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ON THE PROBLEM OF NON-BERWALDIAN LANDSBERG SPACES

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Abstract

We study the long-standing problem of the existence of non-Berwaldian Landsberg spaces from the perspective of conformal transformations. We calculate the Berwald and Landsberg tensors in terms of the T-tensor and show that there are Landsberg spaces with nonvanishing T-tensor. We give a necessary condition for a Landsberg space to be Berwaldian. We find conditions under which the Landsberg spaces cannot be Berwaldian and give examples of (y-local) non-Berwaldian Landsberg spaces.

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1. Introduction

In a Riemannian manifold M, each tangent space T_xM, $x \in M$, is equipped with an inner product. In a Finsler manifold each tangent space is equipped with a Minkowski norm which is not necessarily induced by an inner product. A Finsler function (structure) on a manifold M is a function from the tangent bundle $\mathrm{TM} = \bigcup_{x\in M} T_xM$ to $\mathbb{R}$, which is a norm on each tangent space. Finsler geometry involves the choice of a norm on each tangent space and thus a Finsler function $F : \mathrm{TM} \to \mathbb{R}$ satisfying certain conditions. The usual convention is that x denotes the position in the manifold and y denotes the direction in the tangent space. Finsler geometry arises naturally in various settings as explained in Chern [6] and a remarkable number of features of Riemannian geometry extend to the Finsler case (for more details, see [5]). In applications, for example in relativistic physics, F may only be defined on an open subset of TM, the so-called y-local spaces (see [1]).

Let M be an n-dimensional smooth manifold, (x^i) the coordinate system on the base manifold M and (x^i, y^i) the induced coordinate system on TM. For a Finsler metric $F = F(x, y)$ on M, the geodesic spray $S = y^i\partial/\partial x^i - 2G^i\partial/\partial y^i$ is a vector field on the tangent bundle TM, where the functions $G^i = G^i(x, y)$ are homogeneous of degree two in y and called geodesic coefficients. The G^i are given by

$$G^i = \frac{1}{4} g^{ih}(y^r \partial_r \dot{\partial}_h F^2 - \partial_h F^2), \quad \text{where } \partial_i := \frac{\partial}{\partial x^i},\ \dot{\partial}_h := \frac{\partial}{\partial y^h}. \tag{1.1}$$

The nonlinear connection G^i_j and the coefficients of the Berwald connection G^i_{jk} are defined respectively by

$$G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Berwald tensor G^i_{jkh} and the Landsberg tensor L_{jkh} are given respectively by

$$G^i_{jkh} = \dot{\partial}_h G^i_{jk}, \quad L_{jkh} = -\tfrac{1}{2} F \ell_i G^i_{jkh}, \quad \text{where } \ell_i := \dot{\partial}_i F. \tag{1.2}$$

A Finsler manifold (M, F) is said be *Berwaldian* if the Berwald tensor G^h_{ijk} vanishes identically. In Berwald manifolds, the coefficients of the Berwald connection $G^i_{jk}(x)$ are functions of x only and the spray coefficients G^i are quadratic in y. A Finsler manifold (M, F) is said be *Landsberg* if the Landsberg tensor L_{ijk} vanishes identically.

The regular Landsberg spaces are the most elusive. In 1907, Landsberg [8–10] introduced the Landsberg spaces in a non-Finsler framework. Every Berwald space is a Landsberg space. Whether there are regular Landsberg spaces which are not Berwaldian is a long-standing open question in Finsler geometry.

Asanov [1] obtained examples, arising from Finslerian general relativity, of non-Berwaldian Landsberg spaces of dimension at least three. In Asanov's examples the Finsler functions are not defined for all values of the fibre coordinates y^i (that is, they are y-local). Whether or not there are y-global non-Berwaldian Landsberg spaces remains an open question. Shen [13] studied the class of (α, β) metrics of Landsberg type, of which Asanov's examples are particular cases, and found that there are y-local non-Berwaldian Landsberg spaces with (α, β) metrics, but no y-global ones [3, 13]. Bao [4] tried to construct non-Berwaldian Landsberg spaces by successive approximation. The elusiveness of y-global non-Berwaldian Landsberg spaces led Bao to describe them as the unicorns of Finsler geometry.

In this paper, we study the question of the existence of non-Berwaldian Landsberg spaces from the perspective of conformal transformations. For a Finsler manifold (M, F), a conformal transformation of F is defined by

$$\overline{F} = e^{\sigma(x)} F,$$

where $\sigma(x)$ is a function on the manifold M. We describe the effect of a conformal transformation on the Berwald and Landsberg tensors in terms of the T-tensor, which plays an important role in Finsler geometry. For example, Szabo [14] proved that a positive-definite Finsler metric with vanishing T-tensor is Riemannian.

Hashiguchi [7] showed that a Landsberg space remains Landsberg under *every* conformal transformation if and only if the T-tensor vanishes identically. We show that there are Landsberg spaces (with nonvanishing T-tensor) which remain Landsberg under some conformal transformations. Matsumoto [12] showed that a Berwald space

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(M, F) remains Berwaldian under *every* conformal transformation if and only if $B^{ir}{}_{jkh} := \partial^3(F^2 g^{ir})/\partial y^j \partial y^k \partial y^h = 0$ (that is, the $F^2 g^{ir}$ are quadratic in y^i). We show that there are Berwald spaces (with nonvanishing $B^{ir}{}_{jkh}$) which remain Berwaldian under some conformal transformations.

Starting with a Berwald space (M, F), if the transformed space $(M, \overline{F})$ is Landsberg, we give a necessary condition for $(M, \overline{F})$ to be Berwaldian. We consider some special cases, for example when (M, F) is S_3-like or has vanishing T-tensor and vanishing vertical curvature of the Cartan connection.

The condition for a Berwald space to transform to a Landsberg space is that $\sigma_r T^r_{jkh} = 0$. We show that a positive-definite C-reducible Finsler space does not admit a function $\sigma(x)$ such that $\sigma_r T^r_{jkh} = 0$. Since Randers spaces are C-reducible, a regular Randers space does not admit such a function.

Finally, we show that under a conformal transformation, a C_2-like Berwald space with vanishing T-tensor transforms to a non-Berwaldian Landsberg space and we derive examples of (singular) non-Berwaldian Landsberg spaces.

2. Conformal transformation

Let M be an n-dimensional manifold and (TM, π, M) be its tangent bundle. We denote by (x^i) the local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM, where y^i is called the supporting element.

Definition 2.1. A Finsler function $F : \mathrm{TM} \to \mathbb{R}$ on a manifold M is a continuous function such that:

(i) F is smooth and strictly positive on the slit tangent bundle $\mathcal{T}M := \mathrm{TM}\backslash\{0\}$ and $F(x, y) = 0$ if and only if $y = 0$;
(ii) F is positively homogeneous of degree one in the directional argument y;
(iii) the metric tensor $g_{ij} = \frac{1}{2}\partial^2 F^2/\partial y^i \partial y^j$ has maximal rank on $\mathcal{T}M$.

The pair (M, F) is called a Finsler space and the symmetric bilinear form g given by $g = g_{ij}(x, y)\, dx^i \otimes dx^j$ is called the Finsler metric tensor of the Finsler space (M, F). The function $E := \frac{1}{2}F^2$ is called the energy function associated to F.

A conformal transformation of a Finsler structure F is given by

$$\overline{F} = e^{\sigma(x)} F, \tag{2.1}$$

where $\sigma(x)$ is a smooth function on M. All the geometric objects associated with the transformed space $(M, \overline{F})$ will be denoted by barred symbols. For example, the metric tensor of $(M, \overline{F})$ is denoted by $\overline{g}_{ij}$.

Lemma 2.2. *Under the conformal transformation (2.1):*

(a) $\overline{\ell}_i = e^{\sigma} \ell_i$;
(b) $\overline{g}_{ij} = e^{2\sigma} g_{ij}$;
(c) $\overline{g}^{ij} = e^{-2\sigma} g^{ij}$,

where $\ell_i := \dot{\partial}_i F$ and g^{ij} is the inverse metric tensor.

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The Cartan tensor C_{ijk} is defined by $C_{ijk} := \frac{1}{2}\dot{\partial}_k g_{ij}$ and $C^h_{ij} := C_{ijk}g^{kh}$. The following lemma gives the partial differentiations of C^h_{ij} and other tensors obtained from the Cartan tensor with respect to y^h.

LEMMA 2.3. *The following identities hold:*

(a) $\dot{\partial}_h C^{ir}_j = C^{ir}_{jh} - 2C^r_{sj}C^{is}_h - 2C^i_{sj}C^{rs}_h$;

(b) $\dot{\partial}_h C^r_{sj} = C^r_{sjh} - 2C_{\ell sj}C^{r\ell}_h$;

(c) $\dot{\partial}_h C^{ir}_{jk} = C^{ir}_{jkh} - 2C^r_{sjk}C^{is}_h - 2C^i_{sjk}C^{rs}_h$;

(d) $\dot{\partial}_h C^r_{ijk} = C^r_{ijkh} - 2C^{rs}_h C_{ijkh}$,

where $C_{ijkh} = \dot{\partial}_h C_{ijk}$, $C_{\ell ijkh} = \dot{\partial}_h C_{\ell ijk}$, $C^r_{ijk} = C_{\ell ijk}g^{\ell r}$ *and so on.*

The following lemma obtained from [7] shows the transformations of the geodesic coefficients G^i and the connections G^i_j and G^i_{jk}.

LEMMA 2.4. *Under the conformal transformation (2.1):*

(a) $\overline{G}^i = G^i + B^i$;

(b) $\overline{G}^i_j = G^i_j + B^i_j$;

(c) $\overline{G}^i_{jk} = G^i_{jk} + B^i_{jk}$,

where the G^i *are the geodesic coefficients in (1.1) and*

$$\begin{aligned} B^i &= \sigma_0 y^i - \tfrac{1}{2}L^2\sigma^i, \quad \sigma_0 := \sigma_i y^i, \\ B^i_j &= \sigma_j y^i + \sigma_0\delta^i_j - F\sigma^i\ell_j + F^2\sigma_r C^{ir}_j, \\ B^i_{jk} &= \sigma_j\delta^i_k + \sigma_k\delta^i_j - \sigma^i g_{jk} + 2F\sigma_r C^{ir}_k\ell_j \\ &\quad + 2F\sigma_r C^{ir}_j\ell_k + F^2\sigma_r(C^{ir}_{jk} - 2C^r_{sj}C^{is}_k - 2C^i_{sj}C^{rs}_k). \end{aligned}$$

Hashiguchi [7] showed that a Landsberg space remains Landsberg under any conformal transformation if and only if the T-tensor vanishes identically. Following [11], the T-tensor is defined by

$$T_{rijk} = FC_{rijk} - F(C_{sij}C^s_{rk} + C_{sjr}C^s_{ik} + C_{sir}C^s_{jk}) + C_{rij}\ell_k + C_{rik}\ell_j + C_{rjk}\ell_i + C_{ijk}\ell_r. \tag{2.2}$$

The T-tensor is totally symmetric in all of its indices. Let us write

$$T_{ij} := T_{ijhk}g^{hk}, \quad T := T_{ij}g^{ij}.$$

LEMMA 2.5. *The tensor* C^{ir}_{jkh} *can be rewritten in the following form:*

$$\begin{aligned} C^{ir}_{jkh} &= \frac{1}{F}\dot{\partial}_h T^{ri}_{jk} + \frac{2}{F}(T^r_{sjk}C^{is}_h + T^i_{sjk}C^{sr}_h) - \frac{1}{F}(C^{ri}_j\ell_{kh} + C^{ri}_k\ell_{jh} + C^r_{jk}\ell^i_h + C^i_{jk}\ell^r_h) \\ &\quad + \frac{1}{F}(C^i_{sj}C^r_{\ell k} + C^r_{sj}C^i_{\ell k} + C^{ri}_s C_{\ell jk})\ell_h - 2(C^i_{sj}C^r_{\ell k}C^{s\ell}_h + C^r_{sj}C^i_{\ell k}C^{s\ell}_h + C^{ri}_s C_{\ell jk}C^{s\ell}_h) \\ &\quad + C^i_{sjh}C^{sr}_k + C^i_{sj}C^{sr}_{kh} + C^r_{sjh}C^{si}_k + C^r_{sj}C^{si}_{kh} + C^{ir}_{sh}C^s_{jk} + C^{ri}_s C_{jkh} \\ &\quad - \frac{1}{F}(C^{ir}_{jh}\ell_k + C^{ir}_{kh}\ell_j + C^{ir}_{jk}\ell_h + C^r_{jkh}\ell^i + C^i_{jkh}\ell^r). \end{aligned}$$

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Proof. Differentiating (2.2) with respect to y^h, raising the indices i and r and then using Lemma 2.3 gives the required formula for C^{ir}_{jkh}. $\square$

For a Berwald manifold (M, F), all tangent spaces T_xM with the induced Minkowski norm F_x are linearly isometric. There are many characterisations of Berwald spaces. We give one of them in terms of the Berwald tensor G^i_{jkh} defined in (1.2).

Definition 2.6. A Finsler manifold (M, F) is said to be Berwaldian if the Berwald tensor G^i_{jkh} vanishes identically.

It is known that on a Landsberg manifold M, all tangent spaces T_xM with the induced Riemannian metric $\mathbf{g}_x = g_{ij}(x, y)\, dy^i \otimes dy^j$ are isometric. We give a characterisation of a Landsberg space in terms of the Landsberg tensor L_{jkh} defined in (1.2).

Definition 2.7. A Finsler manifold (M, F) is said to be Landsberg if the Landsberg tensor L_{ijk} vanishes identically.

Straightforward but long calculations using Lemmas 2.3, 2.4 and 2.5 give the following proposition.

Proposition 2.8. *Under the conformal transformation (2.1), the Berwald tensor (defined in (1.2)) transforms by*

$$\overline{G}^i_{jkh} = G^i_{jkh} + B^i_{jkh},$$

where

$$\begin{aligned} B^i_{jkh} = {} & F^2\sigma_r C^{ir}_{jkh} + 2\sigma_r(C^{ir}_j g_{kh} + C^{ir}_h g_{jk} + C^{ir}_k g_{jh}) + 2F\sigma_r(C^{ir}_{jk}\ell_h + C^{ir}_{hj}\ell_k + C^{ir}_{kh}\ell_j) \\ & - 4F\sigma_r((C^r_{sj}C^{si}_k + C^i_{sj}C^{sr}_k)\ell_h + (C^r_{sj}C^{si}_h + C^i_{sj}C^{sr}_h)\ell_k + (C^r_{sk}C^{si}_h + C^i_{sk}C^{sr}_h)\ell_j) \\ & - 2F^2\sigma_r((C^r_{sjk}C^{si}_h + C^i_{sjk}C^{sr}_h) + (C^r_{sjh}C^{si}_k + C^i_{sjh}C^{sr}_k) + (C^r_{skh}C^{si}_j + C^i_{skh}C^{sr}_j)) \\ & + 4F^2\sigma_r((C_{stj}C^{ti}_k + C_{stk}C^{ti}_j)C^{sr}_h + (C_{stj}C^{it}_h + C_{sth}C^{it}_j)C^{sr}_k \\ & + (C_{stk}C^{it}_h + C_{sth}C^{it}_k)C^{sr}_j). \end{aligned}$$

In terms of the T-tensor, B^i_{jkh} is given by

$$\begin{aligned} B^i_{jkh} = {} & F\sigma_r\dot{\partial}_h T^{ri}_{jk} - \sigma_r(T^{ri}_{jh}\ell_k + T^{ri}_{kh}\ell_j - T^{ri}_{jk}\ell_h - T^r_{jkh}\ell^i - T^i_{jkh}\ell^r) \\ & - F\sigma_r(T^i_{sjh}C^{sr}_k + T^r_{skh}C^{si}_j + T^r_{sjh}C^{si}_k + T^i_{skh}C^{sr}_j - T^{ri}_{sh}C^s_{jk} - T^s_{jkh}C^{ri}_s) \\ & + \sigma_r(C^{ri}_j h_{kh} + C^{ri}_k h_{jh} + 2C^{ir}_h h_{jk} - C^r_{jk}h^i_h - C^i_{jk}h^r_h - 2C_{jkh}h^{ir}) \\ & + F^2\sigma_r[C^t_{hj}S_t{}^{ir}{}_k + C^t_{hk}S_t{}^{ri}{}_j - C^{ti}_h S_{tjk}{}^r - C^{tr}_h S_{tkj}{}^i - C^{ti}_j S_{thk}{}^r - C^{tr}_k S_{thj}{}^i], \end{aligned} \tag{2.3}$$

where $S_i{}^h{}_{jk} = C^r_{ik}C^h_{rj} - C^r_{ij}C^h_{rk}$ is the v-curvature of the Cartan connection.

Corollary 2.9. *Under the conformal transformation (2.1), the Landsberg tensor (defined in (1.2)) transforms by*

$$\overline{L}_{jkh} = e^{\sigma}L_{jkh} + e^{2\sigma}F\sigma_r T^r_{jkh}.$$

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3. Necessary condition

Making use of Corollary 2.9, one can see easily that under the conformal transformation (2.1), a Landsberg space remains Landsberg if and only if

$$\sigma_r T^r_{jkh} = 0.$$

Hashiguchi [7] showed that a Landsberg space remains Landsberg under every conformal transformation if and only if the T-tensor vanishes identically. However, there are Landsberg spaces (with nonvanishing T-tensor) which remain Landsberg under some (but not all) conformal transformations. This is shown by the following example.

Example 3.1. Let $M = \mathbb{R}^3$. Let $\overline{F} = \sigma(x^2)F$ with F defined by

$$F(x,y) = \Big(\Big(y^1y^3 + y^3\sqrt{(y^1)^2 + (y^3)^2}\Big)(y^2)^2\Big)^{1/4}.$$

For the space (M, F), we have $\sigma_r T^r_{ijk} = \sigma_2 T^2_{ijk} = 0$ but, generally, $T^h_{ijk} \neq 0$. For example, $T^1_{111} \neq 0$.

When a Landsberg space remains Landsberg under every conformal transformation, by a result of Szabo [14], the space is Riemannian. So, Hashiguchi's result on the vanishing of the T-tensor gives no hope of finding a regular Landsberg space by means of conformal transformations. However, if only some conformal transformations of a Landsberg space preserve the Landsberg property, it may be possible to find a conformal transformation which produces a regular Landsberg space which is not Berwaldian. This prompts the following question.

Question 3.2. Is there a regular Berwald space admitting a function $\sigma(x)$ such that $\sigma_r T^r_{ijk} = 0$?

Matsumoto [12, Corollary 4.1.2.1, page 786] showed that a Berwald space (M, F) remains Berwaldian under every conformal transformation if and only if the tensor $B^{ir}_{jkh} = \dot\partial_j\dot\partial_k\dot\partial_h(F^2g^{ir}) = 0$ (that is, the F^2g^{ir} are quadratic in y^i). Asanov and Kirnasov [2] calculated the tensor B^{ir}_{jkh}, which is related to the tensor B^i_{jkh} by

$$B^i_{jkh} = -\tfrac{1}{2}B^{ir}_{jkh}\sigma_r.$$

On the other hand, there are Berwald spaces (with nonvanishing B^{ir}_{jkh}) which remain Berwaldian under some (but not all) conformal transformations. This is shown by the following example.

Example 3.3. Let $M = \mathbb{R}^4$. Let $\overline{F} = \sigma(x^1, x^3)F$ with F defined by

$$F(x,y) = \Big(\sqrt{y^1y^2y^3y^4}(y^2 + y^4)\Big)^{1/4}.$$

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For the space (M, F), we have $\sigma_r T^r_{ijk} = \sigma_1 T^1_{ijk} = \sigma_3 T^3_{ijk} = 0$ but, generally, $T^h_{ijk} \neq 0$; for example,

$$T^4_{444} = \frac{3y^1(y^2)^3 y^3 y^4 (3y^2 + y^4)}{F^3\sqrt{y^1 y^2}(3(y^2)^2 + 2y^2y^4 + 2(y^4)^2)^2}.$$

Also, $B^i_{jkh} = 0$, $\sigma_r B^{ir}_{jkh} = \sigma_1 B^{i1}_{jkh} = \sigma_3 B^{i3}_{jkh} = 0$, but, generally, $B^{ir}_{jkh} \neq 0$; for example,

$$B^{44}_{444} = \frac{768y^2(y^4)^3((y^2)^3 + (y^2)^2y^4 - 3y^2(y^4)^2 - 2(y^4)^3)}{(3(y^2)^2 + 2y^2y^4 + 2(y^4)^2)^4}.$$

REMARK 3.4. Most of the calculations in the examples are done by using the Maple program and the Finsler package [15]. For simplicity, many of the examples we give are not necessarily regular Finsler spaces but they are at least non-Riemannian.

Start with a Berwald space (M, F) admitting a nonconstant function $\sigma(x)$ such that $\sigma_r T^r_{ijk} = 0$. Under the conformal transformation $\overline{F} = e^{\sigma(x)}F$, the space $(M, \overline{F})$ is Landsberg. But in order to be Berwaldian it has to satisfy some necessary conditions. We try next to determine what kind of conditions the space should satisfy.

THEOREM 3.5. *Let (M, F) be a Berwald space admitting a nonconstant function $\sigma(x)$ such that $\sigma_r T^r_{ijk} = 0$. Under the conformal transformation (2.1), a necessary condition for the Landsberg space $(M, \overline{F})$ to be Berwaldian is*

$$((n-2)C^r + F^2 C^u S_u{}^r - FT_{uv}C^{uvr} - T\ell^r)\sigma_r = 0,$$

where $S_{ik} := S_{ijk\ell}g^{j\ell} = S_{ji\ell k}g^{j\ell}$ is the Ricci tensor of the vertical curvature.

PROOF. Let (M, F) be a Berwald space; then the Berwald tensor G^i_{jkh} vanishes and, so, by (1.2), the Landsberg tensor L_{jkh} is zero. Contracting (2.3) by g^{kh},

$$0 = -\sigma_r T^i_j \ell^r - F\sigma_r(T^i_{sjh}C^{srh} + T^i_s C^{sr}_j - T^s_j C^{ri}_s) + n\sigma_r C^{ir}_j - 2\sigma_r C_j h^{ir} + F^2\sigma_r[C^{tu}_j S_t{}^{ir}{}_u + C^t S_t{}^{ri}{}_j - C^{tiu}S_{tju}{}^r - C^{tur}S_{tuj}{}^i - C^{ti}_j S_t{}^r - C^{tur}S_{tuj}{}^i].$$

Contracting this equation by g^{ij} and using the facts that $C^{hij}S_{ijk\ell} = 0$ and $S_{tui}{}^i = 0$ (which follow because S_{ijkh} is antisymmetric in the first two indices and the last two indices),

$$0 = -\sigma_r T\ell^r - F\sigma_r T_{sh}C^{srh} + \sigma_r(n-2)C^r + F^2\sigma_r C^t S_t{}^r.$$

This completes the proof. □

DEFINITION 3.6. A Finsler space (M, F) is said to be S_3-like if the vertical curvature of the Cartan connection can be written in the form

$$S_{ijkh} = \rho(h_{ik}h_{jh} - h_{ih}h_{jk}), \tag{3.1}$$

where $\rho := S/(n-1)(n-2)$ and $S = S_{ijkh}g^{jh}g^{ik}$ is the vertical scalar curvature.

From (3.1) and Theorem 3.5, we have the following corollary.

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Corollary 3.7. *Let (M, F) be an S_3-like Berwald space admitting a nonconstant function $\sigma(x)$ such that $\sigma_r T^r_{ijk} = 0$. Under the conformal transformation (2.1), a necessary condition for the Landsberg space $(M, \overline{F})$ to be Berwaldian is*

$$((n-2)(1+F^2\rho)C^r - FT_{sh}C^{srh} - T\ell^r)\sigma_r = 0. \tag{3.2}$$

The following result is a direct consequence of Theorem 3.5.

Theorem 3.8. *Let (M, F) be a Berwald space with vanishing T-tensor and vanishing v-curvature. If the Landsberg space $(M, \overline{F})$ is Berwaldian, then either $n = 2$ or $\sigma_r C^r = 0$.*

4. The condition $\sigma_r T^r_{ijk} = 0$

In the previous section, we showed that a Berwald space transforms to a Landsberg space if the T-tensor vanishes or $\sigma_r T^r_{jkh} = 0$. We now focus on the latter condition. The condition $\sigma_r T^r_{jkh} = 0$ can be satisfied in regular Finsler spaces and it is clearly weaker than the vanishing of the T-tensor.

Theorem 4.1. *A positive-definite C-reducible Finsler space does not admit a function $\sigma(x)$ such that $\sigma_r T^r_{jkh} = 0$.*

Proof. If (M, F) is C-reducible, then the T-tensor is given by

$$T_{hijk} = \frac{T}{(n^2-1)}(h_{hi}h_{jk} + h_{ij}h_{hk} + h_{jh}h_{ik}).$$

Contracting the above equation by σ^h gives

$$\frac{T}{(n^2-1)}\sigma^h(h_{hi}h_{jk} + h_{ij}h_{hk} + h_{jh}h_{ik}) = 0.$$

Again, contracting by g^{jk},

$$\sigma^h\frac{T}{(n^2-1)}((n-1)h_{hi} + 2h_{ih}) = 0.$$

Since the metric is positive definite, $T \neq 0$ and so

$$\sigma^h h_{hi} = 0,$$

which gives $\sigma_i - (\sigma_0/F)\ell_i = 0$. But, then, differentiating with respect to y^j gives $(\sigma_0/F^2)\ell_{ij} = 0$, which is a contradiction. In other words, σ_i cannot be proportional to the supporting element y^i. □

Since the metrics of Randers type are C-reducible, we have the following corollary.

Corollary 4.2. *A regular Randers space does not admit a function $\sigma(x)$ such that $\sigma_r T^r_{jkh} = 0$.*

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Proposition 4.3. *Let (M, F) be a non-Riemannian space admitting a function $\sigma(x)$ such that $\sigma_r T^r_{jkh} = 0$. If $\sigma_r C^r_{jk} = 0$, then σ is constant.*

Proof. Let (M, F) be a non-Riemannian space admitting a function $\sigma(x)$ such that $\sigma_r T^r_{jkh} = 0$ and $\sigma_r C^r_{jk} = 0$. From the second condition, σ^r is a function of x only, that is, $\dot{\partial}_j \sigma^r = -2\sigma_h C^{hr}_j = 0$. Contracting (2.2) by σ^r,

$$F^2 \sigma^r C_{rijk} + \sigma_0 C_{ijk} = 0,$$

which can be written in the form

$$F^2 \sigma^r \dot{\partial}_k C_{rij} + \sigma_0 C_{ijk} = 0.$$

Since $\sigma^r(x)$ is a function of x only, $\sigma_0 C_{ijk} = 0$. Since the space is non-Riemannian, $\sigma_0 \neq 0$, which yields $C_{ijk} = 0$ and hence $\dot{\partial}_j \sigma_0 = \sigma_j = 0$. Consequently, σ is constant. □

Definition 4.4. A Finsler space (M, F) of dimension $n \geq 2$ is said to be C_2-like if the Cartan tensor C_{ijk} satisfies

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k, \tag{4.1}$$

where $C_k := C_{ijk} g^{ij}$ and $C^2 := C_i C^i$.

The following theorem gives a condition under which a Landsberg space cannot be Berwaldian.

Theorem 4.5. *Under the conformal transformation (2.1), a C_2-like Berwald space admitting a nonconstant function $\sigma(x)$ such that*

$$\sigma_r T^r_{ijk} = 0 \quad \textit{and} \quad F^2 T_{uv} C^{uvr} \sigma_r + T\sigma_0 = 0$$

transforms to a non-Berwaldian Landsberg space.

Proof. Since $\sigma_r T^r_{ijk} = 0$, the space $(M, \overline{F})$ is Landsberg. For a C_2-like space the v-curvature vanishes and, from Theorem 3.5 and the condition $(F T_{uv} C^{uvr} + T\ell^r)\sigma_r = 0$,

$$C^r \sigma_r = 0.$$

Now, from (4.1),

$$\sigma^i C_{ijk} = 0.$$

By Proposition 4.3, the space is Riemannian or σ is constant, which is a contradiction. □

Since the vanishing of the T-tensor means that the space is not a regular Finsler space, the following corollary gives a condition under which a singular Landsberg space cannot be Berwaldian.

Corollary 4.6. *Under the conformal transformation (2.1), a C_2-like Berwald space with vanishing T-tensor transforms to a non-Berwaldian Landsberg space.*

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5. Examples

Shen [13] obtained a general formula for a class of (y-local) Landsberg spaces which are not Berwaldian. In this class, the Riemannian metric α and the 1-form β must satisfy certain conditions and it is not obvious that there are concrete examples in which α and β satisfy these conditions. Using our results, we will give simple examples of Landsberg spaces which are not Berwaldian. These examples can be seen as special cases of Shen's class [13]. The calculations are done with the Finsler package [15] in Maple.

EXAMPLE 5.1. Let $M = \mathbb{R}^3$. Let $\overline{F} = \sigma(x^3)F$, where F is defined by

$$F(x, y) = \sqrt{(y^3)^2 + y^1y^2 + y^3\sqrt{y^1y^2}}\,\exp\Big(\frac{1}{\sqrt{3}}\arctan\Big(\frac{2y^3}{\sqrt{3y^1y^2}} + \frac{1}{\sqrt{3}}\Big)\Big).$$

For (M, F), we have $\sigma_r T^r_{ijk} = \sigma_3 T^3_{ijk} = 0$, $\sigma_r C^r = \sigma_3 C^3 \neq 0$, but, generally, $T^h_{ijk} \neq 0$; for example, $T^1_{111} \neq 0$. Moreover, the space (M, F) does not satisfy the condition (3.2).

EXAMPLE 5.2. Let $M = \mathbb{R}^3$. Take $\overline{F} = \sigma(x^2)F$, where F is defined by

$$F(x, y) = \sigma(x^2)\sqrt{(y^1)^2 + (y^2)^2 + (y^2)^3 + y^2\sqrt{(y^1)^2 + (y^3)^2}}$$
$$\times \exp\Big(\frac{1}{\sqrt{3}}\arctan\Big(\frac{2y^2}{\sqrt{3((y^1)^2 + (y^3)^2)}} + \frac{1}{\sqrt{3}}\Big)\Big).$$

For (M, F), we have $\sigma_r T^r_{ijk} = \sigma_2 T^2_{ijk} = 0$, $\sigma_r C^r = \sigma_2 C^2 \neq 0$, but, generally, $T^h_{ijk} \neq 0$; for example, $T^1_{111} \neq 0$. Moreover, the space (M, F) does not satisfy the condition (3.2).

The following example shows that the conformal transformation of a non-Berwald space can produce a Berwald space.

EXAMPLE 5.3. Let $M = \mathbb{R}^3$. Take $\overline{F} = e^{\sigma(x^1,x^2)}F$, where F is defined by

$$F(x, y) := \sqrt[4]{((y^1)^2 + (y^2)^2)^2 + e^{-2\sigma(x^1,x^2)}(y^3)^4}.$$

In this example, $\overline{G}^i_{jkh} = 0$, $G^i_{jkh} = -B^i_{jkh}$, $\overline{L}_{jkh} = 0$ and $L_{jkh} = -e^{\sigma(x^1,x^2)}F\sigma_r T^r_{jkh}$.

References

[1] G. S. Asanov, 'Finsleroid–Finsler spaces of positive-definite and relativistic types', *Rep. Math. Phys.* **58** (2006), 275–300.

[2] G. S. Asanov and E. G. Kirnasov, 'On Finsler spaces satisfying the T-condition', *Aequationes Math.* **24** (1982), 66–73.

[3] S. Bácsó, X. Cheng and Z. Shen, 'Curvature properties of (α, β)-metrics', *Adv. Stud. Pure Math.* **48** (2007), 73–110.

[4] D. Bao, 'On two curvature-driven problems in Riemann–Finsler geometry', *Adv. Stud. Pure Math.* **48** (2007), 19–71.

[5] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann–Finsler Geometry* (Springer, Berlin, 2000).
[6] S. S. Chern, 'Finsler geometry is just Riemannian geometry without the quadratic equation', *Notices Amer. Math. Soc.* **43**(9) (1996), 959–963.
[7] M. Hashiguchi, 'On conformal transformations of Finsler metrics', *J. Math. Kyoto Univ.* **16** (1976), 25–50.
[8] G. Landsberg, 'Über die Totalkrümmung', *Jahresber. Dtsch. Math.-Ver.* **16** (1907), 36–46.
[9] G. Landsberg, 'Über die Krümmungstheorie und Variationsrechnung', *Jahresber. Dtsch. Math.-Ver.* **16** (1907), 547–557.
[10] G. Landsberg, 'Über die Krümmung in der Variationsrechnung', *Math. Ann.* **65** (1908), 313–349.
[11] M. Matsumoto, 'V-transformations of Finsler spaces. I. Definition, infinitesimal transformations and isometries', *J. Math. Kyoto Univ.* **12** (1972), 479–512.
[12] M. Matsumoto, 'Finsler geometry in the 20th century', in: *Handbook of Finsler Geometry II* (ed. P. L. Antonelli) (Kluwer Academic, Dordrecht, 2003).
[13] Z. Shen, 'On a class of Landsberg metrics in Finsler geometry', *Canad. J. Math.* **61** (2009), 1357–1374.
[14] Z. Szabo, 'Positive definite Finsler spaces satisfying the T-condition are Riemannian', *Tensor (N.S.)* **35** (1981), 247–248.
[15] N. L. Youssef and S. G. Elgendi, 'New Finsler package', *Comput. Phys. Commun.* **185** (2014), 986–997.

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