BULL. AUSTRAL. MATH. SOC. 30C80, 30D40, 30H05
VOL. 44 (1991) [527–528]

A REMARK ON BOUNDEDNESS OF BLOCH FUNCTIONS

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Two consequences of a theorem of Dahlberg are derived. Let f be a holomorphic function in the unit disk $\mathbf{D}$ of the complex plane, and let E be an F_σ subset of the unit circle $\mathbf{T}$. Suppose that $\overline{\lim}_{r\to 1-} |f(rw)| \leqslant M$, $w \in \mathbf{T}\setminus E$, for some constant M. Then f is bounded in either of the two cases:

(i) if f is in the Bloch space and E is of zero measure with respect to the Hausdorff measure associated with the function $\psi(t) = t\log\log(2\pi e^e/t)$,

(ii) if f is integrable with respect to the planar Lebesgue measure on $\mathbf{D}$ and E is of zero measure with respect to the Hausdorff measure associated with the function $\psi(t) = t\log(2\pi e^e/t)$.

Let $\mathbf{D}$ be the unit disk of the complex plane, and let $\mathbf{T}$ be the unit circle. For a function ψ satisfying the usual conditions, let Λ_ψ be the Hausdorff measure on $\mathbf{T}$ corresponding to ψ. Let $\phi_1(t) = \log(2\pi e^e/t)$, $\phi_2(t) = \log[\phi_1(t)]$, $0 < t \leqslant 2\pi$, and let $\psi_j(t) = t\phi_j(t)$, $j = 1, 2$.

The following theorem is a modification of a theorem of Dahlberg [**2**, Theorem 4], and can be proved in a very similar manner.

THEOREM. (Dahlberg). *Let $j = 1$ or $j = 2$. Let $E \subset \mathbf{T}$ be an F_σ set with $\Lambda_{\psi_j}(E) = 0$. Let u be a sub-harmonic function on $\mathbf{D}$. Suppose that there are constants C and M such that*

$$u(z) \leqslant C\phi_j(1-|z|) \tag{A}$$

for all $z \in \mathbf{D}$, and

$$\overline{\lim_{r\to 1-}}\, u(rw) \leqslant M$$

if $w \in \mathbf{T}\setminus E$. Then the function u is bounded above in $\mathbf{D}$.

Let B denote the Bloch space, that is, the space of those holomorphic functions f on $\mathbf{D}$ which satisfy the condition

$$\sup_{z\in\mathbf{D}} \left(1-|z|^2\right)|f'(z)| < +\infty.$$

Since for any $f \in B$ the function $u = \log|f|$ is sub-harmonic on $\mathbf{D}$ and satisfies (A) with $j = 2$ and some constant C, we have the following corollary, which is a generalisation of [**1**, Theorem 2].

Received 31 January 1991

Corollary 1. *Let $f \in B$. Suppose that there are: a constant M and an F_σ set $E \subset \mathbf{T}$ with $\Lambda_{\psi_2}(E) = 0$ such that*

$$\varlimsup_{r \to 1-} |f(rw)| \leqslant M$$

for $w \in \mathbf{T} \setminus E$. Then f is bounded in $\mathbf{D}$.

Now, let us observe that if f is a holomorphic function in $\mathbf{D}$ and $\iint\limits_{x^2+y^2<1} |f(x+iy)|\, dxdy < +\infty$, then by the mean value property we have

$$\begin{aligned} |f(z)| &= \left| \frac{1}{\pi(1-|z|)^2} \iint\limits_{|x+iy-z|<1-|z|} f(x+iy)\, dxdy \right| \\ &\leqslant \frac{1}{\pi(1-|z|)^2} \iint\limits_{x^2+y^2<1} |f(x+iy)|\, dxdy. \end{aligned}$$

Therefore the subharmonic function $u = \log|f|$ satisfies (A) with $j = 1$ and some constant C. Thus we have

Corollary 2. *Let f be a holomorphic function in $\mathbf{D}$ integrable with respect to the planar Lebesgue measure on $\mathbf{D}$. Suppose that there are: a constant M and an F_σ set $E \subset \mathbf{T}$ with $\Lambda_{\psi_1}(E) = 0$ such that*

$$\varlimsup_{r \to 1-} |f(rw)| \leqslant M$$

for $w \in \mathbf{T} \setminus E$. Then f is bounded in $\mathbf{D}$.

Taking $E = \emptyset$ in Corollary 2 we obtain the affirmative answer to a question asked in [**1**, p.37]. For a finite set E, Corollary 2 proves a conjecture from [**3**].

References

[1] A. Bonilla and F. Perez Gonzales, 'Radial growth and boundedness for Block functions', *Bull. Austral. Math. Soc.* **42** (1990), 33–39.

[2] B.E.I. Dahlberg, 'On the radial boundary values of subharmonic functions', *Math. Scand.* **40** (1977), 301–317.

[3] R.C. Goolsby, 'Boundedness for Bloch functions', *Rocky Mountain J. Math.* **16** (1986), 717–726.

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