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Measures of Noncompactness in Regular Spaces

Nina A. Erzakova

Abstract. Previous results by the author on the connection between three measures of noncompactness obtained for L_p are extended to regular spaces of measurable functions. An example is given of the advantages of some cases in comparison with others. Geometric characteristics of regular spaces are determined. New theorems for (k, β)-boundedness of partially additive operators are proved.

1 Introduction

A condensing operator is a mapping under which the image of any set is, in a certain sense, more compact than the set itself. The degree of noncompactness of a set is measured by means of functions called measures of noncompactness (MNCs for brevity). Condensing operators have properties similar to compact ones. In particular, the theory of rotation of completely continuous vector fields, the Schauder–Tikhonov fixed point principle, and the Fredholm–Riesz–Schauder theory of linear equations with compact operators admit natural generalizations to condensing operators. Therefore, the theory of MNCs and condensing operators has applications in different areas of mathematics. For example, a technique connected with MNCs and condensing operators is used in the study of differential equations in infinite dimensional spaces, function-differential equations of neutral type, integral equations, as well as some types of partial differential equations (see, for example, [1, 3]).

In this paper we investigate the relationships among three different MNCs, and we will illustrate with examples the advantages of some MNCs over the others.

2 Basic Notions

Let E be a Banach space. Given a bounded subset U of E, *the Hausdorff measure of noncompactness* $\chi_E(U) = \chi(U)$ is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite ε-net for U in E.

The measure of noncompactness $\beta_E(U) = \beta(U)$ *of* $U \subset E$ is defined as the supremum of all numbers $r > 0$ such that there exists an infinite sequence in U with $\|u_n - u_m\| \geqslant r$ for every $n \neq m$.

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We denote by $B(u_0, r) = \{u \in E : \|u - u_0\| \leqslant r\}$ the closed ball in E of radius r and with the center u_0, and by $B = B(\theta, 1)$ the unit ball with the center θ where θ is zero element.

The MNCs χ or β (denoted below by φ) satisfy the following properties ([3, 3.1.2], [1, 1.1.4]):

- regularity: $\varphi_E(U) = 0$ if and only U is a totally bounded (a relatively compact) set;
- nonsingularity: $\varphi_E(U)$ is equal to zero on every one-element set;
- semi-homogeneity: $\varphi_E(tU) = |t|\varphi_E(U)$ for any number t;
- semi-additivity: $\varphi_E(U \cup V) = \max\{\varphi_E(U), \varphi_E(V)\}$;
- monotonicity: $\varphi_E(U) \leqslant \varphi_E(V)$, if $U \subseteq V$;
- invariance under translations: $\varphi_E(U + u) = \varphi_E(U)\ \ (u \in E)$;
- Lipschitzianity: $|\varphi_E(U) - \varphi_E(V)| \leqslant 2\rho(U, V)$, where ρ denotes the Hausdorff metric
- (more precisely, semimetric): $\rho(U, V) = \inf\{\varepsilon > 0 : V \subset U + \varepsilon B, U \subset V + \varepsilon B\}$;
- algebraic semi-additivity: $\varphi_E(U + V) \leq \varphi_E(U) + \varphi_E(V)$, where $U + V = \{u + v : u \in U, v \in V\}$;
- invariance under passage to the closure and to the convex hull: $\varphi_E(U) = \varphi_E(\overline{co}\, U)$.

Let Ω be some subset of $\mathbb{R}^n$, and let $\mu(\Omega) < \infty$, μ be a continuous measure; *i.e.*, each subset $D \subset \Omega$, $\mu(D) > 0$, can be split into two subsets of the same measure.

A Banach space E of real-valued measurable functions on Ω is an *ideal space* if it satisfies the following condition: if a function v belongs to E, u is a measurable function, and the inequality $|u| \leqslant |v|$ is fulfilled almost everywhere, then u also belongs to E, and $\|u\|_E \leqslant \|v\|_E$.

An ideal space E is a *regular space* (see [4, 12, 13]) if each function $u \in E$ has an absolutely continuous norm: $\lim_{\mu(D)\to 0} \|P_D u\|_E = 0$. In particular,

$$\lim_{T\to\infty} \|P_{D(u,T,u_0)} u\|_E = 0, \tag{2.1}$$

where $u_0 \in E$ is any fixed function with positive values, conventionally called the *unit* of space E,

$$D(u, T, u_0) = \{s \in \Omega : |u(s)| > Tu_0(s)\}$$

for an arbitrary number $T > 0$, and the symbol $P_D u$ denotes the multiplication operator by characteristic function $\varkappa_D$ of any subset $D \subset \Omega$.

Define $L_\infty(u_0)$ to be a Banach space of all real-valued measurable functions on Ω, with the norm $\|u\|_{L_\infty(u_0)} = \inf\{\lambda : |u| \leq \lambda u_0 \text{ a.e.}\}$ $(L_\infty(1) = L_\infty)$. It is a non-regular space.

We list the following examples of regular spaces for $u_0 \equiv 1$:

- spaces L_p $(1 \leqslant p < \infty)$ with the norm $\|u\|_p = \left(\int_\Omega |u(s)|^p ds\right)^{1/p}$,
- the Lorentz spaces $\Lambda_{1/p}(\Omega, \mu) = \Lambda_{1/p}(\Omega)$ $(1 \leq p < \infty)$ with the norm

$$\|u\|_{\Lambda_{1/p}(\Omega,\mu)} = \int_0^\infty \mu^{1/p}\big(D(u, T, 1)\big)\, dT,$$

- the Orlicz spaces.

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As in [5–10], for any regular space E the symbol $\nu_E(U)$ denotes the measure of the non-uniform absolute equicontinuity of norms $U \subset E$:

$$\nu_E(U) = \overline{\lim_{\mu(D)\to 0}} \sup_{u\in U} \|P_D u\|_E,$$

which is considered an MNC. In particular,

$$\nu_E(U) = \lim_{T\to\infty} \sup_{u\in U} \|P_{D(u,T,u_0)} u\|_E. \tag{2.2}$$

The measure $\nu_E(U)$ has all properties of φ mentioned above, excluding the regularity, since the equality $\nu_E(U) = 0$ is possible on noncompact sets.

Also it has been proved in [5] and [6] that if U is a bounded subset of a regular space E, then $\nu_E(U) \le \chi_E(U)$; if U is, in addition, compact in measure, then $\nu_E(U) = \chi_E(U)$. Below we will prove similar properties for β.

Here *compactness in measure* [1, 4.9.1] means compactness in the normed space S of all measurable, almost everywhere finite functions u, equipped with the norm $\|u\| = inf\{s + \mu\{t : |u(t)| \geqslant s\}\}$.

The following two statements, which will be prove below, are general in nature, *i.e.,* valid for an arbitrary Banach space E.

Lemma 2.1 *Let U be an arbitrary bounded infinite subset of a Banach space E. Then for every $\varepsilon > 0$ there exists an element $u \in U$ such that the ball $B(u, \beta_E(U)+\varepsilon)$ contains an infinite subset of U.*

Proof Let $u_1 \in U$ be an arbitrary element. Choose $\varepsilon > 0$. If the ball $B(u_1, \beta_E(U) + \varepsilon)$ contains an infinite subset of U, then the proof of the lemma is complete; otherwise, there exists an element $u_2 \notin B(u_1, \beta_E(U) + \varepsilon)$, $u_2 \in U$.

Similarly, if the ball $B(u_2, \beta_E(U) + \varepsilon)$ does not contain an infinite subset of U, there exists an element $u_3 \in U$ such that

$$u_3 \notin B\big(u_1, \beta_E(U) + \varepsilon\big) \cup B\big(u_2, \beta_E(U) + \varepsilon\big),$$

etc.

By the definition of $\beta_E(U)$ this process terminates on some step n, since by the construction, for any $i \neq j$ $(1 \leqslant i, j \leqslant n)$,

$$\|u_i - u_j\|_E \geqslant \beta_E(U) + \varepsilon.$$

Lemma 2.1 is proved. ∎

Lemma 2.2 *Let U be an arbitrary bounded infinite subset of a Banach space E. Then for each $\varepsilon > 0$ a set U contains an infinite subset such that the distance between any two elements is less than or equal to $\beta_E(U) + \varepsilon$.*

Proof By Lemma 2.1, for an arbitrary $\varepsilon > 0$ in U there exists an element u_1 such that the ball $B(u_1, \beta_E(U) + \varepsilon)$ contains an infinite subset $U_1 \subset U$.

Now we apply Lemma 2.1 to the set $U_1 \backslash \{u_1\}$. Taking into account the inequality $\beta_E(U_1) \leqslant \beta_E(U)$, we choose an element $u_2 \neq u_1$, such that the ball $B(u_2, \beta_E(U) + \varepsilon)$ contains an infinite set $U_2 \subset U_1$, etc.

Since on n-th step we obtain an infinite subset $U_n \subset U_{n-1}$, this process does not stop and we build an infinite sequence $\{u_n\}$, the distance between any two members of which is not greater than $\beta_E(U) + \varepsilon$.

Lemma 2.2 is proved. ■

3 Connection Between MNCs and Geometrical Characteristics of Regular Spaces

Let E be a regular space.

Definition 3.1 Let $\widetilde{S}$ be the set of all sequences $\{u_n\}$ of elements from E satisfying the following conditions:

(i) u_n, $n \in \mathbb{N}$, have pairwise disjoint supports;
(ii) $\lim_{n\to\infty} \|u_n\|_E = 1$;
(iii) the measure of the support $\operatorname{supp} u_n$ tends to zero as $n \to \infty$;
(iv) there exists a strictly increasing sequence of positive numbers $\{T_n\}$ with $\lim_{n\to\infty} T_n = \infty$ such that the inequality $T_{n-1}u_0(s) \leqslant |u_n(s)| < T_n u_0(s)$ holds for all $n \in \mathbb{N}$ and $s \in \operatorname{supp} u_n$.

Let

(3.1) $$\underline{c}_E = \inf_{\{u_n\}\in\widetilde{S}} \varliminf_{m\to\infty} \varliminf_{n\to\infty} \|u_n - u_m\|_E;$$

(3.2) $$\overline{c}_E = \sup_{\{u_n\}\in\widetilde{S}} \varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \|u_n - u_m\|_E.$$

Remark Note that $1 \leqslant \underline{c}_E \leqslant \overline{c}_E \leqslant 2$. The upper bound follows from the triangle inequality and Condition (ii). The lower bound is a consequence of Conditions (i) and (ii), since E is an ideal space.

We suppose further that the norm in a regular space also satisfies the following condition: for any sequences of subsets $\{D_n\}$, $\{D_n^*\}$ in Ω such that $D_n \cap D_n^* = \varnothing$ for all n and $\lim_{n\to\infty} \max\{\mu(D_n), \mu(D_n^*)\} = 0$, there is no bounded sequence $\{u_n\}$ of functions in E such that

(3.3) $$\lim_{n\to\infty} \|P_{D_n}u_n\|_E = a,\ \lim_{n\to\infty} \|P_{D_n^*}u_n\|_E = b,\ \lim_{n\to\infty} \|P_{D_n\cup D_n^*}u_n\|_E = d,$$

where $a > 0$, $b > 0$, $d = \max\{a, b\}$.

Remark Let $\{u_n\}$ be a bounded sequence of functions in E such that there exist $D_n \subset \Omega$, $n \in \mathbb{N}$, $\lim_{n\to\infty} \mu(D_n) = 0$ such that

$$\nu_E\{u_n\} = \lim_{n\to\infty} \|P_{D_n}u_n\|_E = a > 0.$$

Then $\nu_E\{v_n\} = 0$ for $v_n = u_n - P_{D_n}u_n$.

Proof Indeed, let $\nu_E\{v_n\} = b > 0$. Then there exist $D_{n_k}^* \subset \Omega$ such that

$$D_{n_k} \cap D_{n_k}^* = \varnothing, \quad \lim_{k\to\infty} \mu(D_{n_k}^*) = 0, \quad \lim_{n\to\infty} \|P_{D_{n_k}^*} v_{n_k}\|_E = b > 0.$$

Since

$$P_{D^*_{n_k}} v_{n_k} = P_{D^*_{n_k}}(u_n - P_{D_n} u_n) = P_{D^*_{n_k}} u_n,$$

we have $\lim_{n\to\infty} \|P_{D^*_{n_k}} u_{n_k}\|_E = b > 0$. Recall that E is an ideal space. Thus $\|P_{D_{n_k}} u_{n_k}\| \leqslant \|P_{D_{n_k}\cup D^*_{n_k}} u_{n_k}\|_E$. Since

$$\lim_{n\to\infty} \|P_{D_n} u_n\|_E = a \quad \text{and} \quad \overline{\lim_{k\to\infty}} \|P_{D_{n_k}\cup D^*_{n_k}} u_{n_k}\|_E \leqslant \nu_E\{u_n\} = a,$$

$$\lim_{k\to\infty} \|P_{D_{n_k}\cup D^*_{n_k}} u_{n_k}\|_E = a = \max\{a, b\},$$

and we get a contradiction to (3.3). ■

Lemma 3.2 *Let U be an arbitrary bounded subset of a regular space E with $\nu_E(U) > 0$. Then there exists a sequence $\{u_n\} \subseteq U$ with*

$$\underline{c}_E \nu_E(U) \leqslant \underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} \|u_n - u_m\|_E.$$

If U is compact in measure, we can choose $\{u_n\}$ to satisfy, in addition,

$$\overline{\lim_{m\to\infty}} \, \overline{\lim_{n\to\infty}} \|u_n - u_m\|_E \leqslant \bar{c}_E \nu_E(U).$$

Proof Let U be an arbitrary bounded subset of a regular space E with $\nu_E(U) > 0$. By (2.2), there exists a strictly increasing sequence of numbers $\{T_n\}$, $\lim_{n\to\infty} T_n = \infty$, and a sequence of functions $\{u_n\} \subseteq U$, for which the equality

$$\nu_E(U) = \lim_{n\to\infty} \|P_{D(u_n, T_n, u_0)} u_n\|_E$$

holds.

Note that (2.1) implies $\lim_{n\to\infty} \|P_{D(u_m, T_n, u_0)} u\|_E = 0$ for each fixed m.

Considering a subsequence (for our convenience, we do not change the notation), we may assume that $\nu_E(U) = \lim_{n\to\infty} \|P_{\widetilde{D}_n} u_n\|_E$, where

$$\widetilde{D}_n = \{s \in \Omega : T_n u_0(s) \leqslant |u_n(s)| < T_{n+1} u_0(s)\}.$$

It follows from the boundedness of U [13, Theorem 1], that

$$\lim_{n\to\infty} \operatorname{supp} u \in U \mu\big(D(u, T_n, u_0)\big) = 0.$$

Therefore, $\lim_{n\to\infty} \mu(\widetilde{D}_n) = 0$. Extracting subsequences, we may assume that $\mu\big(\bigcup_{k=n+1}^{\infty} \widetilde{D}_k\big)$ are small enough and the difference between $\|P_{\widetilde{D}_n} u_n\|_E$ and $\|P_{D_n} u_n\|_E$ is slight for $D_n = \widetilde{D}_n \setminus \bigcup_{k=n+1}^{\infty} \widetilde{D}_k$. Eventually, we get a sequence $\{u_n\}$ such that

$$\nu_E(U) = \lim_{n\to\infty} \|P_{D_n} u_n\|_E$$

and the sets D_n are pairwise disjoint.

Note that $\nu_E\{u_n\} = \nu_E(U)$ and by the remark before the lemma, $\nu_E\{v_n\} = 0$ for $v_n = u_n - P_{D_n} u_n$.

As consequence, we obtain

$$\lim_{k\to\infty} \sup_{m,n\geqslant k} \|P_{D_m}(u_n - P_{D_n} u_n) - P_{D_n}(u_m - P_{D_m} u_m)\|_E = 0,$$

$$\text{(3.4)} \qquad \lim_{k\to\infty} \sup_{m,n\geqslant k} \|P_{D_m} u_n - P_{D_n} u_m\|_E = 0.$$

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The constructed sequence of $\widetilde{u}_n = P_{D_n} u_n$ satisfies Conditions (i),(iii), and (iv) from Definition 3.1. Condition (ii) is replaced by the condition $\lim_{n\to\infty} \|\widetilde{u}_n\|_E = \nu_E(U)$.

Therefore

$$\varliminf_{m\to\infty} \varliminf_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}_m\|_E \geqslant \underline{c}_E \nu_E(U).$$

Since E is an ideal space, we have

$$\|u_n - u_m\|_E \geqslant \|P_{D_n\cup D_m}(u_n - u_m)\|_E \geqslant \big| \|\widetilde{u}_n - \widetilde{u}_m\|_E - \|P_{D_m} u_n - P_{D_n} u_m\|_E \big|$$

for any $m \neq n$, and by (3.4),

$$\varliminf_{m\to\infty} \varliminf_{n\to\infty} \|u_n - u_m\|_E \geqslant \underline{c}_E \nu_E(U).$$

The first part of Lemma 3.2 is proved.

Note that by (iii) the sequence $\{\widetilde{u}_n\}$ tends by measure to zero. Let U be compact in measure. Then $\{u_n\}$ is compact in measure too. Therefore, the sequence $\{u_n - \widetilde{u}_n\}$ is compact in measure too.

As it was proved in [5], [6], in this case $\chi_E\{u_n - \widetilde{u}_n\} = \nu_E\{u_n - \widetilde{u}_n\}$. By the remark before the lemma, $\nu_E\{u_n - \widetilde{u}_n\} = 0$. Hence $\chi_E\{u_n - \widetilde{u}_n\} = 0$. By the definition of the Hausdorff MNC, for every $\varepsilon > 0$ there exists a finite ε-net $C = \{c_1, c_2, \dots, c_N\} \subset E$ such that $\{u_n - \widetilde{u}_n\} \subset C + \varepsilon B$. Since C is finite, we can choose an infinite subsequence (with the same notation as before) that satisfies $\{u_n - \widetilde{u}_n\} \subset c^* + \varepsilon B$ for some $c^* \in C$. As a result, we have $|\|u_n - u_m\|_E - \|\widetilde{u}_n - \widetilde{u}_m\|_E| \leqslant 2\varepsilon$. Now we decrease ε and extract a subsequence (which we denote again by $\{u_n\}$) such that

$$\varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \big| \|u_n - u_m\|_E - \|\widetilde{u}_n - \widetilde{u}_m\|_E \big| = 0.$$

The second part of Lemma 3.2 is proved, since $\varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \|\widetilde{u}_n - \widetilde{u}_m\|_E \leqslant \overline{c}_E \nu_E(U)$. ■

Theorem 3.3 *In a regular space E the MNCs ν and β are related by the inequality $\beta_E(U) \geqslant \underline{c}_E \nu_E(U)$ for every bounded U; moreover, if U is compact in measure, then $\underline{c}_E \nu_E(U) \leqslant \beta_E(U) \leqslant \overline{c}_E \nu_E(U)$.*

Proof If $\nu_E(U) = 0$, then the inequality $\underline{c}_E \nu_E(U) \leqslant \beta_E(U)$ is satisfied. If U is compact in measure and $\nu_E(U) = 0$, then by the compactness criterion in regular spaces ([11–13]) U is relatively compact and $\beta_E(U) = 0$. Thus the assertion for the case $\nu_E(U) = 0$ holds. Therefore, we assume that $\nu_E(U) > 0$.

Let a sequence $\{u_n\}$ be as in Lemma 3.2. Choose $\varepsilon > 0$. By Lemma 2.2 we can assume without loss of generality that $\|u_m - u_n\|_E \leqslant \beta_E\{u_n\} + \varepsilon$ for all n and m.

Thus by virtue of the monotonicity of β, we obtain

$$\|u_m - u_n\|_E \leqslant \beta_E\{u_n\} + \varepsilon \leqslant \beta_E(U) + \varepsilon$$

for all n and m.

Since we can take ε arbitrarily small and $\underline{c}_E \nu_E(U) \leqslant \varliminf_{m\to\infty} \varliminf_{n\to\infty} \|u_n - u_m\|_E$ by Lemma 3.2, we get the assertion of the first part of Theorem 3.3.

Let U be compact in measure. By the definition of β, for given $\varepsilon > 0$, there exists a sequence $\{w_n\}$ such that $\|w_n - w_m\| \geqslant \beta_E(U) - \varepsilon$ for all $n \neq m$. Hence by Lemma 3.2, we can extract a subsequence $\{u_n\}$ of $\{w_n\}$ such that

$$\beta_E(U) - \varepsilon \leqslant \varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \|u_n - u_m\|_E \leqslant \bar{c}_E \nu_E(\{w_n\}) \leqslant \bar{c}_E \nu_E(U),$$

which finishes the proof of Theorem 3.3, since ε can be arbitrarily small. ■

Below we shall consider examples of calculation of the constants (3.1) and (3.2).

Example 3.4 $\underline{c}_{L_p} = \bar{c}_{L_p} = 2^{1/p}$ for $1 \leqslant p < \infty$.

Proof Recall that by Definition 3.1(i) functions u_n have disjoint supports and by Definition 3.1(ii) their norms tend to 1. Hence,

$$\varliminf_{m\to\infty} \varliminf_{n\to\infty} \|u_n - u_m\|_E = \varliminf_{m\to\infty} \varliminf_{n\to\infty} \left(\|u_n\|_{L_p}^p + \|u_m\|_{L_p}^p \right)^{1/p} = 2^{1/p}$$

and

$$\varlimsup_{m\to\infty} \varlimsup_{n\to\infty} \|u_n - u_m\|_E = \varlimsup_{m\to\infty} \varlimsup_{n\to\infty} (\|u_n\|_{L_p}^p + \|u_m\|_{L_p}^p)^{1/p} = 2^{1/p}$$

for every $\{u_n\} \in \widetilde{S}$. Therefore, $\underline{c}_{L_p} = \bar{c}_{L_p} = 2^{1/p}$. ■

Example 3.5 $\underline{c}_{\Lambda_{1/p}} = \bar{c}_{\Lambda_{1/p}} = 2$ for $1 \leqslant p < \infty$.

Proof Since by [11, 15.1] the set of all finite-valued functions is dense in $\Lambda_{1/p}$, $1 \leqslant p < \infty$, without loss of generality, we may assume that $\widetilde{S}$ consists of sequences of finite-valued functions.

By Definition 3.1, $\{u_n\}$ is a sequence of functions with disjoint supports such that there exists strictly increasing sequence of positive numbers $\{T_n\}$, such that $T_{n-1} \leqslant |u_n(s)| < T_n$ for all $s \in \operatorname{supp} u_n$.

Since $|||f||| = \|f\|$, we can consider functions of the form: $u_n = \sum_{i=1}^{\ell_n} c_i \varkappa_{D_i}$, $u_m = \sum_{i=\ell_n+1}^{\ell_n+\ell_m} c_i \varkappa_{D_i}$, where

$$c_1 > c_2 > \cdots > c_{\ell_n} > c_{\ell_n+1} > \cdots > c_{\ell_n+\ell_m} > c_{\ell_n+\ell_m+1} = 0, \quad n \geqslant m.$$

Then by [11, Formula (15.3)],

$$\|u_n - u_m\|_{\Lambda_{1/p}} = \sum_{i=1}^{\ell_m+\ell_n} (c_i - c_{i+1}) \mu\Big(\bigcup_{k=1}^{i} D_k \Big)^{1/p}.$$

Hence,

$$\|u_n - u_m\|_{\Lambda_{1/p}} = \|u_n\|_{\Lambda_{1/p}} - c_{\ell_n+1}\, \mu\Big(\bigcup_{k=1}^{\ell_n} D_k \Big)^{1/p} + \sum_{i=\ell_n+1}^{\ell_m+\ell_n} (c_i - c_{i+1}) \mu\Big(\bigcup_{k=1}^{i} D_k \Big)^{1/p}.$$

By Definition 3.1(iii),

$$\lim_{n\to\infty} \mu\Big(\bigcup_{k=1}^{\ell_n} D_k \Big) = \lim_{n\to\infty} \mu(\operatorname{supp} u_n) = 0.$$

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Hence,

$$\begin{aligned}\lim_{n\to\infty}&\|u_n-u_m\|_{\Lambda_{1/p}}\\&=\lim_{n\to\infty}\Big(\|u_n\|_{\Lambda_{1/p}}-c_{\ell_n+1}(\mu(\operatorname{supp}u_n))^{1/p}\\&\qquad+\sum_{i=\ell_n+1}^{\ell_m+\ell_n}(c_i-c_{i+1})\Big(\mu\Big(\bigcup_{k=\ell_n+1}^{i}D_k\Big)+\mu(\operatorname{supp}u_n)\Big)^{1/p}\Big)\\&=1+\|u_m\|_{\Lambda_{1/p}}\end{aligned}$$

since if m is fixed, the numbers $c_{\ell_n+1},\dots,c_{\ell_n+\ell_m}$ do not change.

Therefore,

$$\lim_{m\to\infty}\lim_{n\to\infty}\|u_n-u_m\|_{\Lambda_{1/p}}=\lim_{m\to\infty}(1+\|u_m\|_{\Lambda_{1/p}})=2$$

and $\bar{c}_{\Lambda_{1/p}}=\underline{c}_{\Lambda_{1/p}}=2$. ∎

Example 3.6 Let E be a regular space consisting of all functions on Ω, where $\Omega=G_1\cup G_2$, $G_1\cap G_2=\varnothing$, $\mu(G_i)>0$ $(i=1,2)$, with the norm defined by $\|u\|_E=\|P_{G_1}u\|_{\Lambda_{1/p}}+\|P_{G_2}u\|_{L_p}$ for $1\leqslant p<\infty$. For this space, $\underline{c}_E<\bar{c}_E$.

Corollary *If $E=L_p$ or $\Lambda_{1/p}$, then for any bounded subset $U\subseteq E$ inequalities $\beta_{L_p}(U)\geqslant 2^{1/p}\nu_{L_p}(U)$, $\beta_{\Lambda_{1/p}}(U)\geqslant 2\nu_{\Lambda_{1/p}}(U)$ hold. In particular, $\beta_{\Lambda_{1/p}}(B)=2$.*

If U is compact in measure, then $\beta_{L_p}(U)=2^{1/p}\nu_{L_p}(U)=2^{1/p}\chi_{L_p}(U)$, $\beta_{\Lambda_{1/p}}(U)=2\nu_{\Lambda_{1/p}}(U)=2\chi_{\Lambda_{1/p}}(U)$.

4 MNC β of Bounded Subsets in L_∞

The aim of this section is to show that $\beta_{L_1}(V)\leqslant(2-r/(a\mu(\Omega))r$ follows from the inclusion $V\subset B_{L_\infty}(\theta,a)\cap B_{L_1}(\theta,r)$. We start with some particular cases.

Throughout this section $\widetilde{U}$ denotes the set of all measurable functions on Ω with values in the set $\{-1,0,1\}$.

Below we use the proportionality β and χ in the separable Hilbert space:

$$\beta=\sqrt{2}\chi. \tag{4.1}$$

Lemma 4.1 *Let U be the set of all functions $u\in\widetilde{U}$ satisfying the following condition: there exists $\omega\in\mathbb{R}_+$ such that $\mu(\operatorname{supp}u)=\omega$. Then $\beta_{L_1}(U)\leqslant 2\omega-\omega^2/\mu(\Omega)$.*

Proof By the definition of β, for any $\varepsilon>0$, the set U contains an infinite $(\beta_{L_1}(U)-\varepsilon)$-lattice U_0, *i.e.*, $\|u-v\|_{L_1}\geqslant\beta_{L_1}(U)-\varepsilon$ for all $u\neq v$, $u,v\in U_0$.

First, we show that for the chosen ε we can find an infinite subset $U_1\subset U_0$, such that for any $u,v\in U_1$ we have

$$\xi_{uv}:=\mu(\operatorname{supp}u\triangle\operatorname{supp}v)\leqslant 2\big(\omega-\omega^2/\mu(\Omega)\big)+\varepsilon, \tag{4.2}$$

where $A\triangle B:=(A\cup B)\backslash(A\cap B)$ for sets A and B.

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Let $\widehat{U} := \{u \in \widetilde{U} \mid u(s) \in \{0,1\}$ for all $s \in \Omega\}$. Denote by $(\omega/\mu(\Omega))e$ the constant function with value $\omega/\mu(\Omega)$. Then

$$\big(\chi_{L_2}(\widehat{U})\big)^2 \leqslant \sup_{u\in\widehat{U}} \big\| u - \big(\omega/\mu(\Omega)\big)\, e \big\|^2_{L_2} = \big(1-\omega/\mu(\Omega)\big)^2 \omega + \big(\omega/\mu(\Omega)\big)^2 \big(\mu(\Omega)-\omega\big)$$

by the definition of $\widetilde{U}$.

Hence $(\chi_{L_2}(\widehat{U}))^2 \leqslant \omega - \omega^2/\mu(\Omega)$, and (4.1) implies $\beta_{L_2}(\widehat{U}) \leqslant \sqrt{2(\omega-\omega^2/\mu(\Omega))}$.

Now by Lemma 2.2 we can extract from U_0 for the chosen ε an infinite subset U_1 such that

$$\big\| |u| - |v| \big\|^2_{L_2} \leqslant 2(\omega - \omega^2/\mu(\Omega)) + \varepsilon$$

for any two elements $u, v \in U_1$. Since $|u(s)| - |v(s)| = 0$ for all $s \in \operatorname{supp} u \cap \operatorname{supp} v$, we get $\xi_{uv} = \xi_{|u|,|v|} = \||u| - |v|\|^2_{L_2} \leqslant 2(\omega - \omega^2/\mu(\Omega)) + \varepsilon$, which completes the proof of (4.2).

Next we prove that for the given ε, there exists an infinite subset $U_2 \subset U_1$ such that for any two elements $u, v \in U_2$ we have

$$\text{(4.3)} \qquad \omega_{uv} := \mu\big\{ t \in \Omega \mid |u(t) - v(t)| = 2 \big\} \leqslant \mu(\operatorname{supp} u \cap \operatorname{supp} v)/2 + \varepsilon.$$

Indeed, by (4.1),

$$\beta_{L_2}(U_1) = \sqrt{2}\chi_{L_2}(U_1) \leqslant \sqrt{2} \sup_{u\in U_1} \|u\|_{L_2} = \sqrt{2\omega}.$$

Therefore, by Lemma 2.2, the set U_1 includes an infinite subset U_2 such that $\|u-v\|^2_{L_2} \leqslant 2\omega + \varepsilon$ for all $u, v \in U_2$. Hence

$$\|u - v\|^2_{L_2} = 4\omega_{uv} + 2(\omega - \mu(\operatorname{supp} u \cap \operatorname{supp} v)) \leqslant 2\omega + \varepsilon,$$

which completes the proof of (4.3).

Note that (4.3) implies $\omega_{uv} \leqslant (\omega - \xi_{uv}/2)/2 + \varepsilon$. Thus for every $u, v \in U_2$, $u \neq v$,

$$\beta_{L_1}(U) - \varepsilon \leqslant 2\omega_{uv} + \xi_{uv} \leqslant (\omega - \xi_{uv}/2) + \xi_{uv} + 2\varepsilon \leqslant 2\omega - \omega^2/\mu(\Omega) + 2\varepsilon,$$

whence we obtain the assertion of Lemma 4.1, since ε can be arbitrarily small. ∎

Lemma 4.2 *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\omega_1, \omega_2, \dots, \omega_n$ be two collections of positive numbers $\big(\sum_{i=1}^n \omega_i \leqslant \mu(\Omega)\big)$. Consider the set V of elements $\sum_{i=1}^n \alpha_i u_i$, where $u_i \in \widetilde{U}$, $\mu(\operatorname{supp} u_i) = \omega_i$, and $\operatorname{supp} u_i \cap \operatorname{supp} u_j = \varnothing$, for $1 \leqslant i, j \leqslant n$. Then $\beta_{L_1}(V) \leqslant 2r - r^2/(a\mu(\Omega))$, where $r = \sum_{i=1}^n \alpha_i\omega_i$ and $a = \max_{1\leqslant i\leqslant n} \alpha_i$.*

Proof If $n = 1$, the assertion follows from Lemma 4.1, the semi-homogeneity of β, and the inequality $\alpha_1 > 0$. Therefore, we assume the validity of the assertion for some $n > 1$ and prove that it remains true when we replace n with $n+1$. Without loss of generality, we may assume that $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha_{n+1}$. Then the algebraic additivity of β and the equality $\sum_{i=1}^n \alpha_i u_i = \sum_{i=1}^n \alpha_i u_i + \alpha_n u_{n+1} + (\alpha_{n+1} - \alpha_n) u_{n+1}$

implies

$$\beta_{L_1}(V) \leqslant \beta_{L_1}\Big\{ \sum_{i=1}^{n} \alpha_i \widetilde{u}_i \;\Big|\; u_i \in \widetilde{U},\ \mu(\operatorname{supp} \widetilde{u}_i) = \omega_i \text{ for } 1 \leqslant i \leqslant n-1,$$
$$\mu(\operatorname{supp} \widetilde{u}_n) = \omega_n + \omega_{n+1},\ \operatorname{supp} u_i \cap \operatorname{supp} u_j = \varnothing,\ 1 \leqslant i, j \leqslant n \Big\}$$
$$+ \beta_{L_1}\{(\alpha_{n+1} - \alpha_n) u_{n+1} \mid u_{n+1} \in \widetilde{U},\ \mu(\operatorname{supp} u_{n+1}) = \omega_{n+1}\}.$$

Using the inductive assumption, we get

$$\beta_{L_1}(V) \leqslant 2\widetilde{r} - \widetilde{r}^2 / \big(\alpha_n \mu(\Omega)\big) + 2(\alpha_{n+1} - \alpha_n)\omega_{n+1}$$
$$- \big((\alpha_{n+1} - \alpha_n)\omega_{n+1}\big)^2 / \big((\alpha_{n+1} - \alpha_n)\mu(\Omega)\big),$$

where $\widetilde{r} = \sum_{i=1}^{n} \alpha_i \omega_i + \alpha_n \omega_{n+1}$.

Now

$$\frac{\big\{ \sum_{i=1}^{n} \alpha_i \omega_i + \alpha_n \omega_{n+1} \big\}^2}{\alpha_n} + (\alpha_{n+1} - \alpha_n)\omega_{n+1}^2 \geqslant \frac{\big\{ \sum_{i=1}^{n+1} \alpha_i \omega_i \big\}^2}{\alpha_{n+1}}$$

implies Lemma 4.2. ∎

Now we are ready to prove the main result of the section.

Theorem 4.3 *Let $U \subset B_{L_\infty}(\theta, a) \cap B_{L_1}(\theta, r)$, $r \leqslant a\mu(\Omega)$. Then*

$$\beta_{L_1}(U) \leqslant (2 - r/(a\mu(\Omega))r.$$

Proof By the definition of β, for every $\varepsilon > 0$, the set U contains an infinite sequence $\{u_k\}$, satisfying the inequality $\|u_k - u_m\|_{L_1} \geqslant \beta_{L_1}(U) - \varepsilon$ for all $k \neq m$.

Note that $\{u_k\}$ is bounded in L_∞. Therefore, considering a subsequence, we may assume that there exists $\lim_{k\to\infty} \|u_k\| = r_1 \leqslant r$. Now we consider approximations of $\{u_k\}$ by functions satisfying the assumptions of Lemma 4.2. Considering limit points of sets of values of every ω_k for a fixed k, taking subsequences once again, and using the continuity of the measure μ, we may assume that there exists a sequence $\{\widetilde{u}_k\}$ of elements satisfying the assumptions of Lemma 4.2 with the same α_i and ω_j, such that $\|u_k - \widetilde{u}_k\|_{L_1} < \varepsilon$ for all $k \in \mathbb{N}$. Thus, $\beta_{L_1}(U) - \varepsilon \leqslant \|\widetilde{u}_k - \widetilde{u}_m\|_{L_1} + 2\varepsilon$ for any $k \neq m$. By Lemma 2.2, without loss of generality we may assume that $\|\widetilde{u}_k - \widetilde{u}_m\|_{L_1} \leqslant \beta_{L_1}\{\widetilde{u}_k\} + \varepsilon$.

By Lemma 4.2, $\beta_{L_1}(U) \leqslant \|\widetilde{u}_k - \widetilde{u}_m\|_{L_1} + 3\varepsilon \leqslant \beta_{L_1}\{\widetilde{u}_k\} + 4\varepsilon \leqslant (2 - r_1/(a\mu(\Omega))r_1 + 4\varepsilon$. This completes the proof of Theorem 4.3, since $\varepsilon > 0$ can be arbitrarily small, the function $f(x) = (2 - x/(a\mu(\Omega)))x$ is increasing on $[0; a\mu(\Omega)]$, and $r \leqslant a\mu(\Omega)$. ∎

5 (k, β)-boundedness of Partially Additive Operators

Let E and E_1 be Banach spaces. We recall from [1, 1.5.1] that a continuous operator $A\colon G \subseteq E \to E_1$ (not necessarily linear) is said to be *condensing with respect to MNC* φ, if for any bounded subset $U \subset G$ with noncompact closure, the inequality $\varphi_{E_1}(AU) < \varphi_E(U)$ holds.

A continuous operator $A\colon G \subseteq E \to E_1$ is called (k, φ)-*bounded* with respect to MNC φ, if there exists a constant $k > 0$ such that $\varphi_{E_1}(AU) \leqslant k\varphi_E(U)$ for any bounded subset $U \subset G$.

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If $k < 1$ then (k, φ)-bounded operator A is condensing with respect to MNC φ. The converse, in general, is not true.

Let E be a regular space. We consider partially additive operators $A\colon E \to E_1$ [11, 17.4]. In particular, partially additive operators satisfy the condition

$$A(P_{D(u,T,u_0)}u + P_{A(u,T,u_0)}u) = AP_{D(u,T,u_0)}u + AP_{A(u,T,u_0)}u - A\theta \tag{5.1}$$

for any function $u \in E$.

Let U be any bounded subset from E. We denote the following as in [7–9]:

$$k(U, A, E, E_1) = \overline{\lim_{T\to\infty}} \sup_{\|P_{D(u,T,u_0)}u\|_E \neq 0, u\in U} \frac{\|AP_{D(u,T,u_0)}u\|_{E_1}}{\|P_{D(u,T,u_0)}u\|_E}.$$

Evidently, in the case of a linear bounded operator the constant $k(U, A, E, E_1)$ does not exceed the norm of the operator. For a nonlinear operator, even if it is partially additive and bounded, this constant is either finite or infinite.

Lemma 5.1 *Let $A\colon E \to E_1$ be a continuous partially additive operator, where E is a regular space. In addition, let A be compact as an operator from $L_\infty(u_0)$ to E_1. Let U be an arbitrary bounded subset in E for which the constant k is finite. Then for any $V \subseteq U$ we have $\beta_{E_1}(AV) \leqslant \beta_{E_1}(B(\theta, k(U, A, E, E_1)\nu_E(V)))$.*

Proof By (5.1), the assumption of partially additivity of A, and the algebraic additivity of β, we obtain for any $V \subseteq U$,

$$\beta_{E_1}(AV) \leqslant \beta_{E_1}\left(A\{P_{D(u,T,u_0)}\colon u \in V\}\right) + \beta_{E_1}\left(A\{P_{A(u,T,u_0)}\colon u \in V\}\right) + \beta_{E_1}\left(A(\theta)\right).$$

We have $\beta_{E_1}(A\{P_{A(u,T,u_0)} : u \in V\}) = 0$, since the restriction of A on $L_\infty(u_0)$ is compact. Furthermore, the nonsingularity of β implies $\beta_{E_1}(A(\theta)) = 0$.

Therefore, $\beta_{E_1}(AV) \leqslant \beta_{E_1}(A\{P_{D(u,T,u_0)}\colon u \in V\})$. Note that we have the inclusion of $A\{P_{D(u,T,u_0)}\colon u \in V\}$ into

$$B\Big(\theta, \sup_{\|P_{D(u,T,u_0)}u\|_E \neq 0, u\in V} \frac{\|AP_{D(u,T,u_0)}u\|_{E_1}}{\|P_{D(u,T,u_0)}u\|_E} \sup_{u\in V} \|P_{D(u,T,u_0)}u\|_E\Big)$$

for any $T > 0$. From here, taking into account the monotonicity of β and the inequality $k(V, A, E, E_1) \leqslant k(U, A, E, E_1)$ for every $V \subseteq U$, we obtain the assertion of Lemma 5.1. ∎

Theorem 5.2 *Suppose that A satisfies the conditions of Lemma 5.1. Then the operator A is $((k(U, A, E, E_1)\beta_{E_1}(B))/\underline{c}_E, \beta)$-bounded on U.*

Proof Applying Lemma 5.1, Theorem 3.3, and the semi-homogeneity of β, we obtain

$$\begin{aligned}\beta_{E_1}(AV) &\leqslant k(U, A, E, E_1)\nu_E(V)\beta_{E_1}(B) = k(U, A, E, E_1)\nu_E(V)\beta_{E_1}(B) \\ &\leqslant \frac{k(U, A, E, E_1)\beta_{E_1}(B)}{\underline{c}_E}\underline{c}_E\nu_E(V) \leqslant k(U, A, E, E_1)\frac{\beta_{E_1}(B)}{\underline{c}_E}\beta_E(V).\end{aligned}$$

Theorem 5.2 is proved. ∎

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Corollary (i) *As was proved in* [10], $\beta_{L_p}(B) = \max\{2^{1/p}, 2^{1-1/p}\}$ *for* $1 \leqslant p < \infty$. *Thus a continuous partially additive operator* $A\colon L_p \to L_q$ *compact as an operator* $A\colon L_\infty \to L_q$, *is* (k, β)*-bounded on any set* U *with*

$$k = (k(U, A, L_p, L_q) \max\{2^{1/q}, 2^{1-1/q}\})/2^{1/p}.$$

(ii) *Let* A *be a linear operator, acting from* L_p *in* L_∞ $(1 \leqslant p < \infty)$. *Then* A *as an operator from* L_p *in* L_q *is a* $(2^{(p-q)/pq}\|A\|, \beta)$*-bounded operator for* $1 \leqslant q \leqslant 2$, *and a* $(2^{1-1/p-1/q}\|A\|, \beta)$*-bounded operator for* $2 < q < \infty$ [10, Theorem 2].

Theorem 5.3 *Let* $A\colon L_1(\Omega) \to L_1(\Omega)$ *be continuous partially additive operator with the compact restriction on* $L_\infty(\Omega)$. *Then* A *is*

$$\left(\left(1 - \frac{k(U, A, L_1(\Omega), L_1(\Omega))}{2k(U, A, L_1(\Omega), L_\infty(\Omega))\mu(\Omega)} \right) k(U, A, L_1(\Omega), L_1(\Omega)), \beta \right)\text{-bounded}$$

as an operator from $L_1(\Omega)$ *in* $L_1(\Omega)$.

Proof Let $V \subseteq U$. By the proof of Lemma 5.1,

$$\beta_{L_1}(AV) \leqslant \beta_{L_1}\left(A\{P_{D(u,T,u_0)}\colon u \in V\}\right).$$

Furthermore, we have inclusions

$$A\{P_{D(u,T,u_0)}\colon u \in V\} \subset B_{L_\infty}\left(\theta, k(U, A, L_\infty(\Omega), L_1(\Omega))\nu_{L_1}(V)\right)$$

and

$$A\{P_{D(u,T,u_0)}\colon u \in V\} \subset B_{L_1}\left(\theta, k(U, A, L_1(\Omega), L_1(\Omega))\nu_{L_1}(V)\right)$$

for any $T > 0$. Thus, by Theorem 4.3, we have the inequality

$$\beta_{L_1}(AV) \leqslant \bigl(2 - r/(a\mu(\Omega)\bigr) r, \quad r = k\bigl(U, A, L_1(\Omega), L_1(\Omega)\bigr)\, \nu_{L_1}(V),$$
$$a = k\bigl(U, A, L_1(\Omega), L_\infty(\Omega)\bigr)\, \nu_{L_1}(V).$$

Hence,

$$\beta_{L_1}(AV) \leqslant \left(2 - \frac{k(U, A, L_1(\Omega), L_1(\Omega))}{k(U, A, L_1(\Omega), L_\infty(\Omega))\mu(\Omega)} \right) k(U, A, L_1(\Omega), L_1(\Omega))\nu_{L_1}(V).$$

By Theorem 3.3, $2\nu_{L_1}(V) \leqslant \beta_{L_1}(V)$. This finishes the proof of Theorem 5.3. ■

Note that [11, Lemma 5.3] implies that any linear integral operator is compact as an operator $L_\infty \to L_1$.

Example 5.4 Let $u_n(t)$, $n = 1, 2, \dots$, be the sequence of Rademacher functions in $L_1 := L_1(0, 1)$. Let $\Delta_1, \Delta_2, \dots$ be a sequence of disjoint intervals in $[0, 1]$. Denote by $\varkappa_n(s)$ the characteristic function of Δ_n. Let

$$K(t, s) = \sum_{n=1}^{\infty} u_n(t)\varkappa_n(s).$$

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Clearly, the function $K(t,s)$ is measurable with respect to s and t. Let $t \in [0,1]$ and $u \in L_1$. Then

$$\Big|\int_0^1 K(t,s)u(s)ds\Big| = \Big|\sum_{n=1}^{\infty}\int_{\Delta_n} u_n(t)u(s)ds\Big| \leqslant \sum_{n=1}^{\infty}\Big|\int_{\Delta_n} u(s)ds\Big| \leqslant \sum_{n=1}^{\infty}\int_{\Delta_n}|u(s)|ds \leqslant \|u\|_{L_1}.$$

Note that

$$(Ku)(t) = \int_0^1 K(t,s)u(s)ds$$

is measurable for every $u \in L_1$, and its norm L_1 is less than or equal to $\|u\|_{L_1}$. Therefore, the operator K satisfies all conditions of Theorem 5.3 (see also the remark before the example) and, in addition, $\|K\|_{L_1\to L_\infty} = \|K\|_{L_1\to L_1} = 1$. Thus by Theorem 5.3, K is $(1/2,\beta)$-bounded and, therefore, β-condensing.

Since $\|K\|_{L_1\to L_1} = 1$, the operator K is $(1,\chi)$-bounded. On the other hand, if $v_n(s) = \varkappa_n(s)/\mu(\Delta_n)$, then $(Kv_n)(t) = u_n(t)$. In particular,

$$\chi_{L_1}\{v_n\} = \chi_{L_1}\{\varkappa_n/\mu(\Delta_n)\} = 1, \qquad \chi_{L_1}\{Kv_n\} = \chi_{L_1}\{u_n\} = 1.$$

Thus, the operator K is condensing with respect to β, but not χ-condensing.

Remark MNCs χ and β were cosidered in the works of L. S. Gol'denshtein, I. Gohberg, A. S. Markus, V. Istrătescu, J. Daneš, and others. Detailed description of bibliographic information is given in [1]. In particular, the author has proved the algebraic semi-additivity, the invariance under passage to the convex hull of β and proportionality formula (4.1) (see the references in [1, 1.8.3, 4.9.9]).

The formula (4.1) and the algebraic semi-additivity of β were also obtained independently by the authors of [2].

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Moscow State Technical University of Civil Aviation
e-mail: naerzakova@gmail.com