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Asymptotic analysis of a linearized trailing edge flow

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An Oseén type linearization of the Navier-Stokes equations is made with respect to a uniform shear flow at the trailing edge of a flat plate. Asymptotic expansions are obtained to describe a symmetrical merging flow for distances from the trailing edge that are, in a certain sense, large. Expansions for three regions are found:

(i) a wake region,

(ii) an inviscid region, and

(iii) an upstream lower order boundary layer.

The results are compared with those of Hakkinen and O'Neil (Douglas Aircraft Co. Report, 1967) and Stewartson (*Proc. Roy. Soc. Ser. A* 306 (1968)). They are further related to the results of Stewartson (*Mathematika* 16 (1969)) and Messiter (*SIAM J. Appl. Math.* 18 (1970)).

1. Introduction

A problem of fundamental importance in boundary-layer theory is that of uniform incompressible flow at high Reynolds number past a finite flat plate aligned with the stream. Let L be the length of the plate, U_∞ the unperturbed mainstream velocity, and ν the kinematic viscosity. Choose axes Ox^*y^* with O at the trailing edge and Ox^* along the wake centre line. (Asterisks designate physical quantities.) The Reynolds

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number Re is given by

$$\varepsilon^8 = \mathrm{Re}^{-1} = \frac{\nu}{U_\infty L} \,. \tag{1.1}$$

The drag D on the plate, namely

$$D = 1.328\varepsilon^4 \left(\rho U_\infty^2 L\right) \,, \tag{1.2}$$

gives a good approximation to the drag on a thin aerofoil. Of considerable interest is the error in the drag arising from the trailing edge flow.

Goldstein's [2] near wake solution provides important information on the transition from the Blasius boundary-layer flow to the wake flow. Although the trailing edge flow has received considerable attention during the last twenty years, it was only in the last decade (particularly the last few years) that the essentials of the flow structure were discovered. As late as 1968, it was commonly believed that the Blasius shear flow well within the boundary layer provides the forcing flow for a small region near O in which the full Navier-Stokes equations are needed to describe the flow accurately. It was further believed that the solution for this region could be joined onto the Goldstein solution. Using such a model, we may introduce a scaling of variables, demand that the Navier-Stokes equations are invariant and ask that, as we leave the region of interest, the vorticity approach the Blasius vorticity Ω_B^*, except possibly near the downstream axis. The extent of the region is then found to be $O(\varepsilon^6 L)$.

Using this flow model, Rott and Hakkinen [6, 7] investigated merging shear flows at the trailing edge. In particular, for the case of symmetrical merging shears, they obtained (in numerical form) a wake similarity solution for distances that were large compared with a viscous length $L_\nu = \left(\nu/\Omega_B^*\right)^{1/2} = O\left(\varepsilon^6 L\right)$. This solution was extended by Hakkinen and O'Neil [3] who obtained asymptotic expansions for the flow at the periphery of this trailing edge region. Stewartson [9], using the same scale, also recognized the need for the full equations but introduced an Oseén type linearization with respect to a uniform shear. He solved the resulting approximate problem exactly using Wiener-Hopf techniques. Earlier, Imai [4] had followed the same procedure but, unlike Stewartson, did not permit a pressure gradient. Later (1968) he reworked the problem and included a pressure gradient. Each believed that his solution merged with both the Goldstein and Blasius solutions. Non-dimensionally, since the skin friction is $O(\varepsilon^4)$ over a length $O(\varepsilon^6)$, the error in the drag coefficient thus calculated is $O(\varepsilon^{10}) = O(\mathrm{Re}^{-5/4})$.

Subsequently, Stewartson [10] and Messiter [5] introduced a triple deck structure to describe the flow near the trailing edge. The need for such a region arises from a singularity along the *line* $x^* = 0$ in the Goldstein transverse velocity which implies a velocity of inflow to the wake $O\left\{\varepsilon^4 U_\infty L^{2/3}/x^{*2/3}\right\}$ and a pressure term $O\left\{\varepsilon^4 \rho U_\infty^2 L^{2/3}/x^{*2/3}\right\}$. The singularity may be handled satisfactorily by making the triple deck region $O(\varepsilon^3 L)$ in the x^*-direction. Non-dimensionally, the skin friction, still $O(\varepsilon^4)$, effective over a length $O(\varepsilon^3)$, leads to a correction $O(\varepsilon^7)$ in the drag coefficient. The three decks have scales $\varepsilon^3 L$, $\varepsilon^4 L$, $\varepsilon^5 L$ in the y^*-direction. Deep within the latter or innermost (sublayer) region, the flow on either side of the plate is a uniform shear the order of the vorticity being the same as for the Blasius vorticity, $\lambda\varepsilon^{-4}L^{-1}U_\infty$, where $\lambda = 0.33206$. The Navier-Stokes region is still $O(\varepsilon^6 L)$ and the results of the earlier investigations mentioned above are useful provided Ω_B^* is replaced by Ω^*, the limit of the sublayer shear vorticity as the Navier-Stokes region is approached from upstream. It is expected that $\Omega^* = \lambda_1 \Omega_B^*$, where $\lambda_1 > 1$ through the effect on the Blasius flow of a favourable pressure gradient upstream of O. The contribution to the drag coefficient from the Navier-Stokes region is still $O(\varepsilon^{10})$.

Recently, Talke and Berger [11] followed by Schneider and Denny [8] treated the problem numerically. However, they neglect the nature of the singularity as $x^* \to 0$. Talke and Berger used a series truncation method to determine the extent of the Navier-Stokes region. They retained the full equations and constructed a wake asymptotic expansion of the stream function in terms of parabolic coordinates. Their expansion form was governed by the Goldstein inner expansion with which it was matched. Their apparent success may be due to the fact that, near the wake centre line, the stream function expansion in the Stewartson-Messiter sublayer is formally comparable with the Goldstein inner expansion; further, the actual upstream sublayer shear is of the same order as the Blasius shear.

The aim of the present discussion is to obtain an approximate (asymptotic) solution of the approximate (Oseén-linearized) problem solved exactly by Stewartson. Sufficient progress is made to allow

(i) a quantitative comparison with asymptotic forms of Stewartson's [9] results, and

(ii) a qualitative comparison with the results of Hakkinen and O'Neil [3].

Of particular interest are the eigenfunction problems associated with the wake and inviscid flow expansions, since they throw some light on the corresponding problems overlooked by Hakkinen and O'Neil in the non-linearized problem.

2. Statement of the problem

The scaling of variables for the Navier-Stokes region is given by

(2.1)
$$\begin{cases} \psi^* = \nu\Psi = \varepsilon^8 U_\infty L\Psi , \quad p^* - p^*_\infty = \rho\nu\Omega^* P , \\ x^* = X\sqrt{\dfrac{\nu}{\Omega^*}} , \quad y^* = Y\sqrt{\dfrac{\nu}{\Omega^*}} , \quad r^* = R\sqrt{\dfrac{\nu}{\Omega^*}} , \end{cases}$$

where ψ^* is the stream function, p^* the pressure, and $r^{*2} = x^{*2} + y^{*2}$. When the correct vorticity field $\Omega^* = \lambda_1\Omega^*_B$ is used, we find that $r^* = R\varepsilon^6 L(\lambda\lambda_1)^{-1/2}$. The exact problem for this region is

(2.2)
$$\frac{\partial(\Psi,\nabla^2\Psi)}{\partial(X,Y)} = - \nabla^4\Psi ,$$

where $\nabla^2 = \dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial Y^2}$, with boundary conditions

(2.3)
$$\Psi = \Psi_Y = 0 \quad \text{at} \quad Y = 0 , \text{ for } X < 0 ,$$

(2.4)
$$\Psi = \Psi_{YY} = 0 \quad \text{at} \quad Y = 0 , \text{ for } X > 0 ,$$

(2.5)
$$\Psi \to \frac{1}{2} Y^2 \quad \text{as} \quad Y \to \infty \quad \text{or} \quad X \to -\infty .$$

When (2.2) is linearized with respect to the uniform shear, $\Psi = \frac{1}{2} Y^2$, in the manner of Oseén, we obtain

$$Y \frac{\partial}{\partial X} (\nabla^2 \Psi) = \nabla^4 \Psi . \tag{2.6}$$

The boundary conditions are still given by (2.3) to (2.5). Note that Stewartson's [9] transformation is identical with (2.1) provided the Blasius shear is replaced by the actual shear. The problem stated above is the one solved exactly by him. Since the flow is symmetrical, the solution for $Y < 0$ need not be considered.

3. Asymptotic expansions

Following Hakkinen and O'Neil's treatment of the non-linearized problem, we find an asymptotic solution of (2.3) to (2.6) for $R >> 1$, $-\pi \leq \theta \leq \pi$. For such values of R , the flow field is divided into three regions:

(i) the wake, where inertia and viscous terms are of equal importance;

(ii) an outer region where inertia effects dominate and the flow is essentially inviscid;

(iii) an upstream (lower order) boundary layer to correct for a velocity of slip over the plate as predicted by lower order terms of the inviscid outer expansion.

The corresponding expansions introduced by Hakkinen and O'Neil are

$$\Psi^{w} = X^{2/3} f_0(\eta) + f_1(\eta) + X^{-2/3} f_2(\eta) + X^{-4/3} f_3(\eta) + \dots , \tag{3.1}$$

$$\Psi^{o} = R^2 G_0(\theta) + R^{4/3} G_1(\theta) + R^{2/3} G_2(\theta) + G_3(\theta) + R^{-2/3} G_4(\theta) + \dots + (\ln R) H_3(\theta) + (R^{-2} \ln R) H_6(\theta) + \dots , \tag{3.2}$$

$$\Psi^{\mu} = X^{2/3} h_0(\zeta) + h_1(\zeta) + X^{-2/3} h_2(\zeta) + X^{-4/3} h_3(\zeta) + \dots , \tag{3.3}$$

where $\eta = Y/X^{1/3}$ with $X > 0$, and $\zeta = Y/X^{1/3}$ with $X < 0$. Then (3.1) and (3.2) were matched as $\eta \to \infty$ and $\theta \to 0$, while (3.2) and (3.3)

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were matched as $\theta \to \pi$ and $\zeta \to -\infty$. The forms of the leading terms in (3.1) and (3.3) meet the requirement that the vorticity is independent of X as $\eta \to \infty$ and $\zeta \to -\infty$, respectively. Although Hakkinen and O'Neil found no inconsistency in matching several terms of these expansions, they are not sufficiently general and must be replaced by others which contain (in their final form) arbitrary multiples of eigensolutions. It is important to consider next the eigenfunction problems for the linearized flow.

4. The eigenfunction problems

The boundary-layer equation for the wake vorticity has the parabolic form

$$Y\Omega_X = \Omega_{YY} . \tag{4.1}$$

The boundary conditions at $Y = 0$ and $Y = \infty$, namely

$$\Omega(X, 0) = 0 , \quad \Omega \to -1 \quad \text{as} \quad Y \to \infty , \tag{4.2}$$

are applied to the similarity solution, $\Omega_0 = X^{-2/3}\Psi^{\omega}_{0\eta\eta}$, for which (4.1) reduces to an ordinary differential equation. No boundary conditions are applied at $X = X_0 > 0$ and an eigenfunction problem arises for (4.1) and (4.2). A small symmetrical disturbance near $X = 0$, $Y = 0$ leads to a small perturbation $\varepsilon\tilde{\Omega}(X, \eta) = \varepsilon X^{-2/3}V(X)T(\eta)$ in the wake vorticity and this satisfies (4.1) which, on separation, yields

$$V = bX^{-k} , \quad b = \text{constant}. \tag{4.3}$$

and

$$3T'' + \eta^2 T' + (3k+2)\eta T = 0 . \tag{4.4}$$

Here k is essentially a separation constant. The only boundary conditions are

$$T(0) = 0 , \tag{4.5}$$

$$T \to 0 \quad \text{exponentially as} \quad \eta \to \infty . \tag{4.6}$$

The assumption of exponential decay of vorticity in (4.6) is discussed in the Appendix. After writing $H = e^t T$, where $t = \eta^3/9$, we obtain the confluent hypergeometric equation,

$$tH'' + \left(\frac{2}{3} - t\right)H' + kH = 0 . \tag{4.7}$$

Equations (4.5) and (4.6) are replaced by

$$H(0) = 0 , \tag{4.8}$$

$$e^{-t}H \to 0 \text{ exponentially as } t \to \infty . \tag{4.9}$$

The fundamental solution that vanishes at the origin is

$$H(t) = t^{1/3}\left[1 + \frac{1/3-k}{4/3}\,t + \frac{(1/3-k)(4/3-k)}{(4/3)(7/3)2!}\,t^2 + \ldots\right] . \tag{4.10}$$

Only real values of t are of interest. Asymptotically, as $t \to \infty$,

$$H(t) \sim \frac{\Gamma(4/3)e^{t}t^{-(k+2/3)}}{\Gamma(1/3-k)}\left[1 + O\left(t^{-1}\right)\right] , \tag{4.11}$$

except when $H(t)$ degenerates into a polynomial, which occurs for $k = m + \frac{1}{3}$ with $m = 0, 1, 2, \ldots$. (Apart from these cases, $e^{-t}H \to 0$ as $t \to \infty$ only if $k > -2/3$; thus $-2/3$ is a lower bound on k for solutions that vanish at infinity.) The condition (4.9) is thus seen to be satisfied for the discrete set of eigenvalues

$$k = m + 1/3 \quad (m = 0, 1, 2, \ldots) . \tag{4.12}$$

The first two eigenfunctions corresponding to $k = 1/3$ and $k = 4/3$ are respectively

$$T_0 = \eta e^{-\eta^3/9} , \tag{4.13}$$

$$T_1 = \left(\eta-\eta^4/12\right)e^{-\eta^3/9} . \tag{4.14}$$

The general eigenfunction T_m is obtained by terminating (4.10) at the appropriate term. Now let $\tilde{\Psi} = \chi(X)E(\eta)$. In the wake boundary-layer equations, $\tilde{\Omega} = -\tilde{\Psi}_{YY}$, so that $\chi_m(X) = X^{-(m+1/3)}$ and

$$E_m'' = -b_m T_m . \tag{4.15}$$

Now (4.4) and (4.5) imply that $T_m(\eta)$ is odd and this in turn forces an odd particular integral in (4.15). The coefficient of the odd complementary function η must be determined in terms of b_m so that $E'_m(\infty) = 0$ for undisturbed flow outside the wake. The even complementary function does not appear in the solution. Thus E_m can be found from T_m . In particular,

$$E_0 = b_0\left[\int_0^\eta z(z-\eta)e^{-z^3/9}dz + 3^{1/3}\Gamma(2/3)\eta\right] . \tag{4.16}$$

In the inviscid region, the basic flow satisfying $\nabla^2\Psi = 1$ for $R >> 1$ is a similarity solution $R^2G_0(\theta)$, where θ is the similarity variable. Conditions are imposed at $\theta = 0, \pi$ (with $R >> 1$) but on no other boundary lines. The elliptic nature of the problem leads to an eigenfunction problem. The eigensolutions are harmonic functions which are bounded as $R \to \infty$. They must vanish on $\theta = 0, \pi$, since R^2G_0 satisfies the matching conditions at the edges of the wake and upstream boundary layer. The eigensolutions are, in fact, $R^{-k}\sin k\theta$, where $k = 1, 2, 3, \ldots$.

5. The wake expansion and leading term

The results of Section 4 lead to the following modified form of the wake expansion (3.1):

$$\Psi^w = \sum_{n=0}^{\infty} \Psi_n^w(X, \eta) = X^{2/3}f_0(\eta) + f_1(\eta) + X^{-1/3}f_2(\eta) + X^{-2/3}f_3(\eta) + X^{-4/3}[\ln X f_{41}(\eta)+f_4(\eta)] + X^{-5/3}f_5(\eta) + \ldots . \tag{5.1}$$

The first inner eigensolution $X^{-1/3}f_2(\eta)$ forces new terms later in the expansion; $X^{-5/3}f_5$ is needed to match the first outer eigensolution $R^{-1}\sin\theta$. The logarithmic term is needed to ensure matching and is associated with the second inner eigensolution which appears in $X^{-4/3}f_4(\eta)$. The inner expansion of Hakkinen and O'Neil contains no term $O(X^{-1/3})$ while no term $O(R^{-1})$ appears in their outer expansion. This accounts for the consistent though misleading matching of their expansions.

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Even a term $O(X^{-4/3})$ in (3.1) could be matched successfully by them, the implication being that an eigensolution does not occur at this stage in their expansions for the non-linearized problem.

The equation and boundary conditions for f_0 are

(5.2) $$f_0'''' + \frac{1}{3}\eta^2 f_0''' = 0 ,$$

(5.3) $$f_0(0) = f_0''(0) = 0 ,$$

(5.4) $$f_0'' \to 1 \text{ exponentially as } \eta \to \infty .$$

We find $f_0''' = C_0 e^{-\eta^3/9}$, or alternatively

(5.5) $$f_0''' + \frac{1}{3}\eta^2 f_0'' - \frac{2}{3}\eta f_0' + \frac{2}{3} f_0 = C_0 ,$$

where C_0 is related to a pressure field $P_X = C_0 X^{-1/3}$. The odd solution of (5.2) satisfying (5.4) is

(5.6) $$f_0 = \frac{1}{2} C_0 \int_0^\eta (\eta - z)^2 e^{-z^3/9} dz + \alpha_{01}\eta ,$$

where α_{01} is a constant to be determined by matching; by (5.4) C_0 must have the non-zero value $3^{1/3}/\Gamma(1/3) = 0.5384$.

The analysis leading to (5.6) is related physically to the change in boundary condition from one of no slip upstream of 0 (associated with the uniform shear) to the symmetrical flow condition (associated with the wake boundary layer region within the shear). As a result there is an induced pressure field $P_X = C_0 X^{-1/3}$ in the wake, the scale of the physical variables being $\varepsilon^6 L \ll x^* \ll \varepsilon^3 L$, $y^* \sim \varepsilon^4 x^{*1/3}$. We are therefore describing the wake flow leaving the Navier-Stokes region and entering the sublayer of the triple deck region. The situation where the wake leaves the triple deck region (namely, $\varepsilon^3 L \ll x^* \ll 1$, $y^* \sim \varepsilon^4 x^{*1/3}$) as described by the Goldstein [2] inner solution, is very similar. A linearization with respect to the Blasius shear is formally identical with that above but leads to an intolerable pressure gradient, contradicting

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the presence of a uniform pressure in the uniform flow region outside the Goldstein wake. A correct, if somewhat artificial, linearization here is with respect to a modified uniform shear of the form
$\Psi = \frac{1}{2} Y^2 + \alpha X^{1/3} Y = X^{2/3}\left(\frac{1}{2}\eta^2 + \alpha\eta\right)$. (The scaling, of course would be different.) In principle, the constant α can be found from two relations between α and A' (the analogue of A) obtained by letting the boundary-layer flow merge into the modified shear as $\eta \to \infty$.

The asymptotic form of (5.6) as $\eta \to \infty$ is easily found using the fact that

$$\int_0^\infty z^n e^{-z^3/9} dz = 3^{(2n-1)/3}\Gamma\left(\frac{n+1}{3}\right) .$$

We find

$$f_0 \sim a_{00}\eta^2 + a_{01}\eta + a_{02} + a_{0e}O\left(e^{-\eta^3/9}\right) , \tag{5.7}$$

where

$$\begin{cases} a_{00} = \frac{1}{2} , \quad a_{01} = \alpha_{01} - 3^{1/3}\Gamma\left(\frac{2}{3}\right)C_0 = \alpha_{01} - 1.9529C_0 , \\ a_{02} = \frac{3}{2} C_0 = 0.8076 . \end{cases} \tag{5.8}$$

The dependence of a_{0e} on α_{01} is of no importance. In Section 8, matching is shown to imply $a_{01} = 0$, which is consistent with linearization with respect to an unmodified uniform shear.

6. Lower order wake terms

The equation and boundary conditions for f_1 are

$$f_1'''' + \frac{1}{3}\eta^2 f_1''' + \frac{2}{3}\eta f_1'' = 0 , \tag{6.1}$$

$$f_1(0) = f_1''(0) = 0 . \tag{6.2}$$

The solution,

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(6.3) $$f_1 = C_1 \int_0^\eta ds \int_0^s e^{-t^3/9} dt \int_0^t e^{z^3/9} dz + \alpha_{11}\eta ,$$

contains two arbitrary constants C_1 and α_{11} which can be found by matching. The asymptotic form of f_1, for $\eta >> 1$, is

(6.4) $$f_1 \sim a_{12}\eta + a_{13} + C_1\Big[-3 \ln\eta - 3a_{01}\eta^{-1} + \left(a_{01}^2 - \tfrac{1}{2}a_{02}\right)\eta^{-2} + \left(\tfrac{3}{2} + 2a_{01}a_{02} - 2a_{01}^3\right)\eta^{-3} + \ldots\Big] + O\left(\eta^{-4}e^{-\eta^3/9}\right) ,$$

where

(6.5) $$\begin{cases} a_{12} = 9^{-1/3}C_1\left[\Gamma\left(\frac{1}{3}\right)\right]^2 + \alpha_{11} , \\ a_{13} = -C_1\left[\gamma + \dfrac{\pi}{2\sqrt{3}} - \dfrac{1}{2}\ln 3 + 3\right] , \end{cases}$$

γ being Euler's constant. The equation and boundary conditions for f_2 (the first eigenfunction) are

(6.6) $$f_2'''' + \tfrac{1}{3}\eta^2 f_2''' + \eta f_2'' = 0 ,$$

(6.7) $$f_2(0) = f_2''(0) = 0 .$$

The general form of the odd solution is

(6.8) $$f_2 = \alpha_{21}\eta + \alpha_{22}(2f_0 - \eta f_0') .$$

In preparation for matching, we note that, as $\eta \to \infty$,

(6.9) $$f_2 \sim \eta\left(\alpha_{21} + \alpha_{22}a_{01}\right) + 2a_{02} + O\left(\eta^3 e^{-\eta^3/9}\right) .$$

The precise form of exponentially decaying terms is not required.

7. The upstream inner expansion

Although there is no eigenfunction problem for the upstream boundary layer, (3.3) must be modified to permit matching with new outer expansion terms containing outer eigensolutions as well as others forced by inner and outer eigensolutions. The first additional term is $X^{-1}h_3(\zeta)$ rather than $X^{-1/3}h_2(\zeta)$ as might perhaps have been expected:

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$$(7.1)\quad \Psi^{\mu} = \sum_{n=0}^{\infty} \Psi_n^{\mu}(X, \zeta) = X^{2/3}h_0(\zeta) + h_1(\zeta) + X^{-2/3}h_2(\zeta) + X^{-1}h_3(\zeta) + \dots .$$

Substitution in (2.6) yields equations for h_0, h_1 which are also obtainable from those for f_0, f_1 by replacing η by ζ and f_0, f_1 by h_0, h_1. Of course, the boundary conditions are different:

$$(7.2)\quad h_i(0) = h_i'(0) = 0 \ , \quad i = 1, 2 .$$

Moreover, since $X < 0$, $Y > 0$ and $\zeta = Y/X^{1/3}$, we have $-\infty < \zeta < 0$ and it is the asymptotic behaviour of h_i as $\zeta \to -\infty$ that must be compared with the inviscid expansion as $\theta \to \pi$. We find, after using (7.2),

$$(7.3)\quad h_0 = \frac{1}{2} D_0 \int_0^{\zeta} (\zeta - z)^2 e^{-z^3/9} dz + \frac{1}{2} \beta_{00}\zeta^2 .$$

The exponentially large term cannot be tolerated so that $D_0 = 0$, while $\beta_{00} = 1$ to satisfy (2.5). Then

$$(7.4)\quad h_0 = \frac{1}{2}\zeta^2 .$$

It is easy to show that

$$(7.5)\quad h_1'' = D_1 e^{-\zeta^3/9} \int_{-\infty}^{\zeta} e^{z^3/9} dz + \beta_{10} e^{-\zeta^3/9} .$$

Again $\beta_{10} = 0$ to avoid exponential growth of the solution as $\zeta \to -\infty$. The solution satisfying (7.2) is

$$(7.6)\quad h_1 = D_1 \int_{\zeta}^{0} \int_{s}^{0} e^{-t^3/9} \int_{-\infty}^{t} e^{w^3/9} dw dt ds .$$

This contains just one arbitrary constant D_1 to be determined by matching; in preparation for this we note the asymptotic form of h_1 as $\zeta \to -\infty$:

$$(7.7)\quad h_1 \sim b_{12}\zeta + b_{13} + D_1\left[-3 \ln|\zeta| + \frac{3}{2}\zeta^{-3} + \dots\right] ,$$

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where $b_{12} = -9^{-1/3}D_1\left[\Gamma\left(\frac{1}{3}\right)\right]^2$, $b_{13} = -D_1\left[\gamma - \frac{\pi}{2\sqrt{3}} - \frac{1}{2}\ln 3 + 3\right]$. The remaining terms of Ψ^{u} are not discussed in detail although the correctness of (7.1) is considered in Section 9.

8. The inviscid flow expansion

The modified outer expansion is

$$(8.1)\quad \Psi^o = \sum_{n=0}^{\infty} \Psi_n^o(R, \theta)$$

$$= R^2G_0(\theta) + R^{4/3}G_1(\theta) + R^{2/3}G_2(\theta) + G_3(\theta) + R^{-1/3}G_4(\theta)$$

$$+ R^{-2/3}G_5(\theta) + R^{-1}G_6(\theta) + R^{-4/3}G_7(\theta) + R^{-2}G_8(\theta) + \ldots$$

$$+ \ln RH_3(\theta) + R^{-4/3}\ln RH_7(\theta) + R^{-2}\ln RH_8(\theta) + \ldots .$$

The term $R^{-1/3}G_4(\theta)$ is needed to match the first wake eigensolution, no earlier modifications being necessary. The need for the various terms in (8.1) becomes apparent as the matching proceeds. The polar coordinate form of (2.6) is

$$(8.2)\quad R^2\left[-2\sin\theta\cos\theta\Psi_{\theta\theta}-\sin^2\theta\Psi_{\theta\theta\theta}\right] + R^3\left[-\sin\theta\cos\theta\Psi_R+\sin\theta\cos\theta\Psi_{R\theta\theta}-\sin^2\theta\Psi_{R\theta}\right]$$

$$+ R^4\left[\sin\theta\cos\theta\Psi_{RR}-\sin^2\theta\Psi_{RR\theta}\right] + R^5\left[\sin\theta\cos\theta\Psi_{RRR}\right]$$

$$= \left[\Psi_{\theta\theta\theta\theta}+4\Psi_{\theta\theta}\right] + R\left[\Psi_R-2\Psi_{R\theta\theta}\right] + R^2\left[2\Psi_{RR\theta\theta}-\Psi_{RR}\right] + R^3\left[2\Psi_{RRR}+R\Psi_{RRRR}\right] .$$

Substitution of (8.1) in (8.2) yields firstly the equation for G_0 :

$$(8.3)\quad G_0''' + 4G_0' = 0 ,$$

the general solution of which is expressible most conveniently as

$$(8.4)\quad G_0 = A_{00}\sin^2\theta + A_{01}\cos 2\theta + A_{02}\sin 2\theta .$$

It is convenient to determine progressively the various constants that appear in Ψ^o . At the same time, constants in Ψ^{w} and Ψ^{u} become known. Matching is achieved by comparing Ψ^o with Ψ^{u} as $\theta \to \pi_-$, $\zeta \to -\infty$ and with Ψ^{w} as $\theta \to 0_+$, $\eta \to \infty$. The procedure is initiated upstream, where the initial flow has already led to the value of β_{00} in (7.3). Now

$$\Psi_0^o = R^2 G_0 = A_{01}X^2 + 2A_{02}\zeta X^{4/3} + (A_{00}-A_{01})\zeta^2 X^{2/3}$$

for all θ, and in particular as $\theta \to \pi$. Comparison with

$$\Psi_0^\mu = X^{2/3} h_0(\zeta) = \frac{1}{2}\zeta^2 X^{2/3}$$

shows that $A_{01} = 0 = A_{02}$ and $A_{00} = \frac{1}{2}$. Then

$$\Psi_0^o = \frac{1}{2} Y^2 = \frac{1}{2}\eta^2 X^{2/3} . \tag{8.5}$$

As $\theta \to 0$, a match with the leading term of Ψ_0^ω as $\eta \to \infty$ is prearranged in (5.4). The basic inviscid flow is simply the uniform shear, and the upstream boundary layer is a lower order effect arising, as we shall see, from a velocity of slip associated with the term $R^{2/3}G_2$. The term $R^{4/3}G_1$ is superfluous, its inclusion having been prompted by the possible need to match a term in ζ in h_0 as $\zeta \to -\infty$. Consequently, there is no term $\eta X^{2/3}$ in Ψ_0^o as $\theta \to 0$ with which to match such a term in Ψ_0^ω as $\eta \to \infty$. It follows from (5.7) that $a_{01} = 0$ so that

$$\alpha_{01} = 3^{1/3}\Gamma\left(\frac{2}{3}\right)C_0 = 1.0514 . \tag{8.6}$$

The corresponding coefficient of η in the non-linearized case is also zero: the resulting non-zero pressure gradient, $P_X = C_0 X^{-1/3}$, in both cases is attributed to the change in the boundary condition.

Returning to the linearized problem, note that f_0 is now known completely. The equation for G_2,

$$\sin\theta G_2''' + \frac{2}{3}\cos\theta G_2'' + \frac{4}{9}\sin\theta G_2' + \frac{16}{27}\cos\theta G_2 = 0 , \tag{8.7}$$

has the general solution

$$G_2 = A_{20}\sin^{2/3}\theta + A_{21}\cos 2\theta/3 + A_{22}\sin 2\theta/3 . \tag{8.8}$$

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A non-trivial solution is required here for matching with Ψ^{w} . For merging with the uniform shear upstream, $A_{20} = 0$. The remaining terms lead to a velocity of slip on the plate necessitating the introduction of the upstream boundary layer as anticipated. The expansion of Ψ_2^{o} $\left(= R^{2/3} G_2\right)$ about $\theta = \pi$ may be written in terms of ζ :

$$(8.9)\quad \Psi_2^{o} = \left(-\frac{1}{2} A_{21} + \frac{\sqrt{3}}{2} A_{22}\right) X^{2/3} + A_{20} \zeta^{2/3} X^{2/9} - \left(\frac{1}{\sqrt{3}} A_{21} + \frac{1}{3} A_{22}\right) \zeta + \dots .$$

The first term of (8.9) when matched with Ψ_0^{u} yields

$$(8.10)\quad -A_{21} + \sqrt{3}\, A_{22} = 0 .$$

No term in $X^{2/9}$ appears in Ψ^{u} (or Ψ^{w}) so that $A_{20} = 0$. If such terms are included in these expansions they are found to be zero. For the expansion of Ψ_2^{o} about $\theta = 0$, we then obtain

$$(8.11)\quad \Psi_2^{o} = A_{21}\left(X^{2/3} - \frac{2}{9}\eta^2 X^{-2/3} + \dots\right) + A_{22}\left(\frac{2}{3}\eta - \dots\right) .$$

The coefficient of $X^{2/3}$ in (8.11) is now compared with the constant term in the asymptotic form of Ψ_0^{w} $\left(= X^{2/3} f_0\right)$ for $\eta \gg 1$ - see (5.7), (5.8). We find $A_{21} = \frac{3}{2} C_0 = 0.8076$. Then (8.10) implies $A_{22} = \frac{\sqrt{3}}{2} C_0 = 0.4663$. Thus G_2 is known completely:

$$(8.12)\quad G_2 = A_{21}\left(\cos\frac{2\theta}{3} + \frac{1}{\sqrt{3}}\sin\frac{2\theta}{3}\right) = 0.8076 \cos\frac{2\theta}{3} + 0.4663 \sin\frac{2\theta}{3} .$$

Next compare the coefficient of X^0 in (8.9) with (7.7):

$-9^{-1/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^2 D_1 = b_{12} = -\left(\frac{1}{\sqrt{3}} A_{21} + \frac{1}{3} A_{22}\right) = -\frac{2}{\sqrt{3}} C_0$. Hence

$$(8.13)\quad D_1 = 0.1802 .$$

Thus Ψ_1^{u} $(= h_1)$ is known completely. The term in X^0 in (8.11) matches with the linear term in (6.4), provided

$$a_{12} = 9^{-1/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^2 C_1 + \alpha_{11} = \frac{2}{3}A_{22} = 0.3109 \ . \tag{8.14}$$

Only one degree of freedom remains in f_1 ; this can be removed only after finding new terms in Ψ^o . Now the right side of the equation for G_3 contains G_0 but in such a combination that

$$\sin\theta G_3''' + 2\cos\theta G_3'' = 0 \ , \tag{8.15}$$

for which the general solution is

$$G_3 = A_{30}\ln|\sin\theta| + A_{31} + A_{32}\theta \ . \tag{8.16}$$

The coefficient of $\ln R$ when (8.1) is substituted in (8.2) leads to the equation for H_3 :

$$H_3'' = 0 \ .$$

$$H_3 = B_{31} + B_{32}\theta \ . \tag{8.17}$$

The term $\ln RH_3$ is included in (8.1) through the need to match the logarithmic term in (6.4) and (7.7). Since logarithmic terms occur in both Ψ_3^o $(= G_3)$ and $\ln RH_3$ $(= \Psi_{3L}^o$, say$)$, the matching of these two terms with Ψ^u and Ψ^w is considered together. As $\theta \to \pi$,

$$\Psi_3^o \to A_{30}\ln|\zeta| - \frac{2}{3}A_{30}\ln|X| + (A_{31}+A_{32}\pi) + \ldots \ , \tag{8.18}$$

$$\Psi_{3L}^o \to (B_{31}+B_{32}\pi)\ln|X| + \ldots \ . \tag{8.19}$$

Comparison of (8.18), (8.19) with (7.7) shows that

(i) $A_{30} = -3D_1$,

(ii) $-\frac{2}{3}A_{30} + B_{31} + B_{32}\pi = 0$,

(iii) $A_{31} + A_{32}\pi = b_{13}$.

Similarly, matching Ψ_3^o and Ψ_{3L}^o as $\theta \to 0$ with (6.4) shows that

(iv) $A_{30} = -3C_1$,

(v) $A_{31} = a_{13}$,

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(vi) $-\frac{2}{3}A_{30} + B_{31} = 0$.

When these six equations are solved for the unknowns A_{30}, A_{31}, A_{32}, B_{31}, B_{32}, C_1 , we obtain $A_{30} = -0.5406$, $A_{31} = -0.7090$, $A_{32} = 0.1040$, $B_{31} = -0.3604$, $B_{32} = 0$, $C_1 = 0.1802 = D_1$. From (8.13) and (8.14), it follows that $\alpha_{11} = -\frac{1}{\sqrt{3}}C_0 = -0.311$. Then f_0, f_1, h_0, h_1, G_0, G_1, G_2, G_3, H_3 are known completely. From the discussion in Section 4, the term $R^{-1}G_6(\theta)$ is deemed known also apart from an arbitrary constant that remains after completion of the matching. The term $R^{-1/3}G_4(\theta)$ is discussed in the next section. The term $R^{-2/3}G_5(\theta)$ is not considered at all.

9. Eigensolutions and the matching procedure

Matching f_2 in (6.9) with outer terms shows that $\alpha_{21} = 0$ and $f_2 = \alpha_{22}(2f_0 - \eta f_0')$, which is easily identified as E_0 . (α_{22} persists as an arbitrary constant.) For the express purpose of matching the eigensolution $X^{-1/3}f_2$, the term $R^{-1/3}G_4(\theta)$ has been included in Ψ^o , the general solution for G_4 being

(9.1) $$G_4 = A_{40}\sin^{-1/3}\theta + A_{41}\cos\frac{\theta}{3} + A_{42}\sin\frac{\theta}{3} .$$

Matching with Ψ^{ω} and Ψ^{u} shows that

(9.2) $$G_4 = \frac{1}{2}\alpha_{22}C_0\left(3\cos\frac{\theta}{3} - \sqrt{3}\sin\frac{\theta}{3}\right) .$$

No term in $X^{-1/3}$ appears in Ψ^{u} ; (9.2) must be matched with a term $X^{-1}h_3(\zeta) = \beta X^{-1}\zeta h_1'(\zeta)$, where β depends on α_{22} . The terms Ψ_0^o, Ψ_0^{u} are independent of X and the terms $X^{-1/3}f_2(\eta)$, $R^{-1/3}G_4(\theta)$, $X^{-1}h_3(\zeta)$ are respectively proportional to the X-derivatives of $X^{2/3}f_0(\eta)$, $R^{2/3}G_2(\theta)$, $h_1(\zeta)$: thus, in the usual way, the first eigensolution is related to a shift of origin along OX . The first outer eigensolution leads to modifications in Ψ^{ω} and Ψ^{u} but these are not described here.

10. Discussion

Apart from the eigensolutions, the expansions found above closely resemble those of Hakkinen and O'Neil at least qualitatively. The pattern of matching described in Section 8 is identical with theirs: in particular, the asymptotic behaviour of wake and boundary-layer terms, for $\eta \gg 1$ and $|\zeta| \gg 1$ respectively, is qualitatively the same in the linearized problem as in the non-linearized one. The quantitative agreement cannot be expected to be very good since the Oseén linearization is rather crude within the wake. For example $C_0 = 0.5384$ in the linearized problem while the value in the non-linearized case is 0.4089.

The neglect of eigensolutions by Hakkinen and O'Neil accounts for the important difference in the expansion forms; it is clear from our results, if not from other considerations, that their expansions are not sufficiently general. Moreover, if we take the first inner eigensolution (for the non-linearized problem) to be related to an origin shift, the remarks in Section 9 suggest that the early modifications of their expansions are easy to incorporate. The eigenvalue problems for the non-linearized case and the nature of later modifications are discussed elsewhere.

The results of this paper are also in close agreement both qualitatively and quantitatively (as they should be) with those of Stewartson [9]. From the results above, the velocity on the wake centre line is found to be

$$(10.1)\quad \left.\frac{\partial \Psi^{\omega}}{\partial Y}\right|_{Y=0} = 1.0514X^{1/3} - 0.311X^{-1/3} + 1.0514\alpha_{22}X^{-2/3} + O\left(X^{-1}\right) .$$

The first two numerical coefficients agree with those of Stewartson. In effect, the value of α_{22} can be determined from his coefficient of $X^{-2/3}$. Stewartson actually finds the coefficient of X^{-1} to be zero. For the skin friction on the plate, the results of earlier sections give

$$(10.2) \qquad \left.\frac{\partial^2\Psi^{\mu}}{\partial Y^2}\right|_{Y=0} = 1 + X^{-2/3}h_1''(0) + O\left(X^{-5/3}\right)$$

$$= 1 + 0.3346X^{-2/3} + \ldots .$$

The numerical coefficient again agrees with Stewartson's. Furthermore the terms in (10.2) after the second agree qualitatively with his, the error term $O\left(X^{-5/3}\right)$ corresponding to the modifying term $X^{-1}h_3$ in Ψ^{μ}. Furthermore, Stewartson's pressure gradient in the wake,

$$(10.3) \qquad \frac{\partial P}{\partial X} \sim \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)\mathrm{Ai}'(0)}{2\pi\mathrm{Ai}(0)} X^{-1/3} = 0.5384X^{-1/3} ,$$

agrees with our result.

Thus the expansions are in qualitative and quantitative agreement with Stewartson's results. As we have seen they differ qualitatively from the expansions of Hakkinen and O'Neil through the inclusion of inner and outer eigensolutions.

APPENDIX

Limiting behaviour of the wake vorticity

The boundary-layer approximation of (2.6) for the wake region is

$$(A.1) \qquad YW_X = W_{YY} ,$$

where $W = 1 - \Psi_{YY} \sim 1 - \nabla^2\Psi$. The boundary conditions are

$$(A.2) \qquad W = 1 \quad \text{at} \quad Y = 0 \quad \text{for} \quad X > 0 ,$$

$$(A.3) \qquad W \to 0 \quad \text{as} \quad Y \to \infty \quad \text{for} \quad X \to -\infty .$$

Consider the behaviour of W as $Y \to \infty$ for finite values of X. We make the rather weak assumption that, in the 'similarity region' where $X \to \infty$, $W = o\left(Y^{-1}\right)$ for sufficiently large values of Y to ensure that $\eta \to \infty$. Let the Fourier transform of $W(X, Y)$ be

$$(A.4) \qquad \overline{W}(S, Y) = \int_{-\infty}^{\infty} e^{-iSX}W(X, Y)dX .$$

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Then from (A.1) and (A.3) it follows that

$$\overline{W}(S,\ Y) = B(S)\mathrm{Ai}\{(iS)^{1/3}Y\}\ , \tag{A.5}$$

where $B(S)$ is a function of S only, $\mathrm{Ai}(z)$ is the Airy function, and $(iS)^{1/3}$ is defined so that, when S is real, $\overline{W} \to 0$ as $Y \to \infty$: $(iS)^{1/3} = S^{1/3}e^{\pi i/6}$ for $S > 0$ and $(iS)^{1/3} = |S|^{1/3}e^{-\pi i/6}$ for $S < 0$. Now $W = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iSX}\overline{W}dS$. Using an asymptotic result given by Antosiewicz [1, p. 448] for $\mathrm{Ai}(z)$, we obtain

$$W(X,\ Y) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} B(S)\left[(iS)^{1/3}Y\right]^{-1/4}\exp\left\{iSX - \frac{2}{3}\,(iS)^{1/2}Y^{3/2}\right\}dS\ . \tag{A.6}$$

By suitably deforming the contour of integration into a new contour C passing through the point $S = -\frac{1}{9}\,iY^3/X^2$, at which $\frac{d}{dS}\left\{iSX - \frac{2}{3}\,(iS)^{1/2}Y^{3/2}\right\} = 0$, we finally obtain

$$W(X,\ Y) \sim \frac{1}{2\pi}\exp\left(-\frac{1}{9}\,Y^3/X\right)\int_C B(S)\left[(iS)^{1/3}Y\right]^{-1/4}dS\ . \tag{A.7}$$

Since the exponent in the decay factor as $Y \to \infty$ for finite X contains the similarity variable $\eta = Y/X^{1/3}$, we are led to expect that, in the similarity solution, $W \to 0$ exponentially as $\eta \to \infty$. This is consistent with the earlier assumption that $W = o\left(Y^{-1}\right)$ as $X \to \infty$.

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