(q|\theta| + (q|\theta|P^3)^{-1})^{n/8}.$$

If furthermore $|\theta| \leqslant q^{-1}P^{-3/2}$, then we have

$$S(\alpha) \ll P^{n+\varepsilon} q^{-n/8} \min\{1, (|\theta|P^3)^{-n/8}\}.$$

Proof. This is the essential content of the investigation of Davenport [Dav63] into cubic forms in 16 variables. The bounds are derived in a more succinct manner by Heath-Brown [Hea07, § 2]. The fact that we are working with exponential sums that are differently weighted makes no difference to the validity of the argument, and the reader may wish to consult [BH09b, § 9], where the necessary modifications can be found in the setting of quartic forms. □

Define the complete exponential sum

$$S_{a,q} := \sum_{\mathbf{y} \bmod q} e_q(aC(\mathbf{y})), \tag{3.4}$$

for any coprime integers a, q such that $q > 0$. It can easily be deduced from the proof of Lemma 5 that $S_{a,q} \ll q^{7n/8+\varepsilon}$ for any $\varepsilon > 0$, under the assumption that the cubic form is good. The following improvement is due to Heath-Brown [Hea07, § 7].

Lemma 6. *Let $\varepsilon > 0$ and assume that $C \in \mathbb{Z}[x_1, \dots, x_n]$ is a good cubic form. Then we have $S_{a,q} \ll q^{5n/6+\varepsilon}$.*

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We now come to the real workhorse in our argument. Given $R, \phi > 0$ and $v > 0$ we define

$$\mathcal{M}_v(R, \phi, \pm) := \sum_{R < q \leqslant 2R} \sum_{\substack{a \bmod q \\ \gcd(a,q)=1}} \int_{\phi}^{2\phi} \left| S\left(\frac{a}{q} \pm t\right)\right|^v dt. \tag{3.5}$$

The following result provides an upper bound for this quantity.

LEMMA 7. *Let $\varepsilon > 0$, let $w \in \mathcal{W}_n$ and assume that $C \in \mathbb{Z}[x_1, \dots, x_n]$ is a good cubic form. Let $R, \phi > 0$, with $R \leqslant P^{3/2}$ and $\phi \leqslant R^{-2}$. Then for any $v \in [0, 2]$ and any $H \in [1, P] \cap \mathbb{Z}$ we have*

$$\mathcal{M}_v(R, \phi, \pm) \ll P^3 + R^2 \phi^{1-v/2} \left(\frac{\psi_H P^{2n-1+\varepsilon}}{H^{n-1}} F\right)^{v/2},$$

where

$$\psi_H := \phi + \frac{1}{P^2 H}, \quad F := 1 + (RH^3 \psi_H)^{n/2} + \frac{H^n}{R^{n/2}(P^2 \psi_H)^{(n-2)/2}}.$$

Proof. It is clear that $\mathcal{M}_0(R, \phi, \pm) \ll R^2 \phi$. Hence it follows from Hölder's inequality that

$$\mathcal{M}_v(R, \phi, \pm) \ll (R^2 \phi)^{1-v/2} \mathcal{M}_2(R, \phi, \pm)^{v/2}.$$

On employing Heath-Brown's estimate for $\mathcal{M}_2(R, \phi, \pm)$, which follows from [Hea07, (4.5) and (5.1)], we therefore deduce that

$$\mathcal{M}_v(R, \phi, \pm) \ll (R^2 \phi)^{1-v/2} \left(\psi_H R^2 \left(P^2 H + \frac{P^{2n-1+\varepsilon}}{H^{n-1}} F\right)\right)^{v/2}.$$

As in the deduction of Lemma 5, the fact that we are working with differently weighted exponential sums makes no difference to the final outcome of the argument.

Using the fact that $R \leqslant P^{3/2}$ and $\phi \leqslant R^{-2}$, with $H \leqslant P$, it easily follows that the term involving $P^2 H$ contributes

$$\ll (R^2 \phi)^{1-v/2} (\phi R^2 P^3 + R^2)^{v/2} \ll R^2 \phi P^{3v/2} + R^2 \phi^{1-v/2} \ll P^3,$$

since $0 \leqslant v \leqslant 2$. This completes the proof of the lemma. □

Lemma 7 is based on an averaged version of van der Corput's method and comprises the key innovation in the work of Heath-Brown [Hea07] already alluded to. Although we have presented it in the context of denominators q and values of $\alpha = a/q \pm t$ restricted to dyadic intervals, the general result consists of a bound for $\int |S(\alpha)|^2 \, d\alpha$, where the integral is taken over a certain set of minor arcs. For cubic forms in few variables we will have better results available. When $n = 1$ and $w \in \mathcal{W}_1^{(2)}$, Hua's inequality [Dav05, Lemma 3.2] implies that

$$\int_0^1 |S(\alpha)|^{2^j} \, d\alpha \ll P^{2^j - j + \varepsilon},$$

for any $j \leqslant 3$. The following result is due to Wooley [Woo99], and generalises this to binary forms.

LEMMA 8. *Let $\varepsilon > 0$, let $w \in \mathcal{W}_2^{(2)}$ and let $C \in \mathbb{Z}[x_1, x_2]$ be a binary cubic form, not of the shape $a(b_1 x_1 + b_2 x_2)^3$, for integers a, b_1, b_2. Then we have*

$$\int_0^1 |S(\alpha)|^{2^{j-1}} \, d\alpha \ll P^{2^j - j + \varepsilon},$$

for any $j \leqslant 3$.

Our next selection of results concern the approximation of $S(\alpha)$ on a certain set of arcs in the interval $[0,1]$. For given $A,B,C\geqslant 0$, define $\mathcal{A}=\mathcal{A}(A,B,C)$ to be the set of $\alpha\in[0,1]$ for which there exists $a,q\in\mathbb{Z}$ such that $0\leqslant a<q\leqslant P^A$ and $\gcd(a,q)=1$, with

$$\alpha\in\mathcal{A}_{q,a}:=\left[\frac{a}{q}-\frac{1}{q^BP^{3-C}},\frac{a}{q}+\frac{1}{q^BP^{3-C}}\right]. \tag{3.6}$$

The major arcs in our work will be a subset of these, but it will be useful to maintain a certain degree of generality. When dealing with cubic forms whose singular locus is very small, we have rather good control over the approximation of $S(\alpha)$ on the arcs $\mathcal{A}=\mathcal{A}(A,B,C)$, provided that we work with the class of smooth weight functions $\mathcal{W}_n^{(1)}$. Recall the definition of $S_{a,q}$ from (3.4) and let

$$I_w(\psi):=\int_{\mathbb{R}^n}w(\mathbf{x})e(\psi C(\mathbf{x}))\,d\mathbf{x},$$

for $\psi\in\mathbb{R}$. We will need to work with the familiar quantity

$$S^*(\alpha):=q^{-n}P^nS_{a,q}I_w(\theta P^3), \tag{3.7}$$

concerning which we have the following result.

Lemma 9. *Let $\varepsilon>0$ and $n\geqslant 3$. Assume that $C\in\mathbb{Z}[x_1,\ldots,x_n]$ is a good cubic form defining a projective hypersurface that is not a cone, with singular locus of dimension $\sigma\in\{-1,0\}$. Let $A,B,C\geqslant 0$ such that $A<1$ and $B\in\{0,1\}$, and let $\alpha\in\mathcal{A}_{q,a}$. Then there exists $w\in\mathcal{W}_n^{(1)}$ such that*

$$S(\alpha)-S^*(\alpha)\ll P^{A(n/3+\sigma/2)+(n+1)/2+\varepsilon}+P^{A(1-B)(n+1+\sigma)/2+C(n+1/2)+\varepsilon}.$$

Furthermore, if $kn\geqslant 12$ and $k\leqslant 9$, then we have

$$\int_{\mathcal{A}}|S^*(\alpha)|^k\,d\alpha\ll P^{kn-3+\varepsilon}.$$

Proof. The proof of this result is based on the investigation carried out by Heath-Brown [Hea83] into non-singular cubic forms in 10 variables. One of the key ingredients in his approach is the Poisson summation formula, and it is this part of the argument that we plan to take advantage of.

We begin by choosing $\mathbf{z}_1\in\mathbb{R}^n$ to be a point at which the matrix of second derivatives of C has full rank at $\mathbf{z}_1$. The existence of such a point follows from the work of Hooley [Hoo91, Lemma 26]. With this choice of $\mathbf{z}_1$ we now select w to be the weight function in (3.2), which belongs to $\mathcal{W}_n^{(1)}$. Let

$$S_{a,q}(\mathbf{v}):=\sum_{\mathbf{y}\bmod q}e_q(aC(\mathbf{y})+\mathbf{v}.\mathbf{y}),$$

$$J_w(\psi,\mathbf{v}):=\int_{\mathbb{R}^n}w(P^{-1}\mathbf{x})e(\psi C(\mathbf{x})-\mathbf{v}.\mathbf{x})\,d\mathbf{x},$$

for any $\mathbf{v}\in\mathbb{R}^n$, and let $\alpha=a/q+\theta\in\mathcal{A}_{q,a}$. Then [Hea83, Lemma 8] yields

$$S(\alpha)-S^*(\alpha)\ll 1+q^{-n}\sum_{\substack{\mathbf{v}\in\mathbb{Z}^n\\ 1\leqslant|\mathbf{v}|\ll V}}S_{a,q}(\mathbf{v})J_w(\theta,q^{-1}\mathbf{v}),$$

where $V:=(\log P)^7q(P^{-1}+|\theta|P^2)$, and furthermore,

$$J_w(\theta,\mathbf{w})\ll P^n(\log P)^{7n}\min\{1,(|\theta|P^3)^{-1}\}^{(n-1)/2}, \tag{3.8}$$

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for any $\mathbf{w}\in\mathbb{R}^n$. The main difference between what we have recorded here and the statement of [Hea83, Lemma 8] is that our definition of $S(\alpha)$ does not involve a summation over a. This deviation makes no difference to the final outcome. Note that once the existence of a suitable point $\mathbf{z}_1$ is established for the definition of the weight function, the manipulations involving the exponential integral remain valid even when C is singular.

The summation over $\mathbf{v}$ in our upper bound for $S(\alpha)-S^*(\alpha)$ implies in particular $V\gg 1$. Since $A<1$ we automatically have $(\log P)^7 qP^{-1}\leqslant (\log P)^7P^{A-1}=o(1)$. Hence the condition $V\gg 1$ implies that

$$(\log P)^7 q|\theta|P^2\leqslant V\ll (\log P)^7 q|\theta|P^2$$

and

$$q^{-1}P^{-2}(\log P)^{-7}\ll|\theta|\leqslant q^{-B}P^{-3+C}. \tag{3.9}$$

Putting everything together it follows that

$$S(\alpha)-S^*(\alpha)\ll 1+q^{-n}(\log P)^{7n}P^{(3-n)/2}|\theta|^{(1-n)/2}\mathcal{T}(V),$$

where

$$\mathcal{T}(V):=\sum_{1\leqslant|\mathbf{v}|\ll V}|S_{a,q}(\mathbf{v})|.$$

We will show that

$$\mathcal{T}(V)\ll q^{(n+1+\sigma+\varepsilon)/2}(V^n+q^{n/3}), \tag{3.10}$$

for $\sigma\in\{-1,0\}$. Before doing so let us see how this suffices to complete the proof of the first part of the lemma. Recalling from above that V has order of magnitude $(\log P)^7q|\theta|P^2$, and employing (3.9), we deduce that

$$\begin{aligned}
S(\alpha)-S^*(\alpha) &\ll q^{-n/2+(1+\sigma)/2+2\varepsilon/3}P^{-(n-3)/2}|\theta|^{-(n-1)/2}((q|\theta|P^2)^n+q^{n/3})\\
&\ll q^{(n+1+\sigma)/2+2\varepsilon/3}|\theta|^{(n+1)/2}P^{3(n+1)/2}\\
&\quad+q^{-n/6+(1+\sigma)/2+2\varepsilon/3}|\theta|^{-(n-1)/2}P^{-(n-3)/2}\\
&\ll q^{(1-B)n/2-B/2+(1+\sigma)/2}P^{C(n+1)/2+\varepsilon}+P^{A(n/3+\sigma/2)+(n+1)/2+\varepsilon}.
\end{aligned}$$

If $B=1$ then the first term here is $O(P^{C(n+1)/2+\varepsilon})$, since $\sigma\leqslant 0$. Alternatively, if $B=0$, then the first term is $O(P^{A(n+1+\sigma)/2+C(n+1)/2+\varepsilon})$. This establishes the first part of the lemma subject to (3.10).

To establish (3.10) we return to the manipulations in [BH09b, § 5]. Things are simplified slightly by no longer needing to keep track of the dependence on C in each implied constant. In particular we may take $H\ll 1$ throughout. The sum $S_{a,q}(\mathbf{v})$ satisfies a basic multiplicativity property, as recorded in [BH09b, Lemma 10]. Write $q=bc^2d$, where

$$b:=\prod_{\substack{p^e\| q\\ e\leqslant 2}}p^e,\quad d:=\prod_{\substack{p^e\| q\\ e\geqslant 3,2\nmid e}}p.$$

In particular $d\mid c$ and we deduce from [BH09b, Lemmas 7, 10 and 11] that

$$\mathcal{T}(V)\ll q^{n/2}b^{(1+\sigma)/2+\varepsilon/2}\sum_{1\leqslant|\mathbf{v}|\ll V}\sum_{\substack{\mathbf{a}\bmod c\\ c|(a\nabla C(\mathbf{a})+\mathbf{v})}}N_d(\mathbf{a})^{1/2}.$$

Here $N_m(\mathbf{x})$ is the number of $\mathbf{y}$ modulo m such that $M(\mathbf{x})\mathbf{y}\equiv\mathbf{0}\bmod m$, where $M(\mathbf{x})$ is the matrix of second derivatives of $C(\mathbf{x})$. Recalling the notation of [BH09b, Lemma 12], in which we

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take $\mathbf{v}_0 = \mathbf{0}$ and $g = C$, it follows that there is an absolute constant $\kappa > 0$ such that

$$\mathcal{T}(V) \ll q^{n/2} b^{(1+\sigma+\varepsilon)/2} \mathcal{S}(\kappa V, a).$$

We would now like a version of [BH09b, Lemma 16] which applies to singular forms as well. We claim that

$$\mathcal{S}(\kappa V, a) \ll c^{\varepsilon} d^{(1+\sigma)/2} V^n \left(1 + \frac{c^2 d}{V^3}\right)^{n/2}. \tag{3.11}$$

This relies completely on first establishing suitable analogues of [BH09b, Lemmas 13 and 14]. A little thought reveals that in the present setting we have

$$\sum_{|\mathbf{r}| \leqslant R} N_m(\mathbf{r})^{1/2} \ll m^{n/2} \left(1 + \frac{R^3}{m}\right)^{n/2} R^{\varepsilon},$$

for any $m \in \mathbb{N}$ and $R \geqslant 1$. Here we have used the fact that C is good to bound the number of $|\mathbf{r}| \leqslant R$ such that $\operatorname{rank} M(\mathbf{r}) = t$, rather than using [BH09b, Lemma 2], as there. Furthermore, we have

$$\sum_{\mathbf{a} \bmod d} N_d(\mathbf{a}) \ll d^{n+1+\sigma+\varepsilon}.$$

When $c < V$ it follows from the latter bound and an application of Cauchy's inequality that $\mathcal{S}(\kappa V, a) \ll d^{(1+\sigma+\varepsilon)/2} V^n$, which is acceptable for (3.11). In the alternative case, when $c \geqslant V$, the necessary modifications to the proof of [BH09b, Lemma 16] are straightforward and we omit full details here.

We may now insert (3.11) into the preceding estimate for $\mathcal{T}(V)$ to conclude that

$$\mathcal{T}(V) \ll q^{(n+1+\sigma+\varepsilon)/2} V^n \left(1 + \frac{q}{V^3}\right)^{n/2}.$$

If $q^{1/3} \leqslant V$ then this is clearly satisfactory for (3.10). Alternatively, if $V < q^{1/3}$ then we can only enlarge our bound for $\mathcal{T}(V)$ if we replace V by $q^{1/3}$. But then $\mathcal{T}(V)$ is easily seen to be bounded by (3.10) in this case too. This therefore completes the proof of (3.10).

Our final task is to establish the second part of the lemma. Since C is good, we may combine Lemma 6 with (3.8) to deduce that

$$S^*(\alpha) \ll q^{-n/6} P^n (\log P)^{7n} \min\{1, (|\theta| P^3)^{-1}\}^{(n-1)/2}.$$

Let us write $T = q^{-B} P^{-3+C}$ for convenience. It therefore follows that

$$\begin{aligned}\int_{\mathcal{A}} |S^*(\alpha)|^k \, d\alpha &\ll P^{kn+\varepsilon/2} \sum_{q \leqslant P^A} q^{1-kn/6} \int_{-T}^{T} \min\{1, (|\theta| P^3)\}^{-k(n-1)/2} \, d\theta \\ &\ll P^{kn-3+\varepsilon/2} \sum_{q \leqslant P^A} q^{1-kn/6} \\ &\ll P^{kn-3+\varepsilon},\end{aligned}$$

since $kn \geqslant 12$ and $k \leqslant 9$. □

We remark that when $\sigma = 0$ it seems likely that an even sharper error term is available in Lemma 9 through a more careful analysis of the complete exponential sums $S_{a,q}(\mathbf{v})$, when q is prime. It follows from [Hoo91, Lemma 28] that the form C is automatically good when the corresponding hypersurface has at most isolated singularities and these are suitably mild.

In the setting of one-dimensional exponential sums, we have even better control over $S(\alpha)$ on the arcs $\mathcal{A}=\mathcal{A}(A,B,C)$. Let $C(x)=cx^3$ for some non-zero coefficient $c\in\mathbb{Z}$. Then for any $a,q\in\mathbb{Z}$ such that $0\leqslant a<q\leqslant P^A$ and $\gcd(a,q)=1$, and any $\alpha=a/q+\theta\in\mathcal{A}_{q,a}$, the standard major arc analysis would provide an estimate of the shape

$$S(\alpha)=S^*(\alpha)+O(P^A+P^{A+C-AB}),$$

where $S^*(\alpha)$ is given by (3.7). Our final result in this section improves on this substantially, and is readily derived from the book of Vaughan [Vau97, § 4].

LEMMA 10. *Let $\varepsilon>0$, let $n=1$ and let $w\in\mathcal{W}_1^{(2)}$. Let $A,B,C\geqslant 0$ with $A,B\leqslant 1$. Then for any $\alpha\in\mathcal{A}_{q,a}$ we have*

$$S(\alpha)=S^*(\alpha)+O(P^{A/2+\varepsilon}+P^{(A+C-AB)/2+\varepsilon}).$$

Furthermore, if $k\geqslant 4$, then we have

$$\int_{\mathcal{A}}|S^*(\alpha)|^k\,d\alpha\ll P^{k-3+\varepsilon}.$$

4. Density of rational points on cubic hypersurfaces

Let $X\subset\mathbb{P}^{n-1}$ be a cubic hypersurface, not equal to a cone, that is defined by an absolutely irreducible cubic form $F\in\mathbb{Z}[x_1,\ldots,x_n]$. For $P\geqslant 1$, let

$$N_{n,F}(P):=\#\{\mathbf{x}\in\mathbb{Z}^n:|\mathbf{x}|\leqslant P,\ F(\mathbf{x})=0\}.$$

According to the conjecture of Manin [FMT89] one expects $N_{n,F}(P)\sim cP^{n-3}$ for some constant $c\geqslant 0$ as soon as F is non-singular and $n\geqslant 5$. When F is not necessarily non-singular, or the number of variables is small, there is the dimension growth conjecture due to Heath-Brown. This predicts that

$$N_{n,F}(P)\ll P^{n-2+\varepsilon},\tag{4.1}$$

and has received a great deal of attention in recent years. Let σ denote the projective dimension of the singular locus of X. The dimension growth conjecture has been established by the author [Bro07] when $n\geqslant 6+\sigma$. The following result, which may of independent interest, shows that one can do better than (4.1) if larger values of n are permitted.

LEMMA 11. *We have $N_{n,F}(P)\ll P^{n-5/2+\varepsilon}$ when $n\geqslant 9+\sigma$.*

Proof. Our proof of the lemma is based on the approach developed in [Bro07]. Arguing with hyperplane sections, as in [Bro07, § 2], we see that it will suffice to show that there is an absolute constant $\theta>0$ such that

$$N_w(g;P):=\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n\\ g(\mathbf{x})=0}}w(P^{-1}\mathbf{x})\ll H^\theta P^{n-5/2+\varepsilon},\tag{4.2}$$

for any weight function $w:\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$ belonging to the class of weight functions described at the start of [Bro07, § 2], any $H\geqslant\|g\|_P$, and any cubic polynomial $g\in\mathbb{Z}[x_1,\ldots,x_n]$ such that $n\geqslant 8$ and the cubic part g_0 is non-singular. Here we recall that $\|g\|_P:=\|P^{-3}g(P\mathbf{x})\|$, where $\|h\|$ denotes the height of a polynomial h.

The bulk of [Bro07] goes through verbatim, and we are left with reevaluating the estimation of $\Sigma_2=\Sigma_2(R,\mathbf{R};t)$ and $\Sigma_1=\Sigma_1(R,\mathbf{R};t)$ in [Bro07, § 5.1] and [Bro07, § 5.2], respectively. Beginning with the former, we note from [Bro07, (5.5)] that this breaks into an estimation of $\Sigma_{2,a}$ and $\Sigma_{2,b}$.

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The first of these is estimated as $O(H^\theta P^{3n/4-3/4+\varepsilon}+H^\theta P^{n-3+\varepsilon})$. Both of the exponents of P are clearly at most $n-5/2+\varepsilon$ when $n\geqslant 8$, as required for (4.2). Turning to $\Sigma_{2,b}$, one easily traces through the argument, finding that

$$\Sigma_{2,b}\ll H^\theta P^\varepsilon(P^{3n/4-3/4}+P^{n-2}E_n+P^{13n/16-1}+P^{7n/8-5/3}),$$

where

$$E_n=P^{-1-7n/40}R^{2-3n/20}R_2^{3n/10-3/2}\ll P^{-1-7n/40}R^{5/4}\ll P^{7/8-7n/40}.$$

Here the first term (respectively second term, sum of the final two terms) corresponds to the case $V\geqslant R_2$ (respectively $(R_2^2R_3)^{1/3}\leqslant V<R_2$, $V<(R_2^2R_3)^{1/3}$). A modest pause for thought reveals that all of these exponents are satisfactory when $n\geqslant 8$.

We now turn to the estimation of Σ_1 in [Bro07, § 5.2], which is again written as a sum $\Sigma_{1,a}+\Sigma_{1,b}$. Beginning with $\Sigma_{1,a}$, we easily observe that

$$\Sigma_{1,a}\ll H^\theta P^\varepsilon(P^{3n/4-3/4}+P^{n-2}E_n+P^{n-3}),$$

this time with

$$\begin{aligned}E_n&=P^{2-n/4}t^{1-n/12}R^{11/6-n/4}(R_2^2R_3)^{n/9-1/2}\\&\ll P^{-1/2}R^{-1/9}+P^{-1}R^{2/9}\ll P^{-1/2}.\end{aligned}$$

This therefore shows that $\Sigma_{1,a}\ll H^\theta P^{n-5/2+\varepsilon}$, as required for (4.2). Turning to $\Sigma_{1,b}$, we will need to modify the argument slightly. On noting that $R_2^{3/2}R_3^{1/2}\gg(R_2^2R_3)^{2/3}$, we easily deduce that

$$\Sigma_{1,b}\ll H^\theta P^\varepsilon(P^{3n/4-3/4}+P^{n-2+7/8-7n/40}+T),$$

where we have set

$$T:=P^ntR^{2-n/2}(R_2^2R_3)^{-(2/3)}\min\left\{R_2^n,\left(\frac{R_2^2R_3}{V}\right)^{n/2},R^{3n/8}\min\{1,(tP^3)^{-n/8}\}\right\}.$$

The first and second terms here are satisfactory for $n\geqslant 8$. Moreover the third term is clearly satisfactory for $n\geqslant 16$, on taking $\min\{A,B,C\}=C$. To handle the contribution from the final term when $8\leqslant n<16$, it will be convenient to recall that V has order of magnitude $Rt^{1/2}P^{1/2}$ when $t\geqslant P^{-3}$ and R/P when $t<P^{-3}$.

Suppose first that $R\geqslant P$. When $t\geqslant P^{-3}$ we deduce that

$$\begin{aligned}T&\ll P^{n-3}\min\{R^{2-n}(R_2^2R_3)^{n/2-2/3}P^{n/2},R^{2-n/8}(R_2^2R_3)^{-2/3}\}\\&\ll P^{n-7/3}R^{5/6-n/8}\ll P^{7n/8-3/2},\end{aligned}$$

on taking

$$\min\{A,B\}\leqslant A^{4/(3n)}B^{1-4/(3n)}.\tag{4.3}$$

This is clearly satisfactory for $n\geqslant 8$. When $t<P^{-3}$ we easily deduce that the same bound holds on taking V to be of size R/P in the definition of T.

Suppose now that $R<P$ and $t\geqslant P^{-3}$. If $t>(R^2P)^{-1}$, then it is not hard to see that

$$\begin{aligned}T&\ll P^{n-1}R^{-n/2}(R_2^2R_3)^{-2/3}\min\{(R_2^2R_3)^{n/2},R^{5n/8}P^{-n/4}\}\\&\ll P^{3n/4-2/3}R^{n/8-5/6}\ll P^{7n/8-3/2},\end{aligned}$$

using (4.3). This is satisfactory for $n\geqslant 8$. Alternatively, if $P^{-3}\leqslant t\leqslant(R^2P)^{-1}$, then one finds that

$$T\ll P^nR^{2-n/2}(R_2^2R_3)^{-2/3}\min\{t(R_2^2R_3)^{n/2},R^{3n/8}t^{1-n/8}P^{-3n/8}\}.$$

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Using (4.3) it easily follows that

$$T \ll P^{5n/8+1/2} t^{7/6-n/8} R^{3/2-n/8}.$$

Since $t \leqslant (R^2P)^{-1}$ and $R < P$ this is clearly satisfactory when $n = 8$. If instead $n \geqslant 9$ then we deduce that

$$T \ll P^{n-3} R^{5-n/2} (R_2^2 R_3)^{n/2-14/3},$$

on taking $\min\{A, B\} \leqslant A^{1-8/n} B^{8/n}$ rather than (4.3). Since $R < P$ we easily conclude that $T \ll P^{n-5/2}$ in this case too. Finally, when $R < P$ and $t < P^{-3}$, we see that

$$\begin{aligned} T &\ll P^{n-3} \min\{R^{2-n/2}(R_2^2R_3)^{n/2-2/3}, R^{2-n/8}(R_2^2R_3)^{-2/3}\} \\ &\ll P^{n-3} R^{3/2-n/8} \ll P^{n-5/2}, \end{aligned}$$

using (4.3). This therefore concludes the proof of the lemma. □

It is clear from the proof of Lemma 11 that one actually achieves an estimate of the shape

$$N_{n,F}(P) \leqslant c_{\varepsilon,n} \|F\|^{\theta} P^{n-5/2+\varepsilon},$$

for a constant $\theta > 0$, when $n \geqslant 9 + \sigma$. It seems likely that one can push the analysis further, obtaining $N_{n,F}(P) \ll P^{n-3+\varepsilon}$ for $n \geqslant 11 + \sigma$, as predicted by Manin.

A key step in our argument involves generating good estimates for the moments

$$M_n(P) := \int_0^1 |S(\alpha)|^2 \, d\alpha, \tag{4.4}$$

where $S(\alpha)$ is the cubic exponential sum (3.1), for an appropriate weight $w \in \mathcal{W}_n$. By the orthogonality of the exponential function we have

$$M_n(P) = \sum_{\substack{\mathbf{x},\mathbf{y} \in \mathbb{Z}^n \\ C(\mathbf{x}) = C(\mathbf{y})}} w(P^{-1}\mathbf{x}) w(P^{-1}\mathbf{y}).$$

It is clear that there exists a constant $c > 0$ depending on w such that the overall contribution to $M_n(P)$ from $\mathbf{x}, \mathbf{y}$ such that $\max\{|\mathbf{x}|, |\mathbf{y}|\} > cP$ is $O(1)$, if P is taken to be sufficiently large. Hence it follows that

$$M_n(P) \ll N_{2n, C-C}(cP).$$

When C is a non-singular form in n variables it is obvious that $C - C$ is a non-singular form in $2n$ variables, defining a hypersurface of dimension $2n - 2$. When C has a finite non-empty singular locus it is not hard to see that $C - C$ has singular locus of dimension one. The following result now flows very easily from (4.1) and Lemma 11.

LEMMA 12. *Let $\varepsilon > 0$ and let $n \geqslant 3$. Assume that $C \in \mathbb{Z}[x_1, \ldots, x_n]$ is a cubic form defining a projective hypersurface whose singular locus has dimension $\sigma \in \{-1, 0\}$. Then we have*

$$M_n(P) \ll \begin{cases} P^{4+\varepsilon} & \text{if } n = 3 \text{ and } \sigma = -1, \\ P^{2n-5/2+\varepsilon} & \text{if } n \geqslant 5 + \sigma. \end{cases}$$

5. Cubics splitting off a form

In this section we establish Theorem 1. Let $n_1, n_2 \geqslant 1$ such that

$$n_1 + n_2 = n \geqslant 13.$$

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It will be convenient to write $\mathbf{x}=(x_1,\ldots,x_{n_1})$ and $\mathbf{y}=(y_1,\ldots,y_{n_2})$. We henceforth fix our attention on cubic forms of the shape

$$C(\mathbf{x},\mathbf{y})=C_1(\mathbf{x})+C_2(\mathbf{y}),$$

with $C_1\in\mathbb{Z}[\mathbf{x}]$ and $C_2\in\mathbb{Z}[\mathbf{y}]$. In what follows we may always suppose that $C=C_1+C_2$ is non-degenerate, by which we mean that it is not equivalent over $\mathbb{Z}$ to a cubic form in fewer variables, since such forms have obvious non-zero integral solutions. Recall the definition of 'good' cubic forms from § 2. It follows from [Dav05, § 14] that when $n_1\geqslant 3$ either C_1 is good or else the cubic hypersurface $C_1=0$ has a rational point. The analogue is true for the cubic forms C_2 and C_1+C_2. Since the existence of a rational point on any of these hypersurfaces is enough to ensure that $X(\mathbb{Q})\neq\emptyset$ in the statement of Theorem 1, so we may proceed under the assumption that C_1, C_2 and C_1+C_2 are all good when they possess at least three variables.

Let $w=w_1w_2\in\mathcal{W}_n$, as introduced in § 3. When C_1 satisfies the hypotheses in Lemma 9 we will assume that $w_1\in\mathcal{W}_{n_1}^{(1)}$ is the weight function constructed there. Our argument revolves around establishing an asymptotic formula for the sum

$$N(P):=\sum_{\substack{(\mathbf{x},\mathbf{y})\in\mathbb{Z}^n\\ C_1(\mathbf{x})+C_2(\mathbf{y})=0}} w_1(P^{-1}\mathbf{x})w_2(P^{-1}\mathbf{y}),$$

as $P\to\infty$. As is usual in applications of the Hardy–Littlewood circle method, the starting point is the simple identity

$$N(P)=\int_0^1 S_1(\alpha)S_2(\alpha)\,d\alpha,$$

where

$$S_i(\alpha):=\sum_{\mathbf{x}\in\mathbb{Z}^{n_i}} w_i(P^{-1}\mathbf{x})e(\alpha C_i(\mathbf{x}))$$

for $i=1,2$. It will be convenient to define

$$S(\alpha):=S_1(\alpha)S_2(\alpha)=\sum_{(\mathbf{x},\mathbf{y})\in\mathbb{Z}^n} w(P^{-1}(\mathbf{x},\mathbf{y}))e(\alpha(C_1(\mathbf{x})+C_2(\mathbf{y}))),$$

where $w=w_1w_2$. Then $S(\alpha)=S_w(\alpha;C_1+C_2,P)$ is a cubic exponential sum of the sort introduced in (3.1).

In the usual way one divides the interval $[0,1]$ into a set of major arcs and minor arcs. For major arcs we will take the union of intervals

$$\mathfrak{M}:=\bigcup_{q\leqslant P^{\Delta}}\ \bigcup_{\substack{a=0\\ \gcd(a,q)=1}}^{q-1}\left[\frac{a}{q}-P^{-3+\Delta},\frac{a}{q}+P^{-3+\Delta}\right],$$

which is equal to $\mathcal{A}(\Delta,0,\Delta)$ in the notation of (3.6). The corresponding set of minor arcs is defined modulo 1 as $\mathfrak{m}=[0,1]\backslash\mathfrak{M}$. Here $\Delta>0$ is an arbitrary small parameter. It turns out the choice

$$\Delta:=\tfrac{1}{10}$$

is acceptable. We may deduce from Lemma 15.4 and §§ 16–18 in [Dav05] (see also [Hea07, Lemma 2.1]) that

$$\int_{\mathfrak{M}} S(\alpha)\,d\alpha=\mathfrak{S}\mathfrak{I}P^{n-3}+o(P^{n-3}),$$

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where

$$\mathfrak{S} := \sum_{q=1}^{\infty} \sum_{\substack{a \bmod q \\ \gcd(a,q)=1}} q^{-n} S_{a,q}^{(1)} S_{a,q}^{(2)},$$

$$\mathfrak{I} := \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} w(\mathbf{x}, \mathbf{y}) e(\theta(C_1(\mathbf{x}) + C_2(\mathbf{x})))\, d\mathbf{x}\, d\mathbf{y}\, d\theta$$

are both absolutely convergent. Here, the absolute convergence of $\mathfrak{S}$ follows from Lemma 6, and we have written

$$S_{a,q}^{(i)} := \sum_{\mathbf{u} \in (\mathbb{Z}/q\mathbb{Z})^{n_i}} e_q(aC_i(\mathbf{u})),$$

for $i = 1, 2$. Since $\mathfrak{S}$ is absolutely convergent and $C = C_1 + C_2$ is non-degenerate, it follows from standard arguments (see [Dav59, Lemma 7.3], for example) that $\mathfrak{S} > 0$. The treatment of the singular integral is routine and we omit giving the details here, all of which can be supplied by consulting [Dav05, § 16] and [Hea83, § 4]. Assuming that neither C_1 nor C_2 has a linear factor defined over $\mathbb{Q}$ it is possible to choose $(\mathbf{z}_1, \mathbf{z}_2) \in \mathbb{R}^n$ in the definition of $w = w_1 w_2$, so that each $\mathbf{z}_i$ is a non-singular real solution to $C_i = 0$. On selecting a sufficiently small value of $\rho > 0$ in the definition of w_2 we can then ensure $\mathfrak{I} > 0$. The case in which C_1 or C_2 does factorise over $\mathbb{Q}$ clearly enables us to deduce the statement of Theorem 1 very easily.

In order to conclude the proof of Theorem 1 it remains to show that the overall contribution from the minor arcs

$$E := \int_{\mathfrak{m}} S_1(\alpha) S_2(\alpha)\, d\alpha, \tag{5.1}$$

is satisfactory. This is where the bulk of our work lies and we will find it necessary to undertake a lengthy case by case analysis to handle the different values of n_1 and n_2. In doing so it will suffice to handle the case $n_1 + n_2 = 13$, the case $n_1 + n_2 > 13$ being taken care of by [Hea07]. Without loss of generality we assume henceforth that $1 \leqslant n_1 \leqslant 6$.

Let $Q \geqslant 1$ and let $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem we may find coprime integers $1 \leqslant a \leqslant q$ such that $q \leqslant Q$ and $|q\alpha - a| \leqslant 1/Q$. The value of Q should satisfy $1 \leqslant Q \leqslant P^{3/2}$ and is chosen to optimise the final stages of the argument. The obvious approach involves applying estimates for each individual exponential sum $S_1(\alpha)$ and $S_2(\alpha)$ for $\alpha \in \mathfrak{m}$, before then deriving an estimate for the integral over the full set of minor arcs. While we have rather good control over these sums when n_1 and n_2 are both large, the case in which one of n_1 or n_2 is small presents more of an obstacle. Instead we apply Hölder's inequality to deduce that

$$|E| \leqslant \left(\int_{\mathfrak{m}} |S_1(\alpha)|^u\, d\alpha \right)^{1/u} \left(\int_{\mathfrak{m}} |S_2(\alpha)|^v\, d\alpha \right)^{1/v}, \tag{5.2}$$

for any $u, v > 0$ such that $1/u + 1/v = 1$. This will allow us to separate out the behaviour of the exponential sums $S_1(\alpha)$ and $S_2(\alpha)$ on the minor arcs.

Before embarking on the case by case analysis alluded to above, it will save needless repetition if we give some reasonably general estimates here that can be applied in various contexts. Our principal means for dealing with small values of n_1 relies on taking the inequality

$$\int_{\mathfrak{m}} |S_1(\alpha)|^u\, d\alpha \leqslant \int_0^1 |S_1(\alpha)|^u\, d\alpha \tag{5.3}$$

in (5.2). This will in turn be estimated as $O(P^{k+\varepsilon})$ for an appropriate $k>0$, whence a typical scenario entails studying

$$I_{u,v}(k;\mathfrak{n}) := P^{k/u+\varepsilon}\left(\int_{\mathfrak{n}} |S_2(\alpha)|^v \, d\alpha\right)^{1/v}, \tag{5.4}$$

for $u,v>0$ such that $1/u+1/v=1$ and certain subsets $\mathfrak{n}\subseteq\mathfrak{m}$. We will always assume that $6/5<v\leqslant 2$.

Let $\alpha\in\mathfrak{n}$ and let $Q\geqslant 1$. There exist coprime integers $0\leqslant a<q\leqslant Q$ such that $|q\alpha-a|\leqslant 1/Q$. An argument based on dyadic summation reveals that

$$I_{u,v}(k;\mathfrak{n}) \ll P^{k/u+\varepsilon}(\log P)^2 \max_{R,\phi,\pm} \mathcal{M}_v(R,\phi,\pm)^{1/v}, \tag{5.5}$$

where $\mathcal{M}_v(R,\phi,\pm)$ is given by (3.5), and the maximum is over the possible sign changes and R,ϕ such that

$$0<R\leqslant Q, \quad 0<\phi\leqslant (RQ)^{-1}. \tag{5.6}$$

Furthermore, R,ϕ should satisfy whatever conditions are appropriate to ensure we are dealing with points on $\mathfrak{n}$. In particular, since $\mathfrak{n}\subseteq\mathfrak{m}$ the inequalities $R\leqslant P^{\Delta}$ and $\phi\leqslant P^{-3+\Delta}$ cannot both hold simultaneously.

Let u,v,k be given. Define

$$\rho_n := \frac{n(vn-8-v)}{vn^2-(3v+4)n+2v}, \quad \pi_n := \frac{-2v(n^2-(18-2k/u)n-2)}{vn^2-(3v+4)n+2v}, \tag{5.7}$$

and

$$\rho_n' := \frac{2v}{2-v}\left(\frac{n^2}{2(3n-2)}-\frac{2}{v}\right), \quad \pi_n' := \frac{2v}{2-v}\left(n-\frac{23}{2}+\frac{k}{u}\right). \tag{5.8}$$

Let

$$\delta := \frac{1}{10^4}. \tag{5.9}$$

Recall that our task is to show that $E=o(P^{10})$ when $n=n_1+n_2=13$. The following result provides us with easily checked conditions on u,v,k and n_2 under which $I_{u,v}(k;\mathfrak{n})$ makes a satisfactory contribution.

Lemma 13. *Let $6/5<v<2$. Assume that $n_2\geqslant 6$ and*

$$\rho_{n_2}+\rho_{n_2'}\geqslant 1. \tag{5.10}$$

Define $\mathfrak{m}_0$ to be the set of $\alpha\in\mathfrak{m}$ for which there exist coprime integers $0\leqslant a<q$ such that

$$q\leqslant P^{(\pi_{n_2}'-\pi_{n_2})/(\rho_{n_2}'+\rho_{n_2})+2\delta}, \quad q^{\rho_{n_2}'}P^{-\pi_{n_2}'-\delta}\leqslant \left|\alpha-\frac{a}{q}\right| \leqslant q^{-\rho_{n_2}}P^{-\pi_{n_2}+\delta}.$$

Then

$$I_{u,v}(k;\mathfrak{n}\backslash\mathfrak{m}_0)=o(P^{10}),$$

for any $\mathfrak{n}\subseteq\mathfrak{m}$, provided that

$$\frac{2}{v}+\frac{21}{2}-n_2\leqslant\frac{k}{u}<\frac{103}{10}-\frac{4n_2}{5} \tag{5.11}$$

and

$$\pi_{n_2}'\geqslant 3. \tag{5.12}$$

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Proof. We will commence under the assumption that $6/5 < v \leqslant 2$, saving the restriction $v < 2$ until later in the argument. It is clear that $I_{u,v}(k; \mathfrak{n}\backslash\mathfrak{m}_0) \leqslant I_{u,v}(k; \mathfrak{m}\backslash\mathfrak{m}_0)$. Let us consider the consequences of applying Lemma 7 in our estimate (5.5) for $I_{u,v}(k; \mathfrak{m}\backslash\mathfrak{m}_0)$. Throughout the proof of Lemma 13 we will denote $\mathfrak{m}\backslash\mathfrak{m}_0$ by $\mathfrak{a}$ and we will set $n = n_2$. We may deduce from Lemma 7 and (5.4) that

$$I_{u,v}(k; \mathfrak{a}) \ll P^{k/u+\varepsilon}\left(P^{3/v} + \max_{R,\phi} R^{2/v}\phi^{1/v-1/2}\left(\frac{\psi_H P^{2n-1}}{H^{n-1}}F\right)^{1/2}\right),$$

where ψ_H and F are as in the statement of the lemma and $H \in [1, P] \cap \mathbb{Z}$ is arbitrary. Furthermore the maximum is over R, ϕ such that (5.6) holds with any choice of $Q \geqslant 1$ that we care to choose. We write $Q = P^\kappa$, with

$$\kappa := \frac{3(2n - 21 + 2k/u)}{n-1} + 3\varepsilon. \tag{5.13}$$

In particular one easily checks that $0 \leqslant \kappa \leqslant 3/2$ if (5.11) holds and $\varepsilon > 0$ is sufficiently small. It follows from (5.11) that $k/u < 11/2$ since $n \geqslant 6$. Hence the term involving $P^{3/v}$ contributes $O(P^{8+\varepsilon})$, which is satisfactory.

Let us now turn to the contribution from the term involving F in our estimate for $I_{u,v}(k; \mathfrak{a})$. Define

$$\phi_0 := (R^{-2/v}P^{-(2n-23/2+k/u)})^{2v/(v(n-1)+2)}. \tag{5.14}$$

Then our investigation will be optimised by taking

$$H := \begin{cases} \lfloor P^\varepsilon \max\{1, R^{2/v}\phi^{1/v}P^{n-21/2+k/u}\}^{2/(n-1)}\rfloor & \text{if } \phi > \phi_0,\\ \lfloor P^\varepsilon \max\{1, R^{2/v}\phi^{1/v-1/2}P^{n-23/2+k/u}\}^{2/n}\rfloor & \text{if } \phi \leqslant \phi_0. \end{cases}$$

One checks that $\psi_H \ll \phi$ when $\phi > \phi_0$ and $\psi_H \ll (P^2H)^{-1}$ when $\phi \leqslant \phi_0$. If $F \ll 1$ then

$$\frac{P^{n-1/2+k/u+\varepsilon}R^{2/v}\phi^{1/v-1/2}\psi_H^{1/2}}{H^{(n-1)/2}}F^{1/2} \ll P^{10-3\varepsilon/2},$$

since $n \geqslant 6$. We deduce from (5.6) that

$$(R^{2/v}\phi^{1/v}P^{n-21/2+k/u})^{2/(n-1)} \leqslant P^{2(n-21/2+k/u)/(n-1)},$$
$$(R^{2/v}\phi^{1/v-1/2}P^{n-23/2+k/u})^{2/n} \leqslant P^{2(n-10+k/u)/n}.$$

In either case the final exponent of P is less than 1, by (5.11). Hence H is an integer in the interval $[1, P]$ and it remains to show that $F \ll 1$ with this choice of H. Recall the definition of F from Lemma 7.

Suppose first that $\phi > \phi_0$, with ϕ_0 given by (5.14). Then $\psi_H \ll \phi$ and it follows that

$$RH^3\psi_H \ll R\phi P^{3\varepsilon}(1 + R^{2/v}\phi^{1/v}P^{n-21/2+k/u})^{6/(n-1)} \ll 1 + P^{-\kappa}P^{6(n-21/2+k/u)/(n-1)+3\varepsilon},$$

by (5.6). It follows from our expression (5.13) for κ that this is $O(1)$. Turning to the third term in the definition of F, we see that

$$\frac{H^n}{R^{n/2}(P^2\psi_H)^{(n-2)/2}} \ll \frac{P^{n\varepsilon}}{R^{n/2}\phi^{(n-2)/2}P^{n-2}}(1 + R^{2/v}\phi^{1/v}P^{n-21/2+k/u})^{2n/(n-1)}.$$

The exponent of ϕ in the second term is $2n/(v(n-1)) - (n-2)/2$, which is negative since $v > 6/5$ and $n \geqslant 6$. Hence this quantity is $O(1)$ provided that $\phi \geqslant \phi_1$, with

$$\phi_1 := R^{-\rho_n}P^{-\pi_n+\delta}, \tag{5.15}$$

with ρ_n, π_n given by (5.7) and δ given by (5.9). Here, as is customary, we have assumed that ε is sufficiently small. One also checks that taking $\phi \geqslant \phi_1$ is enough to ensure that the first term is $O(1)$, in view of the lower bound for k/u in (5.11).

Suppose now that $\phi \leqslant \phi_0$. Then $\psi_H \ll (P^2H)^{-1}$ and it follows that

$$\begin{aligned} RH^3\psi_H &\ll \frac{RP^{2\varepsilon}}{P^2}(1+R^{2/v}\phi^{1/v-1/2}P^{n-23/2+k/u})^{4/n} \\ &\ll 1+R^{1+8/(vn)}(RQ)^{-(1/v-1/2)(4/n)}P^{(n-23/2+k/u)(4/n)-2+2\varepsilon} \\ &\ll 1+P^{\kappa(1+4/n)}P^{2-46/n+4k/(un)+2\varepsilon}, \end{aligned}$$

since $Q=P^\kappa$. It follows from (5.11) and (5.13) that this is $O(1)$. Thus the second term makes a satisfactory contribution in F. Turning to the third term, we find that

$$\frac{H^n}{R^{n/2}(P^2\psi_H)^{(n-2)/2}} \ll \frac{P^{(3n-2)\varepsilon/2}}{R^{n/2}}(1+R^{2/v}\phi^{1/v-1/2}P^{n-23/2+k/u})^{(3n-2)/n}.$$

We now make the assumption $6/5<v<2$. Hence the overall contribution from the second term is $O(1)$ provided that $\phi \leqslant \phi_2$, with

$$\phi_2 := R^{\rho_n'}P^{-\pi_n'-\delta}, \tag{5.16}$$

with ρ_n', π_n' given by (5.8) and δ given by (5.9). Assuming (5.12) we note that if $\phi \leqslant \phi_2$ and $R \leqslant P^{(3n-2)\varepsilon/n}$ then we would have a point on the major arcs if ε is sufficiently small in terms of Δ, which we have seen to be impossible. Hence the inequality $\phi \leqslant \phi_2$ is also enough to ensure that the first term is $O(1)$.

When $v=2$ the exponent of ϕ is zero in the above and we will have an overall contribution of $O(1)$ unless

$$R \leqslant P^{(3n-2)(2n-23+k)/(n^2-6n+4)+\delta}. \tag{5.17}$$

We will return to this case shortly. Recall the definitions (5.14)–(5.16) of ϕ_0, ϕ_1, ϕ_2. It follows from the inequality $\phi_2<\phi_1$ that $R^{\rho_n'+\rho_n}<P^{\pi_n'-\pi_n+2\delta}$. We now employ the assumption (5.10) on the size of $\rho_n'+\rho_n$. Combining the above we conclude that there is an overall contribution of $o(P^{10})$ to $I_{u,v}(k;\mathfrak{a})$ from all of the relevant values of R, ϕ, apart from those which satisfy the inequalities

$$R<P^{(\pi_n'-\pi_n)/(\rho_n'+\rho_n)+2\delta}, \quad \phi_2<\phi<\phi_1.$$

But then the relevant point is forced to lie in the set $\mathfrak{m}_0$ that was defined in the statement of the lemma. This is impossible, and so completes the proof of Lemma 13. □

Our next result deals with the corresponding case in which $u=v=2$. In this setting (5.7) becomes

$$\rho_n=\frac{n(n-5)}{n^2-5n+2}, \quad \pi_n=\frac{-2(n^2-(18-k)n-2)}{n^2-5n+2}. \tag{5.18}$$

Define

$$\rho_n'' := \frac{(3n-2)(2n-23+k)}{n^2-6n+4} \tag{5.19}$$

and

$$\psi_n := \rho_n''\bigg(1+\frac{n}{8}-\frac{(4+n)\rho_n}{8}\bigg)+n+\frac{k}{2}-\frac{(4+n)\pi_n}{8}, \tag{5.20}$$

for any k and n, and recall the definition (5.9) of δ. Then we have the following result.

Lemma 14. *Let $u = v = 2$. Assume that $n_2 \geqslant 6$. Define $\mathfrak{m}_0$ to be the set of $\alpha \in \mathfrak{m}$ for which there exist coprime integers $0 \leqslant a < q$ such that*

$$q \leqslant P^{\rho''_{n_2}+\delta}, \quad \left|\alpha - \frac{a}{q}\right| \leqslant q^{-(n_2-8)/(n_2-4)} P^{-(80-5n_2-4k)/(n_2-4)+\delta}.$$

Then

$$I_{2,2}(k; \mathfrak{n}\backslash\mathfrak{m}_0) = o(P^{10}),$$

for any $\mathfrak{n} \subseteq \mathfrak{m}$, provided that (5.11) holds and

$$\psi_{n_2} \leqslant 10 - \tfrac{1}{10}. \tag{5.21}$$

Proof. We continue to write $\mathfrak{a} = \mathfrak{m}\backslash\mathfrak{m}_0$ and $n = n_2$ throughout the proof, in order to improve the appearance of our expressions. Our starting point is the proof of Lemma 13, which on passing to dyadic intervals via (5.5), shows that $I_{2,2}(k; \mathfrak{a}) = o(P^{10})$ unless $\phi < \phi_1$, in the notation of (5.15), and the inequality (5.17) holds for R. This much is valid subject to (5.11).

We now consider the effect of applying Lemma 5 in (5.5) when R and ϕ are in the remaining ranges, with $u = v = 2$. This gives

$$\begin{aligned} I_{2,2}(k; \mathfrak{a}) &\ll P^{n+k/2+2\varepsilon} \max_{R,\phi} (R^2\phi)^{1/2}(R\phi + (R\phi P^3)^{-1})^{n/8} \\ &\ll P^{2\varepsilon} \max_{R,\phi} (R^{1+n/8}\phi^{(4+n)/8}P^{n+k/2} + R^{1-n/8}\phi^{(4-n)/8}P^{5n/8+k/2}). \end{aligned}$$

Taking $\phi < \phi_1$ and recalling the assumed inequality (5.17) for R we see that the first term here is

$$\begin{aligned} &\ll R^{1+n/8}(R^{-\rho_n}P^{-\pi_n+\delta})^{(4+n)/8}P^{n+k/2+2\varepsilon} \\ &\ll R^{1+n/8-(4+n)\rho_n/8}P^{n+k/2-(4+n)\pi_n/8+2\varepsilon+((4+n)/8)\delta} \\ &\ll P^{\psi_n+2\varepsilon+(1+n/8-(4+n)\rho_n/8+(4+n)/8)\delta}, \end{aligned}$$

where ψ_n is given by (5.20). According to (5.21) this contribution is satisfactory. Turning to the second term in the above estimate for $I_{2,2}(k; \mathfrak{a})$, we will have $O(P^{10-\varepsilon})$ as an upper bound for this quantity provided that $\phi > \phi_3$, with

$$\phi_3 := R^{-(n-8)/(n-4)}P^{-(80-5n-4k)/(n-4)+\delta},$$

since $n \geqslant 6$ by assumption.

Our investigation has therefore allowed us to handle all α apart from those for which $R \leqslant P^{\rho''_n+\delta}$ and $\phi < \phi_3$, where ρ''_n is given by (5.19). Such points are forced to lie on the set of arcs defined in $\mathfrak{m}_0$. This therefore completes the proof of Lemma 14. □

The ideal scenario is when we can apply Lemma 14 with

$$k = 2n_1 - 3 = 23 - 2n_2,$$

and we will find this is possible for certain ranges of n_1, n_2 such that $n_1 + n_2 = 13$. When this comes to pass it follows from (5.18), (5.19) that $\pi_{n_2} = 2$ and $\rho''_{n_2} = 0$, and furthermore, $\psi_{n_2} = 21/2 - n_2/4$. One easily checks that the conditions in (5.11) and (5.21) are satisfied for $n_2 \geqslant 6$. Finally we note that $\mathfrak{m}\backslash\mathfrak{m}_0 = \mathfrak{m}$ in the statement of Lemma 14 since clearly any element of $\mathfrak{m}_0$ is forced to lie on the major arcs. We may conclude as follows.

Lemma 15. *Assume that $n_2 \geqslant 6$. Then we have*

$$I_{2,2}(2n_1 - 3; \mathfrak{m}) = o(P^{10}).$$

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We are now ready to apply this collection of estimates in our case by case analysis of the minor arc integral E in (5.1).

5.1 The case $n_1 = 1$

We will assume that $w \in \mathcal{W}_n^{(2)}$ throughout this section. One of the ingredients in our treatment of this case is the use of 'pruning'. We will find it convenient to sort the minor arcs into subsets

$$\emptyset = \mathfrak{n}_3 \subseteq \mathfrak{n}_2 \subseteq \mathfrak{n}_1 \subseteq \mathfrak{n}_0 := \mathfrak{m}.$$

Recall the definition (5.9) of δ. We define $\mathfrak{n}_1$ to be the set of $\alpha \in \mathfrak{m}$ for which there exists $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{17/24+2\delta}$ and $\gcd(a, q) = 1$, with

$$q^{42/17} P^{-4-\delta} \leqslant \left|\alpha - \frac{a}{q}\right| \leqslant q^{-1} P^{-37/24+2\delta}. \tag{5.22}$$

We denote by $\mathfrak{n}_2$ the corresponding set of $\alpha \in \mathfrak{n}_1$ with the property that whenever (5.22) holds with $\gcd(a, q) = 1$ and $0 \leqslant a < q \leqslant P^{17/24+\delta}$, then

$$q \leqslant P^{27/50}.$$

We will write E_i for the overall contribution to E from integrating over the set $\mathfrak{n}_i \backslash \mathfrak{n}_{i+1}$, for $i = 0, 1, 2$. Our task is to show that $E_i = o(P^{10})$ for each i.

To handle the case $i = 0$ we begin as in (5.2) and (5.3) with $(u, v) = (4, 4/3)$. It easily follows that

$$\int_0^1 |S_1(\alpha)|^4 \, d\alpha \ll P^{2+\varepsilon},$$

on interpreting the integral as a sum over the solutions of the equation $x_1^3 + x_2^3 = x_3^3 + x_4^3$, with $x_i \ll P$, and applying standard estimates for the divisor function. Hence we have

$$E_0 \ll I_{4,4/3}(2; \mathfrak{m} \backslash \mathfrak{n}_1),$$

in the notation of (5.4). When $(u, v) = (4, 4/3)$, $k = 2$ and $n_2 = 12$ we have

$$\rho_{12} = \tfrac{30}{37}, \quad \pi_{12} = \tfrac{62}{37}, \quad \rho'_{12} = \tfrac{42}{17}, \quad \pi'_{12} = 4,$$

in (5.7) and (5.8). In particular (5.10), (5.11) and (5.12) are satisfied. Now it is easily to see that $\mathfrak{m} \backslash \mathfrak{n}_1 \subset \mathfrak{m} \backslash \mathfrak{m}_0$, where $\mathfrak{m}_0$ is as in the statement of Lemma 13, since for $\alpha \in \mathfrak{m}_0$ we have

$$q^{42/17} P^{-4-\delta} \leqslant \left|\alpha - \frac{a}{q}\right| \leqslant q^{-30/37} P^{-62/37+\delta} = q^{-1} q^{7/37} P^{-62/37+\delta} \leqslant q^{-1} P^{-37/24+2\delta}.$$

It therefore follows from Lemma 13 that $E_0 = o(P^{10})$, as required.

Turning to the case $i = 1$, we begin as above with the observation that

$$E_1 \ll I_{4,4/3}(2; \mathfrak{n}_1 \backslash \mathfrak{n}_2).$$

This time we appeal to Lemma 5. On observing that

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-1} P^{-37/24+2\delta} \leqslant q^{-1} P^{-3/2},$$

for any $\alpha \in \mathfrak{n}_1$, we deduce from the second part of this result that

$$E_1 \ll P^{25/2+2\varepsilon} \max_{R,\phi} (R^2\phi)^{3/4} R^{-3/2} \min\{1, (\phi P^3)^{-3/2}\} \ll P^{8+2\varepsilon} \max_{R,\phi} \phi^{-3/4},$$

where the maximum is over all $R, \phi > 0$ such that

$$P^{27/50} < R \leqslant P^{17/24+2\delta}, \quad R^{42/17} P^{-4-\delta} < \phi < R^{-1} P^{-37/24+2\delta}.$$

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Taking the lower bounds for ϕ and R that emerge from these inequalities therefore implies that

$$E_1 \ll P^{11+3\delta/4+2\varepsilon} \max_R R^{-63/34} = o(P^{10}),$$

on recalling that $\delta = 10^{-4}$ from (5.9) and $\varepsilon > 0$ is arbitrary.

The key idea in our treatment of E_2 is to take advantage of the fact that we have rather good control of the one-dimensional exponential sum $S_1(\alpha)$ on suitable sets of 'major arcs'. Recall the definition (3.6) of $\mathcal{A} = \mathcal{A}(A, B, C)$. We will take $(A, B, C) = (24/50, 1, 35/24 + 2\delta)$, whence we may deduce from Lemma 10 that

$$S_1(\alpha) = S_1^*(\alpha) + O(P^{35/48+\delta+\varepsilon}),$$

for any $\alpha \in \mathcal{A}_{a,q}$, where $S_1^*(\alpha)$ is given by (3.7). It follows that

$$E_2 \ll \int_{\mathfrak{n}_2} |S_1^*(\alpha) S_2(\alpha)|\, d\alpha + P^{35/48+\delta+\varepsilon} \int_{\mathfrak{n}_2} |S_2(\alpha)|\, d\alpha = I_1 + I_2, \tag{5.23}$$

say. We will show that I_1 and I_2 are both $o(P^{10})$.

Let us begin by analysing the first term in this bound. Now it follows from the second part of Lemma 10 and Hölder's inequality that

$$I_1 \ll I_{4,4/3}(1, \mathfrak{n}_2),$$

in the notation of (5.4). A straightforward application of Lemma 13 reveals that $I_{4,4/3}(1, \mathfrak{n}_2 \backslash \mathfrak{n}_*) = o(P^{10})$, where $\mathfrak{n}_*$ is the set of $\alpha \in \mathfrak{n}_1$ for which there exists $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{17/48+2\delta}$ and $\gcd(a, q) = 1$, with

$$q^{42/17} P^{-3-\delta} \leqslant \left|\alpha - \frac{a}{q}\right| \leqslant q^{-30/37} P^{-68/37+\delta}.$$

To estimate $I_{4,4/3}(1, \mathfrak{n}_*)$ we employ the second part of Lemma 5 in much the same way that we did in our analysis of E_1. This implies that

$$I_{4,4/3}(1, \mathfrak{n}_*) \ll P^{49/4+2\varepsilon} \max_{R,\phi} (R^2\phi)^{3/4} R^{-3/2} \min\{1, (\phi P^3)^{-3/2}\},$$

where the maximum is over all $R, \phi > 0$ such that

$$R \leqslant P^{17/48+2\delta}, \quad R^{42/17} P^{-3-\delta} < \phi < R^{-30/37} P^{-68/24+\delta},$$

with the inequalities $R \leqslant P^{\Delta}$ and $\phi \leqslant P^{-3+\Delta}$ not both holding simultaneously. Taking the lower bound for ϕ we obtain the contribution

$$\ll P^{31/4+2\varepsilon} \phi^{-3/4} \ll P^{10+3\delta/4+2\varepsilon} R^{-63/34}.$$

This is $o(P^{10})$ if $R \geqslant P^{\delta}$. If on the other hand $R < P^{\delta} \leqslant P^{\Delta}$ we must automatically have $\phi \geqslant P^{-3+\Delta}$, whence we still obtain a satisfactory contribution. This completes the treatment of $I_{4,4/3}(1, \mathfrak{n}_*)$, and so that of I_1.

We now turn to the contribution from I_2 in (5.23). Breaking the ranges for q and $|\alpha - a/q|$ into dyadic intervals as usual, and applying the second part of Lemma 5, we have

$$I_2 \ll P^{12+35/48+\delta+2\varepsilon} \max_{R,\phi} R^{1/2} \phi \min\{1, (\phi P^3)^{-3/2}\},$$

where the maximum is over all R, ϕ such that

$$0 < R \leqslant P^{27/50}, \quad 0 < \phi < R^{-1} P^{-37/24+2\delta},$$

with the inequalities $R \leqslant P^{\Delta}$ and $\phi \leqslant P^{-3+\Delta}$ not both holding simultaneously. If $\phi > P^{-3}$ then this is

$$\ll R^{1/2} P^{9+35/48+\delta+2\varepsilon} \ll P^{10-1/1200+\delta+2\varepsilon} = o(P^{10}),$$

whereas if on the other hand $\phi \leqslant P^{-3}$, then the same basic conclusion holds.

Once taken all together, this therefore completes the treatment of the minor arcs when $(n_1, n_2) = (1, 12)$.

5.2 The case $n_1 = 2$

We will continue to assume that $w \in \mathcal{W}_n^{(2)}$ throughout this section. In what follows we may assume that C_1 does not take the shape $a(b_1x_1 + b_2x_2)^3$, for integers a, b_1, b_2, since otherwise the resolution of Theorem 1 is trivial.

Recall the manipulations in (5.2) and (5.3). Taking $(u, v) = (4, 4/3)$ it follows from Lemma 8 that the latter inequality is bounded by $O(P^{5+\varepsilon})$. Thus we are led to estimate $I_{4,4/3}(5; \mathfrak{m})$, as given by (5.4). We clearly have

$$\rho_{11} = \tfrac{44}{57}, \quad \pi_{11} = \tfrac{103}{57}, \quad \rho'_{11} = \tfrac{56}{31}, \quad \pi'_{11} = 3,$$

in (5.7) and (5.8). One easily checks that the conditions (5.10), (5.11) and (5.12) in the statement of Lemma 13 are satisfied with our choice of u, v, k and n_2. Recall the definition (5.9) of δ. Define $\mathfrak{m}_0$ to be the set of $\alpha \in \mathfrak{m}$ for which there exist coprime integers $0 \leqslant a < q$ such that

$$q \leqslant P^{31/67+2\delta}, \quad q^{56/31} P^{-3-\delta} \leqslant \left|\alpha - \frac{a}{q}\right| \leqslant q^{-44/57} P^{-103/57+\delta}.$$

Then we may conclude from Lemma 13 that $I_{4,4/3}(5; \mathfrak{m} \backslash \mathfrak{m}_0) = o(P^{10})$.

It remains to deal with the contribution from $\alpha \in \mathfrak{m}_0$. We will use Lemma 5 to handle this remaining range. Now it follows that

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-1} q^{13/57} P^{-103/57+\delta} < q^{-1} P^{-3/2}.$$

Recalling the definition (5.4) of $I_{4,4/3}(5; \mathfrak{m}_0)$, we therefore deduce from the second part of Lemma 5 with $n = 11$ that

$$I_{4,4/3}(5; \mathfrak{m}_0) \ll P^{49/4+2\varepsilon} \left(\sum_q q^{-5/6} \int \min\{1, (|\theta| P^3)^{-11/6}\} \, d\theta \right)^{3/4},$$

where the integral is over $q^{56/31} P^{-3-\delta} \leqslant |\theta| \leqslant q^{-44/57} P^{-103/57+\delta}$ and the sum is over integer $q \leqslant P^{31/67+2\delta}$, with the inequalities $q \leqslant P^{\Delta}$ and $|\theta| \leqslant P^{-3+\Delta}$ not both holding simultaneously. The contribution from $q \leqslant P^{\Delta}$ is therefore

$$\ll P^{10+2\varepsilon-5\Delta/8} \left(\sum_{q \leqslant P^{\Delta}} q^{-5/6} \right)^{3/4} \ll P^{10+2\varepsilon-\Delta/2},$$

which is satisfactory. Taking $|\theta| \geqslant q^{56/31} P^{-3-\delta}$, we see that the corresponding contribution from $q > P^{\Delta}$ is

$$\ll P^{10+5\delta/8+2\varepsilon} \left(\sum_{q > P^{\Delta}} q^{-145/62} \right)^{3/4} \ll P^{10+5\delta/8-\Delta+2\varepsilon}.$$

This too is satisfactory, and so completes our analysis of the case $(n_1, n_2) = (2, 11)$.

5.3 The case $n_1 = 3$

According to Lemma 2 we may proceed under the assumption that either C_1 is non-singular or else our cubic form C splits off a ternary norm form. In the latter case Theorem 3 readily ensures that $X(\mathbb{Q}) \neq \emptyset$, and so we may focus our efforts on the case C_1 is non-singular. We will assume that $w = w_1 w_2$, with $(w_1, w_2) \in \mathcal{W}_3^{(1)} \times \mathcal{W}_{10}^{(2)}$, throughout this section.

Our argument relies upon the same notion of pruning that was put to good effect in § 5.1. Let us define $\mathfrak{m}_1$ to be the set of $\alpha \in \mathfrak{m}$ for which there exists $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{16/25}$ and $\gcd(a, q) = 1$, with

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-1} P^{-143/75}.$$

It will be convenient to refer to the set $\mathfrak{m} \backslash \mathfrak{m}_1$ as the set of 'proper minor arcs', and $\mathfrak{m}_1$ will be the set of 'improper minor arcs'. Let us write E_{prop} and E_{improp} for the corresponding contributions to E.

We begin by estimating E_{prop}. Taking $u = v = 2$ in (5.2) and (5.3), and applying Lemma 12, we deduce that $E_{\text{prop}} \ll I_{2,2}(4; \mathfrak{m} \backslash \mathfrak{m}_1)$. When $u = v = 2$, $k = 4$ and $n_2 = 10$ we have

$$\rho_{10} = \tfrac{25}{26}, \quad \pi_{10} = \tfrac{21}{13}, \quad \rho''_{10} = \tfrac{7}{11}, \quad \psi_{10} = \tfrac{839}{88} = 9.53\ldots,$$

in (5.18), (5.19) and (5.20). Now it is easily to see that $\mathfrak{m} \backslash \mathfrak{m}_1 \subset \mathfrak{m} \backslash \mathfrak{m}_0$, where $\mathfrak{m}_0$ is as in the statement of Lemma 14, since for $\alpha \in \mathfrak{m}_0$ we have

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-1/3} P^{-7/3+\delta} = q^{-1} q^{2/3} P^{-7/3+\delta} \leqslant q^{-1} P^{-21/11+2\delta} < q^{-1} P^{-143/75},$$

where δ is given by (5.9). On observing that ψ_{10} satisfies (5.21), and that the inequalities in (5.11) are trivially satisfied, it therefore follows from Lemma 14 that $E_{\text{prop}} = o(P^{10})$.

We now turn to the argument needed to control the overall contribution to E from the improper minor arcs, which we denote by E_{improp}. We select $(A, B, C) = (16/25, 1, 82/75)$ in the definition (3.6) of $\mathcal{A} = \mathcal{A}(A, B, C)$. It now follows from taking $n = 3$ and $\sigma = -1$ in Lemma 9 that

$$S_1(\alpha) - S_1^*(\alpha) \ll P^{A/2+2+\varepsilon} + P^{2C+\varepsilon} \ll P^{58/25+\varepsilon},$$

for any $\alpha \in \mathcal{A}_{q,a}$, where $S_1^*(\alpha)$ is given by (3.7). Hence

$$E_{\text{improp}} \ll \int_{\mathfrak{m}_1} |S_1^*(\alpha) S_2(\alpha)| \, d\alpha + P^{58/25+\varepsilon} \int_{\mathfrak{m}_1} |S_2(\alpha)| \, d\alpha = I_1 + I_2,$$

say. Our goal is to show that $E_{\text{improp}} = o(P^{10})$.

We begin by handling the contribution from I_2. Using dyadic summation it follows from Lemma 5 that

$$I_2 \ll P^{10+58/25+2\varepsilon} \max_{R,\phi} R^{3/4} \phi \min\{1, (\phi P^3)^{-5/4}\} \ll P^{7+58/25+2\varepsilon} R^{3/4} \ll P^{10-1/5+2\varepsilon},$$

where the maximum is over R, ϕ dictated by the definition of $\mathfrak{m}_1$. This is plainly satisfactory and so completes our treatment of I_2.

We now turn to an upper bound for I_1. Combining Hölder's inequality with the second part of Lemma 9 gives

$$|I_1| \leqslant \left(\int_{\mathfrak{m}_1} |S_1^*(\alpha)|^4 \, d\alpha\right)^{1/4} \left(\int_{\mathfrak{m}_1} |S_2(\alpha)|^{4/3} \, d\alpha\right)^{3/4} \ll I_{4,4/3}(9; \mathfrak{m}_1),$$

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in the notation of (5.4). Let us dissect $\mathfrak{m}_1$ into $\mathfrak{m}_1^a \cup \mathfrak{m}_1^b$, where $\mathfrak{m}_1^a$ is the set of $\alpha \in \mathfrak{m}_1$ for which there exist coprime integers $0 \leqslant a < q$ such that $q \leqslant P^{16/25}$ and $|\alpha - a/q| \leqslant q^{1/2} P^{-3+\delta}$, and $\mathfrak{m}_1^b = \mathfrak{m}_1 \backslash \mathfrak{m}_1^a$. An application of Lemma 5 yields

$$\begin{aligned} I_{4,4/3}(9; \mathfrak{m}_1^b) &\ll P^{10+9/4+2\varepsilon} \bigg(\sum_{q \leqslant P^{16/25}} q^{-2/3} \int_{q^{1/2}P^{-3+\delta}}^{q^{-1}P^{-143/75}} (\theta P^3)^{-5/3}\, d\theta \bigg)^{3/4} \\ &\ll P^{10+9/4+2\varepsilon} \bigg(\sum_{q} q^{-2/3} P^{-5} (q^{1/2}P^{-3+\delta})^{-2/3} \bigg)^{3/4} \\ &\ll P^{10-\delta/2+2\varepsilon} \log P, \end{aligned}$$

which is satisfactory. Turning to $I_{4,4/3}(9; \mathfrak{m}_1^a)$, we note that when $(u, v) = (4, 4/3)$ and $k = 9$ one has

$$\rho_{10} = \tfrac{5}{7}, \quad \pi_{10} = \tfrac{37}{21}, \quad \rho'_{10} = \tfrac{8}{7}, \quad \pi'_{10} = 3,$$

in (5.7) and (5.8). Since (5.10), (5.11) and (5.12) are evidently satisfied in Lemma 13, a modest pause for thought reveals that $I_{4,4/3}(9; \mathfrak{m}_1^a) = o(P^{10})$, as required.

5.4 The case $n_1 = 4$

We follow the strategy of the preceding section. According to Lemma 3 we may assume that either C_1 is non-singular or else the surface $C_1 = 0$ contains precisely 3 conjugate double points. In the latter case (2.2) implies that our cubic form C can be written as

$$\mathrm{N}_{K/\mathbb{Q}}(x_1\omega_1 + \cdots + x_3\omega_3) + a x_4^2 \,\mathrm{Tr}_{K/\mathbb{Q}}(x_1\omega_1 + \cdots + x_3\omega_3) + b x_4^3 + C_2(x_5, \ldots, x_{13}),$$

for appropriate coefficients $\omega_1, \omega_2, \omega_3 \in K$ and $a, b \in \mathbb{Z}$, and where K is a certain cubic number field. Setting $x_4 = 0$ we arrive at a cubic form in 12 variables which is exactly of the type considered in Theorem 3. Hence $X(\mathbb{Q}) \neq \emptyset$ in this case, and so we are free to proceed under the assumption that C_1 is non-singular. Throughout this section we will take $w = w_1 w_2$ as our weight function, with $w_1 \in \mathcal{W}_4^{(1)}$ and $w_2 \in \mathcal{W}_9^{(2)}$.

Let $\mathfrak{m}_1$ be the set of $\alpha \in \mathfrak{m}$ for which there exists $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{21/50}$ and $\gcd(a, q) = 1$, with

$$\left| \alpha - \frac{a}{q} \right| \leqslant q^{-1} P^{-113/50}.$$

As previously we define $\mathfrak{m} \backslash \mathfrak{m}_1$ to be the proper minor arcs and $\mathfrak{m}_1$ to be the improper minor arcs, with the same notation $E_{\mathrm{prop}}, E_{\mathrm{improp}}$ for the corresponding contributions to E.

We begin by estimating E_{prop}. Taking $u = v = 2$ in (5.2) and (5.3), and applying Lemma 12, we find that $E_{\mathrm{prop}} \ll I_{2,2}(11/2; \mathfrak{m} \backslash \mathfrak{m}_1)$. When $u = v = 2$, $k = 11/2$ and $n_2 = 9$ we have

$$\rho_9 = \tfrac{18}{19}, \quad \pi_9 = \tfrac{67}{38}, \quad \rho''_9 = \tfrac{25}{62}, \quad \psi_9 = 9 + \tfrac{15}{124} = 9.12\ldots,$$

in (5.18), (5.19) and (5.20). Furthermore, it is easily checked that $\mathfrak{m} \backslash \mathfrak{m}_1 \subset \mathfrak{m} \backslash \mathfrak{m}_0$, where $\mathfrak{m}_0$ is as in the statement of Lemma 14. On observing that (5.21) and (5.11) are satisfied, it therefore follows from Lemma 14 that $E_{\mathrm{prop}} = o(P^{10})$.

It remains to estimate E_{improp}, for which we select

$$(A, B, C) = (\tfrac{21}{50}, 1, \tfrac{37}{50})$$

in the definition (3.6) of $\mathcal{A}$. Taking $n = 4$ and $\sigma = -1$ in Lemma 9 therefore gives

$$S_1(\alpha) - S_1^*(\alpha) \ll P^{5C/2+\varepsilon} + P^{5A/6+5/2+\varepsilon} \ll P^{57/20+\varepsilon},$$

for any $\alpha \in \mathcal{A}_{q,a}$. It now follows that

$$E_{\text{improp}} \ll \int_{\mathfrak{m}_1} |S_1^*(\alpha)S_2(\alpha)|\, d\alpha + P^{57/20+\varepsilon} \int_{\mathfrak{m}_1} |S_2(\alpha)|\, d\alpha = I_1 + I_2,$$

say. We begin by handling the contribution from I_2. Using dyadic summation it follows from Lemma 5 that

$$I_2 \ll P^{9+57/20+2\varepsilon} \max_{R,\phi} R^{7/8}\phi \min\{1, (\phi P^3)^{-(9/8)}\} \ll P^{6+57/20+2\varepsilon} R^{7/8} \ll P^{9.2175+2\varepsilon},$$

where the maximum is over R, ϕ such that

$$0 < R \leqslant P^{21/50}, \quad 0 < \phi < R^{-1}P^{-113/50}.$$

This is plainly satisfactory and so completes our treatment of I_2.

We now turn to an upper bound for I_1. Since C_1 is assumed to be good as well as non-singular, we may apply the second part of Lemma 9 with $k = 3$ and $n = 4$ to conclude that $I_1 \ll I_{3,3/2}(9, \mathfrak{m}_1)$. As in the case $n_1 = 3$ we write $\mathfrak{m}_1 = \mathfrak{m}_1^a \cup \mathfrak{m}_1^b$, where now $\mathfrak{m}_1^a$ is the set of $\alpha \in \mathfrak{m}_1$ for which there exist coprime integers $0 \leqslant a < q$ such that $q \leqslant P^{21/50}$ and $|\alpha - a/q| \leqslant qP^{-3+\delta}$, and $\mathfrak{m}_1^b = \mathfrak{m}_1 \backslash \mathfrak{m}_1^a$. It follows from Lemma 5 that

$$\begin{aligned} I_{3,3/2}(9; \mathfrak{m}_1^b) &\ll P^{12+2\varepsilon}\bigg(\sum_{q \leqslant P^{21/50}} q^{1-27/16} \int_{qP^{-3+\delta}}^{q^{-1}P^{-113/50}} (\theta P^3)^{-27/16}\, d\theta\bigg)^{2/3} \\ &\ll P^{12+2\varepsilon}\bigg(\sum_{q \leqslant P^{21/50}} q^{-11/16}P^{-81/16}(qP^{-3+\delta})^{-11/16}\bigg)^{2/3} \\ &\ll P^{10-11\delta/24+2\varepsilon}. \end{aligned}$$

To handle $I_{3,3/2}(9; \mathfrak{m}_1^a)$ we note that when $(u, v) = (3, 3/2)$ and $k = 9$ we have

$$\rho_9 = \tfrac{3}{4}, \quad \pi_9 = \tfrac{29}{16}, \quad \rho_9' = \tfrac{43}{25}, \quad \pi_9' = 3,$$

in (5.7) and (5.8). Lemma 13 easily gives $I_{3,3/2}(9; \mathfrak{m}_1^a) = o(P^{10})$, as required. This completes the treatment of the improper minor arcs when $(n_1, n_2) = (4, 9)$.

5.5 The case $n_1 = 5$

An application of Lemma 4 reveals that we are free to assume that C_1 defines a projective cubic hypersurface whose singular locus is either empty or finite. Throughout this section we will take $w = w_1w_2$ as our weight function, with $(w_1, w_2) \in \mathcal{W}_5^{(1)} \times \mathcal{W}_8^{(2)}$.

We let the improper minor arcs $\mathfrak{m}_1$ be the set of $\alpha \in \mathfrak{m}$ for which there exists $a, q \in \mathbb{Z}$ such that $0 \leqslant a < q \leqslant P^{11/20+\delta}$ and $\gcd(a, q) = 1$, with

$$\left|\alpha - \frac{a}{q}\right| \leqslant P^{-5/2+\delta},$$

with δ given by (5.9), and we let $\mathfrak{m}\backslash\mathfrak{m}_1$ be the proper minor arcs. As above, let $E_{\text{prop}}, E_{\text{improp}}$ denote the corresponding contributions to E.

Taking $u = v = 2$ in (5.2) and (5.3), and applying Lemma 12 with $n = 5$ and $\sigma \leqslant 0$, we find that $E_{\text{prop}} \ll I_{2,2}(15/2; \mathfrak{m}\backslash\mathfrak{m}_1)$. When $u = v = 2$, $k = 15/2$ and $n_2 = 8$ we have

$$\rho_8 = \tfrac{12}{13}, \quad \pi_8 = \tfrac{22}{13}, \quad \rho_8'' = \tfrac{11}{20}, \quad \psi_8 = 9.55,$$

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in (5.18), (5.19) and (5.20). Furthermore, it is easily checked that $\mathfrak{m}\backslash\mathfrak{m}_1 \subset \mathfrak{m}\backslash\mathfrak{m}_0$, where $\mathfrak{m}_0$ is as in the statement of Lemma 14. On observing that (5.11) and (5.21) are satisfied, it therefore follows from Lemma 14 that $E_{\text{prop}} = o(P^{10})$.

It remains to estimate E_{improp}, for which we select

$$(A, B, C) = (\tfrac{11}{20} + \delta, 0, \tfrac{1}{2} + \delta)$$

in the definition (3.6) of $\mathcal{A}$. Taking $n = 5$ and $\sigma \leqslant 0$ in Lemma 9 therefore gives

$$S_1(\alpha) - S_1^*(\alpha) \ll P^{3+5A/3+\varepsilon} + P^{3(A+C)+\varepsilon} \ll P^{3.92},$$

for any $\alpha \in \mathcal{A}_{q,a}$. It now follows that

$$E_{\text{improp}} \ll \int_{\mathfrak{m}_1} |S_1^*(\alpha)S_2(\alpha)|\, d\alpha + P^{3.92} \int_{\mathfrak{m}_1} |S_2(\alpha)|\, d\alpha = I_1 + I_2,$$

say.

We begin by handling the contribution from I_2. Using dyadic summation it follows from Lemma 5 that

$$I_2 \ll P^{3.92+\varepsilon} \max_{R,\phi} R\phi P^8 \min\{1, (\phi P^3)^{-1}\} \ll P^{8.92+\varepsilon} R \ll P^{9.47+\delta+\varepsilon},$$

where the maximum is over R, ϕ such that

$$0 < R \leqslant P^{11/20+\delta}, \quad 0 < \phi < P^{-5/2+\delta}.$$

This is plainly satisfactory and so completes our treatment of I_2.

We now turn to an upper bound for I_1. Since C_1 is assumed to be good, we may apply the second part of Lemma 9 with $k = 12/5$ and $n = 5$ to conclude that $I_1 \ll I_{12/5,12/7}(9; \mathfrak{m}_1)$, in the notation of (5.4). When $(u, v) = (12/5, 12/7)$, $k = 9$ and $n_2 = 8$ we have

$$\rho_8 = \tfrac{4}{5}, \quad \pi_8 = \tfrac{66}{35}, \quad \rho_8' = \tfrac{38}{11}, \quad \pi_8' = 3,$$

in (5.7) and (5.8). Furthermore one easily checks that $k/u = 15/4$ satisfies the inequalities in (5.11). It therefore follows from Lemma 13 that $I_{12/5,12/7}(9; \mathfrak{m}_1\backslash\mathfrak{m}_2) = o(P^{10})$, where $\mathfrak{m}_2$ is the set of $\alpha \in \mathfrak{m}_1$ for which there exist coprime integers $0 \leqslant a < q$ such that

$$q \leqslant P^{11/42+2\delta}, \quad q^{38/11} P^{-3-\delta} \leqslant \left|\alpha - \frac{a}{q}\right| \leqslant q^{-4/5} P^{-66/35+\delta}. \tag{5.24}$$

Note that $q^{-4/5}P^{-66/35+\delta} \leqslant q^{-1}P^{-3/2}$. To handle $I_{12/5,12/7}(9; \mathfrak{m}_2)$ we appeal to Lemma 5, deducing that

$$\begin{aligned} I_{12/5,12/7}(9; \mathfrak{m}_2) &\ll P^{8+15/4+2\varepsilon} \max_{R,\phi} (R^2\phi)^{7/12} R^{-1} \min\{1, (\phi P^3)^{-1}\} \\ &\ll \max_{R,\phi} P^{5+15/4+2\varepsilon} R^{1/6} \phi^{-5/12} \\ &\ll P^{10+\delta/2+2\varepsilon} R^{-14/11}, \end{aligned}$$

on taking $\phi \geqslant R^{38/11}P^{-3-\delta}$. Here the maximum is over the relevant R, ϕ determined by (5.24), with the inequalities $R \leqslant P^{\Delta}$ and $\phi \leqslant P^{-3+\Delta}$ not both holding simultaneously. Now either $R > P^{\Delta}$ and this estimate is satisfactory, or else $R \leqslant P^{\Delta}$ and it follows that we may actually take the lower bound $\phi > P^{-3+\Delta}$ in the second term, giving instead

$$\ll P^{10-5\Delta/12+2\varepsilon} R^{1/6} \ll P^{10-\Delta/4+2\varepsilon}.$$

This too is satisfactory, and so completes the proof that $I_1 = o(P^{10})$, thereby completing the treatment of the minor arcs in the case $(n_1, n_2) = (5, 8)$.

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5.6 The case $n_1 = 6$

We now come to the final case that we need to analyse in our proof of Theorem 1. We take $w = w_1w_2$ with $(w_1, w_2) \in \mathcal{W}_6^{(2)} \times \mathcal{W}_7^{(2)}$, and seek an estimate for

$$M_6(P; \mathfrak{m}) := \int_{\mathfrak{m}} |S_1(\alpha)|^2 \, d\alpha.$$

Note that the integral is now taken over the set of minor arcs, rather than the entire interval $[0, 1]$ as in (4.4). As usual we assume that C_1 and C_2 are good. We will show that

$$M_6(P; \mathfrak{m}) \ll P^{9+\varepsilon}. \tag{5.25}$$

Once in place, we may take $u = v = 2$ in (5.2) and (5.4) to conclude that $E \ll I_{2,2}(9; \mathfrak{m})$, whence the desired conclusion is given by Lemma 15.

It remains to establish (5.25). Let us consider the consequences of applying Lemma 7 to estimate $M = M_6(P; \mathfrak{m})$, following the general approach in the proof of Lemmas 13 and 14. We have

$$M \ll P^3 + P^\varepsilon \max_{R,\phi} \frac{\psi_H R^2 P^{11}}{H^{10}} F,$$

where ψ_H and F are as in the statement of Lemma 7 and $H \in [1, P] \cap \mathbb{Z}$ is arbitrary. Furthermore the maximum is over R, ϕ such that (5.6) holds with any choice of $Q \geqslant 1$ that we care to choose. We will take $Q = P^{3/2}$. In our deduction of (5.25) it will be convenient to allow the value of $\varepsilon > 0$ to take different values at different parts of the argument.

We define $\phi_0 := R^{-2/11} P^{-2}$ and take

$$H := \begin{cases} \lfloor P^\varepsilon \max\{1, (\phi R^2 P^2)^{1/10}\} \rfloor & \text{if } \phi > \phi_0, \\ \lfloor P^\varepsilon R^{2/11} \rfloor & \text{if } \phi \leqslant \phi_0. \end{cases}$$

If we can show that $F \ll P^\varepsilon$ with this choice of H then (5.25) will follow. It is clear that H is an integer in the interval $[1, P]$.

Suppose first that $\phi > \phi_0$. Then $\psi_H \ll \phi$ and it follows that

$$RH^3\psi_H \ll P^\varepsilon R\phi(1 + \phi R^2 P^2)^{3/10} \ll 1 + Q^{-1} P^{3/5+\varepsilon} \ll 1,$$

by (5.6) and the fact that $Q = P^{3/2}$. The third term in F is

$$\frac{H^6}{R^3(P^2\psi_H)^2} \ll \frac{P^\varepsilon}{R^3 P^4 \phi^2}(1 + \phi R^2 P^2)^{3/5} \ll \frac{P^\varepsilon}{R^3 P^4 \phi_0^2} + \frac{P^\varepsilon}{R^{9/5} P^{14/5} \phi_0^{7/5}} \ll P^\varepsilon,$$

which is satisfactory.

Suppose now that $\phi \leqslant \phi_0$. Then $\psi_H \ll (P^2 H)^{-1}$ and it follows that

$$RH^3\psi_H \ll \frac{RH^2}{P^2} \ll \frac{RP^\varepsilon}{P^2}(1 + R^{2/11}) \ll 1 + \frac{Q^{13/11} P^\varepsilon}{P^2} \ll 1.$$

Turning to the third term in F, we find that

$$\frac{H^6}{R^3(P^2\psi_H)^2} \ll \frac{H^8}{R^3} \ll \frac{P^\varepsilon}{R^3}(1 + R^{16/11}) \ll P^\varepsilon,$$

which is also satisfactory. This therefore completes the proof of (5.25), and so our treatment of the case $(n_1, n_2) = (6, 7)$.

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Appendix. Groupe de Brauer non ramifié des hypersurfaces cubiques singulières (d'après P. Salberger)

J.-L. Colliot-Thélène

En réponse à une question de R. Heath-Brown, P. Salberger en 2006 a indiqué les grandes lignes de la démonstration de l'énoncé suivant, qui étend un résultat connu dans le cas lisse [Col04]. Nous donnons le détail de la démonstration. On utilise les notations usuelles dans ce domaine. Pour X un schéma on note $\text{Pic } X = H^1_{\text{ét}}(X, \mathbf{G}_m)$ son groupe de Picard et $\text{Br } X = H^2_{\text{ét}}(X, \mathbf{G}_m)$ son groupe de Brauer. Pour les propriétés usuelles de ces groupes, nous renvoyons le lecteur à [CS87].

Théorème. *Soit k un corps de caractéristique zéro, $\overline{k}$ une clôture algébrique, $\mathcal{G}$ le groupe de Galois de $\overline{k}$ sur k. Soit $X \subset \mathbf{P}^n_k$ une intersection complète géométriquement intègre de dimension au moins 3. Supposons le lieu singulier vide ou de codimension au moins égale à 4 dans X. Alors pour tout k-modèle projectif et lisse Y de X :*

(a) *le groupe de Picard de $\overline{Y} = Y \times_k \overline{k}$ est un module galoisien $\mathbf{Z}$-libre de type fini qui est stablement de permutation ;*

(b) *on a $H^1(\mathcal{G}, \text{Pic } \overline{Y}) = 0$;*

(c) *on a $\text{Br } k \xrightarrow{\simeq} \text{Ker}[\text{Br } Y \to \text{Br } \overline{Y}]$.*

Démonstration. Les anneaux locaux de X en codimension au moins 3 sont réguliers, donc factoriels (théorème d'Auslander–Buchsbaum, voir [Gro05, § XI Theorem 3.13]). Comme X est une intersection complète, un théorème de Grothendieck [Gro05, § XI Corollary 3.14], ex-conjecture de Samuel, implique que tous les anneaux locaux de X sont factoriels. Ainsi les diviseurs de Weil sur X sont tous des diviseurs de Cartier. Ceci implique que pour tout ouvert $U \subset X$ la flèche de restriction $\text{Pic } X \to \text{Pic } U$ est surjective. Cette flèche est aussi injective. Soit en effet D un diviseur sur X qui est le diviseur d'une fonction rationnelle f sur U. Comme le complémentaire de U dans X est de codimension au moins 2 et que sur X diviseurs de Weil et diviseurs de Cartier coïncident, on conclut que D est le diviseur de f sur X. Le même argument montre que toute fonction rationnelle sur X définie et inversible sur U est définie et inversible sur X, et comme X est projectif et géométriquement intègre, toute telle fonction est une constante, elle appartient à k^*.

L'hypothèse sur la codimension du lieu singulier est géométrique, elle vaut pour X_K pour toute extension K/k de corps, par exemple $\overline{k}/k$. Les mêmes conclusions s'appliquent donc à $\overline{X}$. En particulier la flèche de restriction $\text{Pic } \overline{X} \to \text{Pic } \overline{U}$ est un isomorphisme.

Par ailleurs, le Corollaire 3.7 de [Gro05, § XII] montre que la flèche de restriction

$$\mathbf{Z} = \text{Pic } \mathbf{P}^n_k \to \text{Pic } X$$

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qui envoie la classe de $1 \in \mathbf{Z}$ sur la classe du faisceau inversible $\mathcal{O}_X(1)$ est un isomorphisme. Il en est de même de $\mathbf{Z} = \mathrm{Pic}\, \mathbf{P}^n_{\overline{k}} \to \mathrm{Pic}\, \overline{X}$, et l'action du groupe de Galois sur $\mathbf{Z} \simeq \mathrm{Pic}\, \overline{X}$ est triviale.

En conclusion, sous les hypothèses du théorème, le module galoisien $\mathrm{Pic}\, \overline{U}$ est le module galoisien trivial $\mathbf{Z}$ et l'on a $\overline{k}^* \xrightarrow{\simeq} \overline{k}[U]^*$, où $\overline{k}[U]$ est l'anneau des fonctions définies sur $\overline{U}$. L'argument ci-dessus montre aussi que l'application naturelle $\mathrm{Pic}\, U \to \mathrm{Pic}\, \overline{U}$ est un isomorphisme.

D'une suite exacte bien connue (cf. [CS87, p. 386]) on déduit que la flèche naturelle $\mathrm{Br}\, k \to \mathrm{Ker}[\mathrm{Br}\, U \to \mathrm{Br}\, \overline{U}]$ est un isomorphisme. Par des arguments standards sur le groupe de Brauer (pureté et injection par passage d'une variété lisse à un ouvert) un tel énoncé implique le même énoncé pour toute k-variété projective et lisse k-birationnelle à X : c'est l'énoncé (c).

Soit $U \subset Y$ une k-compactification lisse de U (le théorème de Hironaka assure l'existence d'une telle compactification). On a alors la suite exacte de modules galoisiens

$$0 \to \mathrm{Div}_\infty \overline{Y} \to \mathrm{Pic}\, \overline{Y} \to \mathrm{Pic}\, \overline{U} \to 0,$$

où le groupe $\mathrm{Div}_\infty \overline{Y}$ est le module de permutation sur les points de codimension 1 de $\overline{Y}$ en dehors de $\overline{U}$, et le zéro à gauche tient au fait que l'on a $\overline{k}^* \simeq \overline{k}[U]^*$. La suite de modules galoisiens ci-dessus est scindée, car tout groupe $H^1(\mathcal{G}, P)$ à valeurs dans un module de permutation est nul (lemme de Shapiro). Ainsi $\mathrm{Pic}\, \overline{Y}$ est la somme directe de deux modules de permutation, et est donc un module de permutation. Il en résulte que pour tout autre modèle projectif et lisse Y', le module galoisien $\mathrm{Pic}\, \overline{Y}'$ est stablement de permutation [CS87, Proposition 2.A.1 on p. 461]. L'énoncé (b) en résulte. □

COROLLAIRE. *Soit $X \subset \mathbf{P}^n_k$ une hypersurface cubique géométriquement intègre de dimension au moins 3 qui n'est pas un cône. Supposons le lieu singulier vide ou de codimension au moins égale à 4 dans X. Alors pour tout modèle projectif et lisse Y de X, on a $\mathrm{Br}\, k \xrightarrow{\simeq} \mathrm{Br}\, Y$.*

Démonstration. Au vu du théorème ci-dessus, il suffit de montrer $\mathrm{Br}\, \overline{Y} = 0$.

Si l'hypersurface X est lisse, on a $\mathrm{Br}\, \overline{X} = 0$ comme il est établi dans [Col04] sans restriction sur le degré de l'hypersurface. Par l'invariance birationnelle du groupe de Brauer pour les variétés projectives et lisses ceci implique $\mathrm{Br}\, \overline{Y} = 0$.

Si l'hypersurface cubique X est singulière, comme elle n'est pas un cône, en utilisant les droites passant par un $\overline{k}$-point singulier on obtient une équivalence birationnelle de $\overline{X}$ avec l'espace projectif $\mathbf{P}^{n-1}_{\overline{k}}$, dont le groupe de Brauer est nul. Par l'invariance birationnelle du groupe de Brauer pour les variétés projectives et lisses ceci implique $\mathrm{Br}\, \overline{Y} = 0$. □

Remarque 1. Il serait intéressant de voir si le corollaire vaut pour les hypersurfaces de degré supérieur à 3. C'est le cas lorsque les hypersurfaces sont lisses [Col04].

Remarque 2. La condition que la codimension du lieu singulier est au moins égale à 4 est nécessaire. Dans [CS89] on trouve des hypersurfaces cubiques géométriquement intègres non coniques dans $\mathbf{P}^4_k$, dont le lieu singulier est un ensemble fini non vide de points, et qui admettent un modèle projectif et lisse Y avec $\mathrm{Br}\, Y / \mathrm{Br}\, k \neq 0$.

Remarque 3. Lorsque k est un corps de nombres, le corollaire permet de conjecturer que, sous les hypothèses données, le principe de Hasse et l'approximation faible valent pour le lieu lisse de l'hypersurface cubique X.

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