A CHARACTERISATION OF THE MOEBIUS AND SIMILARITY GROUPS

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Let $PGL(2, F)$ denote the group of all Moebius transformations

$$z \to \frac{az+b}{cz+d} \qquad (ad-bc \neq 0)$$

over a field F. In a recent paper [2], the author has given a characterisation of the groups $PGL(2, F)$, F finite, char $F \neq 2$. It is the purpose of this paper to give a similar characterisation of the group $PGL(2, F)$, char $F = 2$, F finite or infinite.

The similarity transformations $z \to az+b(a \neq 0)$ form a subgroup $S(2, F)$ of $PGL(2, F)$. A simple direct calculation shows that, for any field F of characteristic 2, $S(2F)$ is an S_1-subgroup of $PGL(2, F)$ in the sense of the following definition.

Definition. A subgroup H of a group G is called an S_1-subgroup of G if, whenever $a \notin H$ and $b^{-1}ab \notin H$, there exists a unique $h \in H$ such that $b^{-1}ab = h^{-1}ah$.

S_1-subgroups were first studied by H. W. E. Schwerdtfeger in [3], where he discusses T_1-subgroups, being those S_1-subgroups which are normal.

The main result of this paper is

THEOREM 1. *Let H be a non-normal S_1-subgroup of a group G and suppose that $G-H$ contains an involution t such that $H \cap H^t \neq 1$. Then G is isomorphic to a group $PGL(2, F)$, char $F = 2$.*

There is a similar characterisation of the groups $S(2, F)$. It is easy to verify that, if F is any field, then the transformations

$$z \to az(a \neq 0)$$

form an S_1-subgroup of $S(2, F)$. We prove

THEOREM 2. *Let $H(\neq 1)$ be an S_1-subgroup of the group G, and suppose that $G-H$ contains an involution t such that $H \cap H^t = 1$. Then G is isomorphic to a group $S(2, F)$.*

Notations. Upper case Latin letters stand for groups and fields, lower

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case Latin letters for their elements. $H \lhd G$ means that H is a normal subgroup of G. $N(H)$ is the normaliser of the subgroup H, $C(h)$ is the centraliser of the element h, $|H|$ is the order of the group H, and $(G; H)$ is the index of the subgroup H in the group G. $G-H$ is the set of elements of G not in H and $a^x = x^{-1}ax$, $H^x = x^{-1}Hx$.

By the results of Zassenhaus [6] and Tits [5] it is sufficient to prove

THEOREM 1'. *Under the conditions of Theorem* 1, *G is isomorphic to a triply transitive permutation group in which only the identity fixes three symbols and in which the group fixing two symbols is abelian.*

THEOREM 2'. *Under the conditions of Theorem* 2, *G is isomorphic to a doubly transitive permutation group in which only the identity fixes two symbols, and in which the group fixing one symbol is abelian.*

In both theorems, H is a non-normal S_1-subgroup of the group G. We prove a series of lemmas under this assumption.

LEMMA 1. *If* $h(\neq 1) \in H$, *then* $C(h) \subseteqq H$.

PROOF. If $a \notin H$, then by the property S_1, the only element of H which commutes with a is the unit element.

LEMMA 2. *If* H *and* $\bar{H}$ *are* S_1*-subgroups of* G, *then* $H \cap \bar{H}$ *is an* S_1*-subgroup of* H.

PROOF. Suppose a, $a^b \in H-(H \cap \bar{H})$ where $b \in H$. We wish to show that there exists exactly one $h \in H \cap \bar{H}$ such that $a^h = a^b$.

By the property S_1, there exists exactly one $h \in \bar{H}$ such that $a^h = a^b$. Then $a^{bh-1} = a$. i.e. $bh^{-1} \in C(a) \subseteqq H$ by Lemma 1. Hence $h^{-1} \in b^{-1}H = H$, and so $h \in H \cap \bar{H}$.

LEMMA 3. $N(H) = H$.

PROOF. We assume $H \neq N(H)$ and deduce the contradiction that H is normal in G. By lemma 1, $N(H)$ is a Frobenius group with Frobenius kernel H, so that H is a characteristic subgroup of $N(H)$. Hence it is sufficient to prove that $N(H)$ is normal in G, i.e. $N(H)^x = N(H)$ for each $x \in G$.

Let $n \in N(H)-H$. If $n^x \notin N(H)$, then, since H is an S_1-subgroup of G, $n^x = n^h$ for some $h \in H$; but then $n^x \in N(H)$, which is a contradiction. Hence $n^x \in N(H)$. Since $N(H)-H$ generates $N(H)$, we have $N(H)^x = N(H)$ as required.

LEMMA 4. *If* $x \in G-H$, *then* $G = H+HxH$.

PROOF. By lemma 3 it suffices to prove that if $H \neq H^y$, then $y \in HxH$. Since $x \notin H$, $H \neq H^x$. Hence $H-H^x$ generates H and so, since $H \neq H^y$,

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we have $H - H^x \nsubseteq H^y$. Thus we can choose $h \in H$ such that $h \notin H^x$ and $h \notin H^y$, i.e. $h^{x^{-1}} \notin H$ and $h^{y^{-1}} \notin H$. Since H is an S_1-subgroup of G, $h^{y^{-1}} = h^{x^{-1}h_1}$ for some $h_1 \in H$. Then $y^{-1}h_1^{-1}x \in C(h)$ and so, by lemma 1, $y^{-1}h_1^{-1}x = h_2 \in H$. Thus $y = h_1^{-1}xh_2^{-1} \in HxH$ as required.

We can now prove theorem 2'. Suppose $\bar{H} = H^x$ and consider the permutation representation P of G on the left cosets of H. By lemma 4, P is doubly transitive. Since $H \cap H^x = 1$, P is faithful and only the identity fixes the symbols H and xH. Finally, the group fixing the symbol H, viz. H itself is abelian; for by Lemma 2, $H \cap H^x = 1$ is an S_1-subgroup of H. This completes the proof of Theorem 2'.

LEMMA 5. *The group $K = H \cap H^t$ is abelian and $k^t = k^{-1}$ for each $k \in K$.*

PROOF. $K^t = (H \cap H^t)^t = H^t \cap H = K$. Thus t maps K onto itself and, as $t^2 = 1$, $K \cup tK$ is a subgroup of G. Further $K \lhd K \cup tK$.

Suppose $k \in K$. Then $(tk)^2 \in K$, say $(tk)^2 = k_1$. Then $k^t = k_1k^{-1}$ and so $(k^{-1})^t = kk_1^{-1}$. Thus $k = k^{t^2} = k_1^t(k^{-1})^t = k_1^tkk_1^{-1}$ and so $k_1^t = k_1^{-1}$ i.e. $tk \in C(k_1) \subseteq K$ by lemma 1, which is a contradiction unless $k_1 = 1$. Hence for every $k \in K$, $(tk)^2 = 1$, i.e. $k^t = k^{-1}$. Thus $k \to k^{-1}$ is an automorphism of K and hence K is abelian.

LEMMA 6. *If H^x is a conjugate of H different from H and H^t, then $H \cap H^t \cap H^x = 1$.*

PROOF. By lemma 4, $x = h_1th_2 \in HtH$, so that $H^x = H^u$ where u is the involution t^{h_2}. Suppose $h(\neq 1) \in H \cap H^t \cap H^u$. By lemma 5, $h^t = h^u = h^{-1}$. Therefore by lemma 1, $tu^{-1} = h_1 \in H$. Then $H^t = H^{h_1u} = H^u$ contrary to assumption.

We can now prove theorem 1'. Set $K = H \cap H^t$ and let H^x be a conjugate of H different from H, H^t. Consider the permutation representation P of G on the left cosets of H. By lemma 6, P is faithful and only the identity fixes the three symbols H, tH, xH. It remains to prove that P is triply transitive.

Now, by lemma 4, $H^x = H^{th}$ for some $h \in H$. Then, by lemma 6, $K \cap K^h = H \cap H^t \cap H^x = 1$ so that K is not normal in H. By lemma 2, K is an S_1-subgroup of H. By lemma 4, $H = K + KaK$ for some $a \in K$. Then, since $Kt = tK$ by lemma 5, we have

$$\begin{aligned} G &= H + HtH = H + (K + KaK)tH, \\ &= H + tH + KatH. \end{aligned}$$

This shows that the group K which fixes the cosets H and tH permutes the remaining cosets transitively. Since, by lemma 4, P is doubly transitive, it is hence triply transitive. This completes the proof.

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