Proceedings of the Edinburgh Mathematical Society (1987) **30**, 41–49 ©

ON THE OTHER $p^\alpha q^\beta$ THEOREM OF BURNSIDE

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(Received 10th August 1985)

A. Introduction

The "other" $p^\alpha q^\beta$ theorem of Burnside states the following:

Theorem A.1. *Let G be a group of order $p^\alpha q^\beta$, where p and q are distinct primes. If $p^\alpha > q^\beta$, then $O_p(G) \neq 1$ unless*

(a) *p is a Mersenne prime and $q=2$;*

(b) *$p=2$ and q is a Fermat prime; or*

(c) *$p=2$ and $q=7$.*

Burnside's proof [3] was incorrect; he omitted exception (c). However, M. Coates, M. Dwan and J. Rose gave a correct proof of Burnside's theorem, see [5]. Independently, V. S. Monakhov gave a correct proof as well, see [8] and [9]. In [12], T. R. Wolf proved the following Theorem A.2, which handles the exceptional cases of Theorem A.1 as well.

Theorem A.2. *Let G be a group of order $p^\alpha q^\beta$, where p and q are distinct primes. If $p^\alpha > q^{\beta c}/2$, where $c = (\log 32/\log 9)$, then $O_p(G) \neq 1$.*

G. Glauberman, see [6], took a different approach. For a finite group G and a positive integer k, or $k=\infty$, let $d(k,G)$ denote the maximum of the orders of all nilpotent subgroups of G of class at most k. Using this notation, Glauberman's theorem states the following:

Theorem A.3. *If G is a group of order $p^\alpha q^\beta$ and P and Q are p-Sylow and q-Sylow subgroups of G, respectively, then $d(2,P) > d(2,Q)$ implies that $O_p(G) \neq 1$.*

For groups of odd order, the author generalized Glauberman's theorem and in [1] proved:

Theorem A.4. *Let $G = HK$ be a group of odd order, where H and K are π-Hall and π'-Hall subgroups of G, respectively. Then $d(\infty, H) > d(2,K)$ implies that $O_\pi(G) \neq 1$.*

*This paper forms part of the Proceedings of the conference Groups–St Andrews 1985.

In the present paper, we continue Glauberman's approach and prove the following stronger version of Theorem A.3.

Theorem A.5 (Main Theorem). *Let G be a finite group of order $p^\alpha q^\beta$ and let P and Q be a p-Sylow subgroup and q-Sylow subgroup of G, respectively. For various primes p and q, sufficient conditions under which $O_p(G) \neq 1$ are given below:*

(a) $d(m, P) > d(2, Q)$ *for* $p=2$ *and* $q=2^m+1$, $m \geqq 2$

(b) $d(2, P) > d(2, Q)$ *for* $p=2$ *and* $q=3$

(c) $d(15, P) > d(2, Q)$ *for* $p=2$ *and* $q=7$

(d) $d(p-1, P) > d(2, Q)$ *for* $p=2^l-1$ *and* $q=2$

(e) $d(\infty, P) > d(2, Q)$ *for p and q not as above.*

The proof of the above theorem is carried out in two main steps. First, in Section B, we evaluate $d(k, S_p(GL(n,q)))$ where $(p,q)=1$ and make use of it to prove a main lemma about p-groups in $GL(n,q)$. For the structure of $S_p(GL(n,q))$ where $(p,q)=1$, the reader is referred to [**4**] and [**11**]. The exponent and the nilpotency class of these groups are used frequently and the reader is referred to [**2**]. In the second step, we follow Glauberman [**6**], and prove a theorem about a product of two nilpotent groups which can be combined with the main lemma to yield our main theorem.

B. Evaluation of $d(k, S_p(GL(n,q)))$ where $(p,q)=1$ and the main lemma

First we evaluate $d(k, S_p(GL(n,q)))$ where $(p,q)=1$ in the general case, excluding the case $p=2$ and $q=3(\text{mod } 4)$. We can assume that $p \mid q-1$, and let $s, s \geqq 1$ be such that $p^s \| q-1$. On the one hand, given a prime p, a power of prime q such that $p^s \| q-1$, and positive integers n and k, we have to find a suitable candidate $A, A \subseteq S_p(GL(n,q))$ for which $\text{class}(A) \leqq k$. On the other hand, we have to prove that $d(k, S_p(GL(n,q))) \leqq |A|$. If $1 \leqq k < (p-1)s+1$, then it is natural to define A as a direct product of n cyclic groups of order p^s each, thus $|A| = p^{sn}$ and $\text{class}(A)=1$. However, if $k \geqq (p-1)s+1$, then there exists a minimal $\alpha, \alpha \geqq 1$ such that $k < ((p-1)s+1)p^\alpha$ and the construction of A is as follows: Let $n = p^\alpha t + u$ where $0 \leqq u < p^\alpha$, then we can write the underlying vector space V as a direct sum $V = V_0 \oplus V_1 \oplus \cdots \oplus V_t$, where $\dim(V_i) = p^\alpha$ for $1 \leqq i \leqq t$ and $\dim(V_0) = u$. As $\text{class}(S_p(GL(u,q))) \leqq \text{class}(S_p(GL(p^\alpha, q))) = ((p-1)s+1)p^{\alpha-1} \leqq k$ for $\alpha \geqq 1$, we define A to be the direct product of $S_p(GL(V_i))$, $0 \leqq i \leqq t$ and it follows that $\text{class}(A) = ((p-1)s+1)p^{\alpha-1} \leqq k$ and that $|A| = |S_p(GL(p^\alpha, q))|^t |S_p(GL(u,q))|$. Now we prove:

Theorem B.1. *Let p be a prime and let q be a power of a prime such that $p^s \| q-1$ for $s \geqq 1$. Assume that $p \neq 2$ if $q = 3(\text{mod } 4)$ and let k and n be positive integers. If $k \geqq (p-1)s+1$, then define $\alpha, \alpha \geqq 1$ as the minimal integer satisfying $k < ((p-1)s+1)p^\alpha$ and let t and u be determined by $n = p^\alpha t + u$ where $0 \leqq u < p^\alpha$. Then $d(k, S_p(GL(n,q))) = f(k,n)$ where:*

$$f(k,n) = \begin{cases} p^{sn} & \text{if } k < (p-1)s+1 \\ |S_p(GL(p^\alpha, q))|^t |S_p(GL(u,q))| & \text{if } k \geqq (p-1)s+1 \end{cases}$$

Proof. By the observation which precedes the theorem, it follows that $d(k, S_p(GL(n,q))) \geqq f(k,n)$. We will prove that $d(k, S_p(GL(n,q))) \leqq f(k,n)$. Consider the following two properties of $f(k,n)$ which can be easily verified.

(a) $f(k,n_1)\cdot f(k,n_2) \leqq f(k, n_1+n_2)$, for all positive integers k, n_1 and n_2.

(b) $p\cdot(f(l,m))^{\bar{p}} \leqq f(lp, mp)$, for $l \geqq (p-1)s+1$ and every positive integer m.

Suppose that the theorem does not hold for a certain p and q, and fixing those p and q, let $P \subseteq S_p(GL(n,q))$ be a counterexample for which $n+k$ is minimal. Thus, $|P| = d(k, S_p(GL(n,q))$ and $|P| > f(n,k)$. As

$$\text{class}(S_p(GL(n,q))) = \begin{cases} 1 & \text{if } n<p \\ (p-1)s+1 & \text{if } n=p \end{cases}$$

the theorem holds for $n \leqq p$ and every positive integer k. Hence we can assume that $n>p$. If P is reducible, then it is decomposable and $V = V_1 \oplus V_2$ where the V_i's are non-trivial P-invariant subspaces of V for $i=1,2$. It follows that $P \subseteq P_1 \times P_2$, where $P_i \subseteq GL(V_i)$ is the projection of P on V_i, $i=1,2$, and hence in view of property (a) of $f(k,n)$ and minimality of $n+k$, we obtain a contradiction. Thus we can assume that P is irreducible and hence n is a power of p.

Suppose that $k=1$. Then P is abelian and since P is irreducible, it follows that P is cyclic. But this is impossible since, if P is one of the following: cyclic, dihedral, semi-dihedral or generalized quaternion, then it is not difficult to derive a contradiction for an arbitrary k. Indeed, if P is one of the above-mentioned types, and $\exp(P) = p^\beta$, then $|P| \leqq p^{\beta+1}$. Notice that, in view of the fact that n is a power of p, Proposition B.2 of **[2]** implies that $n \geqq p^{\beta-s}$ and since $n>p$, it follows that $n \geqq \max\{p^2, p^{\beta-s}\}$. It is not difficult to show that under these conditions, keeping in mind that $p=2$ and $s=1$ is not allowed, we have $p^{sn} > p^{\beta+1}$. Hence, $p^{sn} > p^{\beta+1} > |P|$. But since $S_p(GL(n,q))$ contains a direct product of n cyclic groups which is of order p^{sn}, the inequality $p^{sn} > |P|$ contradicts $|P| = d(k, S_p(GL(n,q)))$.

Thus, we can assume that P is irreducible, P does not belong to the four exceptional families, $n>p$ and $k>1$. Now Theorem 19.2 of **[10]** can be applied, yielding:

(1) P contains a subgroup H such that $|P:H| = p$.

(2) The underlying vector space V can be written as $V = V_1 \oplus \cdots \oplus V_p$, where the subspace V_i, $1 \leqq i \leqq p$ are H-invariant and if $x \in P \backslash H$, then x permutes the V_i's in a p-cycle.

Let $\dim(V_i) = m$, for $1 \leqq i \leqq p$ (thus $n = mp$) and let $H_i \subseteq GL(V_i)$ be the projection of H on V_i for $1 \leqq i \leqq p$. The direct product $H_1 \times \cdots \times H_p$ is a group in which H can be embedded and if $H \neq H_1 \times \cdots \times H_p$, then by the minimality of $n+k$, we can apply the theorem to the H_i's with parameters k and n and, in view of property (a) of $f(k,n)$, it follows that $|P| = p|H| \leqq |H_1|^p \leqq (f(k,m))^p \leqq f(k,n)$, a contradiction. Thus, we can assume that $H = H_1 \times \cdots \times H_p$. Now we consider two cases:

Case (a). Assume that $k < (p-1)s+1$. As $|P| = d(k, GL(n,q))$, the scalar transformations are contained in P and since $p^s \| p-1$, it follows that $Z(P)$ contains a scalar transformation y of order p^s. By (2) $y \in Z(H)$ and hence its projections on the H_i's belong to

$Z(H_i)$ for $1 \leqq i \leqq p$ and are of order p^s. Take $x \in P \backslash H$ and consider the group generated by $y_1, \ldots, y_p$ and x. It follows that $\langle y_1, \ldots, y_p, x \rangle = C_{p^s} \sim C_p$ and Proposition B.3(b) of [2] implies that $\text{class}(\langle y_1, \ldots, y_p, x \rangle) = (p-1)s+1$ and hence $\text{class}(P) \geqq (p-1)s+1$ contradicting our assumption that $\text{class}(P) \leqq k < (p-1)s+1$.

Case (b). Assume that $k \geqq (p-1)s+1$. Let $\text{class}(H_1) = l$, hence by Proposition B.3(a) of [2], it follows that $k \geqq \text{class}(P) \geqq lp$. The minimality of $n+k$ yields $|H_1| \leqq f(l, m)$ and hence in view of property (b) of $f(k, n)$, it follows that $|P| = p|H_1|^p \leqq p(f(l, m))^p \leqq f(lp, mp) \leqq f(k, n)$, a contradiction, and Theorem B.1 is proved. □

Now we evaluate $d(k, S_p(GL(n, q)))$ in the case $p = 2$ and $q = 3 (\text{mod } 4)$. As in the previous case, on the one hand, given a power of a prime q, $q = 3 (\text{mod } 4)$, and positive integers k and n, we have to find a suitable candidate A, $A \subseteq S_2(GL(n, q))$, for which $\text{class}(A) \leqq k$. On the other hand, we have to prove that $d(k, S_2(GL(n, q))) \leqq |A|$. If $k < s$, where $2^s \| q^2 - 1$, it is natural to define A as the direct product of $[n/2]$ cyclic groups of order 2^s each, and to join to the product a cyclic group of order 2 if n is odd. Thus $\text{class}(A) = 1$ and $|A| = 2^{s[n/2] + \varepsilon(n)}$ where $\varepsilon(n) = 0$ if n is even and $\varepsilon(n) = 1$ if n is odd. However, if $k \geqq s$, then there exists a minimal $\alpha, \alpha \geqq 1$, such that $k < s2^\alpha$, and the construction of A is as follows: Let $n = 2^\alpha t + u$ where $0 \leqq u < 2^\alpha$, then we can write the underlying vector space V as $V = V_0 \oplus V_1 \oplus \cdots \oplus V_t$ where $\dim(V_0) = u$ and $\dim(V_i) = 2^\alpha$ for $1 \leqq i \leqq t$. As $\text{class}(S_2(GL(u, q))) \leqq \text{class}(S_2(GL(2^\alpha, q))) = s2^{\alpha-1} \leqq k$ for $\alpha \geqq 1$, we define A to be the direct product of $S_p(GL(V_i))$, $0 \leqq i \leqq t$, and it follows that $\text{class}(A) = s2^{\alpha-1} \leqq k$ and that $|A| = |S_2(GL(2^\alpha, q))|^t |S_2(GL(u, q))|$. Before stating and proving the theorem we need a certain lemma.

Lemma B.2. *Let n be a positive integer and let q be a power of a prime such that $q = 3(\text{mod } 4)$ and $2^s \| q-1$. Suppose that γ is a positive integer, $2 \leqq \gamma \leqq s$, and suppose that P is a 2-subgroup of $GL(n, q)$, which is of maximal order among all 2-subgroups A of $GL(n, q)$, which satisfy the following conditions:*

(a) $\exp(A) \leqq 2^\gamma$; (b) $\text{class}(A) \leqq \gamma - 1$.

Then $|A| = 2^{\gamma[n/2] + \varepsilon(n)}$ *where* $\varepsilon(n) = \begin{cases} 0 & \text{if } n \text{ is even.} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

Proof. By taking a direct product of $[n/2]$ cyclic groups of order 2^γ each and joining to the product a cyclic group of order 2 if n is odd, we get that $|P| \geqq 2^{\gamma[n/2] + \varepsilon(n)}$. Thus, it suffices to prove the opposite inequality. Suppose that the lemma does not hold for a certain q and fixing that q let p be a counterexample for which $n + \gamma$ is minimal. As $S_2(GL(1, q))$ is of order 2 and as $S_2\ (GL(2, q))$ is semidihedral of order 2^{s+1}, it follows by [7, p. 191] that the lemma holds for $n = 1, 2$ and every γ, $2 \leqq \gamma \leqq s$.

Thus we can assume that $n > 2$. If P is reducible, then it is decomposable and $V = V_1 \oplus V_2$ where the V_i's are nontrivial P-invariant subspaces of V for $i = 1, 2$. It follows that $P \subseteq P_1 \times P_2$, where P_1 and P_2 are the projections of P on V_i of dimensions n_i, $i = 1, 2$. The minimality of $n + \gamma$ yields that $|P_i| \leqq 2^{\gamma[n_i/2] + \varepsilon(n_i)}$ for $i = 1, 2$, hence

$$|P| \leqq |P_1||P_2| = 2^{\gamma[n_1/2] + \varepsilon(n_1)} \cdot 2^{\gamma[n_2/2] + \varepsilon(n_2)} \leqq 2^{\gamma[n/2] + \varepsilon(n)}.$$

Thus, we can assume that P is irreducible and hence n is a power of 2. If $\gamma=2$, then P is abelian and since it is irreducible, it is cyclic.

Now (a) implies that $|P|\leqq 4\leqq 2^{2[n/2]}$ for $n>2$, thus we can also assume that $\gamma>2$. If P is one of the following: cyclic, dihedral, semidihedral or generalized quaternion, then $|P|\leqq 2^{\gamma+1}\leqq 2^{\gamma(n/2)}$ for $n>2$ and $3\leqq\gamma\leqq s$. Thus we can assume that P does not belong to one of the four exceptional families and Theorem 19.2 of **[10]** can be applied yielding:

(1) P contains a subgroup H such that $|P:H|=2$.

(2) The underlying vector space V can be written as $V=V_1\oplus V_2$ where the subspaces V_i, $i=1,2$ are H-invariant and if $x\in P\backslash H$, then x interchanges V_1 and V_2.

Let $\dim(V_i)=m$ (thus $n=2m$) and let H_i be the projection of H on V_i, $i=1,2$. If $H\neq H_1\times H_2$, then by the minimality of $n+\gamma$ we can apply the lemma with parameters m and γ to H_1 and H_2, yielding its validity for n and γ. Thus we can assume that $H=H_1\times H_2$, where $H_i=2^{\gamma(m/2)}$ for $i=1,2$. The minimality of $n+\gamma$ implies that either $\exp(H_i)\geqq 2^\gamma$ or $\mathrm{class}(H_i)\geqq\gamma-1$. We deal with the two cases separately.

(1) Assume that $\exp(H_1)\geqq 2$. If H_1 is abelian, then since it is irreducible, it is cyclic, and hence by the Proposition B.3(b) of [2], it follows that $\mathrm{class}(P)\geqq\gamma+1$, contradicting (b). If H_1 is not abelian, then since it contains a cyclic subgroup of order 2^γ at least, it follows by [7, p. 193] that H_1 contains one of the following subgroups: Dihedral, semidihedral or generalized quaternion of order $2^{\gamma+1}$ at least. Thus HZ^1 contains a subgroup of class γ at least and it follows that $\mathrm{class}(P)\geqq\gamma$, contradicting (b) again.

(2) Assume that $\mathrm{class}(H_1)\geqq\gamma-1$. By Proposition B.3(a) of **[2]**, it follows that $\mathrm{class}(P)\geqq 2\gamma-2>\gamma-1$, contradicting (b). Thus, our lemma is proved. □

Theorem B.3. *Let n and k be positive integers. Let q be a power of a prime such that $q=3(\mathrm{mod}\,4)$ and $2^s\|q^2-1$. If $k\geqq s$ define $\alpha\geqq 1$ as the minimal integer satisfying $k<s2^\alpha$ and let t and u be determined by $n=2^\alpha t+u$, where $0\leqq u<2^\alpha$.*

Then $d(k,S_2(GL(n,q)))=g(k,n)$ where

$$g(k,n)=\begin{cases}2^{s[n/2]+\varepsilon(n)} & \text{if } k<s\\ |S_2(GL(2^\alpha,q))|^t\cdot|S_2(GL(u,q))| & \text{if } k\geqq s\end{cases}$$

where $\varepsilon(n)=0$ if n is even and $\varepsilon(n)=1$ if n is odd.

Proof. The proof is similar to that of Theorem B.1. By the observation which precedes Lemma B.2, it follows that $d(k,S_2(GL(n,q)))\geqq g(k,n)$. We will prove that $d(k,S_2(GL(n,q)))\leqq g(k,n)$. Consider the following two properties of $g(k,n)$ which can be easily verified.

(a) $g(k,n_1)\cdot g(k,n_2)\leqq g(k,n_1+n_2)$ for all positive integers k, n_1 and n_2.
(b) $2(g(l,m))^2\leqq g(2l,2m)$ for $l\geqq s$ and every positive integer m.

Suppose that the theorem does not hold for a certain q and fixing q, let $P\subseteq S_2(GL(n,q))$ be a counter-example for which $n+k$ is minimal.

Thus, $|P| = d(k, S_2(GL(n,q)))$ and $|P| > g(k,n)$. As

$$\text{class}(S_2(GL(n,q))) = \begin{cases} 1 & \text{if} \quad n=1 \\ s & \text{if} \quad n=2,3 \end{cases}$$

the theorem holds for $n=1,2,3$ and every positive integer k. Thus we can assume that $n>3$. If P is reducible, then it is decomposable and $V = V_1 \oplus V_2$, where the V_i's are P-invariant subspaces for $i=1,2$. It follows that $P \subseteq P_1 \times P_2$, where P_1 and P_2 are the projections of P on V_i, $i=1,2$. Hence, in view of property (a) of $g(k,n)$ and the minimality of $n+k$, we derive a contradiction. Thus, we can assume that P is irreducible and hence n is a power of 2. Suppose that $k=1$, then P is abelian and since P is irreducible, it follows that P is cyclic. But this is impossible since if P is one of the following: cyclic, dihedral, semidihedral or generalized quaternion, then it is not difficult to derive a contradiction for an arbitrary k. Indeed, if P is one of the above-mentioned types and $\exp(P) = 2^\beta$, then $|P| \leqq 2^{\beta+1}$. Notice that in view of the fact that n is a power of 2, Proposition B.2 of **[2]** implies that $(n/2) \geqq 2^{\beta - s}$ and since $n>3$, it follows that $(n/2) \geqq \max\{2, 2^{\beta - s}\}$. It is not difficult to show that under these conditions, keeping in mind that $s>2$, we have $2^{s(n/2)} > 2^{\beta+1}$. But since $S_2(GL(n,q))$ contains a direct product of $n/2$ cyclic groups which is of order $2^{s(n/2)}$, the inequality $2^{s(n/2)} > |P|$ contradicts $|P| = d(k, S_2(GL(n,q)))$. Thus we can assume that P is irreducible, P does not belong to any of the exceptional four families, $n>3$ and $k>1$. Now Theorem 19.2 of **[10]** can be applied, yielding:

(1) P contains a subgroups H such that $|P:H| = 2$.

(2) The underlying vector space V can be written as $V_1 \oplus V_2$, where the subspaces V_i, $i=1,2$ are H-invariant and if $x \in P \setminus H$, then x interchanges V_1 and V_2.

Let $\dim(V_i) = m$ for $i=1,2$ (thus $n=2m$), and let $H_i \subseteq GL(V_i)$ be the projection of H on V_i for $i=1,2$. The direct product $H_1 \times H_2$ is a group in which H can be embedded, and if $H \neq H_1 \times H_2$, then by the minimality of $n+k$, we can apply the theorem to the H_i's with parameters k and m, and in view of property (a) of $g(k,n)$, it follows that $|P| = 2|H| \leqq |H_1|^2 \leqq (g(k,m))^2 \leqq g(k,n)$, a contradiction. Thus we can assume that $H = H_1 \times H_2$. Now we consider two cases:

Case (a). Assume that $k<s$. By the minimality of $n+k$, it follows that $|H_1| = 2^{s(n/2)}$ and applying Lemma B.2, we get that either $\exp(H_1) \geqq 2^s$ or $\text{class}(H_1) \geqq s-1$. As in the corresponding part of the proof of Lemma B.2, each of the above two inequalities implies that $\text{class}(P) \geqq s$, contradicting our assumption that $\text{class}(P) \leqq k < s$.

Case (b). Assume that $k \geqq s$. Let $\text{class}(H_1) = l$, hence by Proposition B.3(b) of **[2]**, it follows that $k \geqq \text{class}(P) \geqq 2l$. The minimality of $n+k$ yields $|H_1| \leqq g(l,m)$ and hence, in view of property (b) of $g(k,n)$, it follows that $|P| = 2|H_1|^2 \leqq 2(g(l,m))^2 \leqq g(2l,2m) \leqq g(k,n)$, a contradiction, and Theorem B.3 is proved. □

Now we state and prove our main lemma.

Lemma B.4 (Main Lemma). *Let p and q be two distinct primes and let k and n be positive integers.*

(a) *If $p=2$ and $q=2^m+1$ where $m\geqq 2$, then $d(k, S_2(GL(n,q)))\leqq q^n$ for every n iff $k\leqq m$.*

(b) *If $p=2$ and $q=3$, then $d(k, S_2(GL(n,q)))\leqq 3^n$ for every n iff $k\leqq 2$.*

(c) *If $p=2$ and $q=7$, then $d(k, S_2(GL(n,q)))\leqq 7^n$ for every n iff $k\leqq 15$.*

(d) *If $p=2^l-1$ and $q=2$, then $d(k, S_p(GL(n,2)))\leqq 2^n$ for every n iff $k\leqq p-1$.*

(e) *If p and q are not as above, then $d(k, S_p(GL(n,q)))\leqq q^n$ for every n and k.*

Proof. Case (e) is exactly Burnside's Lemma whose corrected version appears in [5] and will not be proved here. Case (b) follows from Glauberman's Lemma [6], but for completeness, we shall prove it.

(a) As $2^m \| q-1$ Theorem B.1 implies that $d(m, S_2(GL(n,q)))=2^{mn}<q^n$. On the other hand, $d(m+1, S_2(GL(2,q)))=2^{m+1}>(2^m+1)^2=q^2$.

(b) As $2^3\|3^2-1$ Theorem B.3 implies that $d(2, S_2(GL(n,q)))=2^{3[n/2]+\varepsilon(n)}\leqq 2^{3n/2}<3^n$. On the other hand, $d(3, S_2(GL(2,3)))=2^4>3^2$.

(c) As $2^4\|7^2-1$ Theorem B.3 implies that $d(15, S_2(GL(n,7)))=2^{5[n/2]+[n/4]+\varepsilon(n)}\leqq 2^{11n/4}<7^n$. On the other hand, $d(16, S_2(GL(8,7)))=2^{23}>7^8$.

(d) As $S_p(GL(n,2))=S_p(GL([n/l],2^l))$ Theorem B.1 implies that $d(p-1, S_p(GL(n,2)))=d(p-1, S_p([n/l],2^l))=p^{[n/l]}=(2^l-1)^{[n/l]}<2^n$. On the other hand, $d(p, GL(p,2^l))=p^{p+1}=p^{2^l}>2^{lp}$.

This completes the proof of Lemma B.4. □

C. Proof of the Main Theorem

In this section, we use the notation of A_p for the p-Sylow subgroup of A in the case where A is a nilpotent group. The Fitting subgroup of G and the Frattini subgroup of G are denoted by $F(G)$ and $\Phi(G)$, respectively. We need the following theorem.

Theorem C.1. *Let p and q be distinct primes and let k satisfy $d(k, S_p(GL(n,q)))<q^n$ for every n. Moreover, let G be a $\{p,q\}$-group and let A be a nilpotent subgroup of G of maximal order among all nilpotent subgroups C of G satisfying:*

(1) $\text{class}(C_p)\leqq k$,

(2) $\text{class}(C_q)\leqq 2$.

If B is a nilpotent subgroup of G normalized by A, then AB is nilpotent.

Proof. Let G be a counterexample and choose G and B such that $|G|+|B|$ is minimal. Clearly, we can assume that $G=AB$. We proceed in three steps.

(a) We prove that $B=[G,A_p]$ and $\text{class}(B)\leqq 2$. By the minimality of $|B|$ it follows that A_p centralizes every proper subgroup of B which is normalized by A. In particular $\Phi(B)$ is such subgroup, so A_p operates on $V=B/\Phi(B)$. It follows from Theorem 5.3.2 [7, p. 177] that $V=C_v(A_p)\times[V,A_p]$. By the minimality of B, it follows that V cannot be A-

decomposable. If $V=C_v(A_p)$, then AB is nilpotent, so $C_v(A_p)=1$ and $V=[V,A_p]$ yielding $B=[B,A_p]$. As B' is a proper A-invariant subgroup of B, it is centralized by A_p. Using the three subgroups lemma we get from $[B',B,A_p]=1$ and $[A_p,B',B]=1$, that $[B,A_p,B']=1$. But $B=[B,A_p]$, so it follows that class$(B)\leqq 2$.

(b) We prove that A_q centralizes B. Consider the group A_qV which is an extension of V by A_q. Since A_qV is a q-group, by a known property of nilpotent groups it follows that $[V,A_q]\neq V$. Since $[V,A_q]$ is A-invariant, it follows by the minimality of $|B|$ that A_p centralizes $[V,A_q]$. We have proved that $C_V(A_p)=1$, hence $[V,A_q]$ is trivial yielding $[B,A_q]\subseteq\Phi(B)$. Applying the three subgroups lemma again, we get from $[B,A_q,A_p]=1$ and $[A_q,A_p,B]=1$ that $[A_p,B,A_q]=1$. But $[A_p,B]=B$, so A_q centralizes B.

(c) We derive a contradiction. Define $\bar{A}=A/C_A(B)$. If $|V|=|B/\Phi(B)|=q^n$, then $\bar{A}\subseteq GL(n,q)$. By (b) $\bar{A}$ is a p-group and by the definition of A, we have class$(\bar{A})\leqq k$, hence $|\bar{A}|\leqq d(k,S_p(GL(n,q)))<q^n=|V|$. Define $A^*=C_A(B)B$. Since A^* is nilpotent satisfying class$(A_p^*)\leqq k$ and class$(A_q^*)\leqq 2$, we have $|A^*|<|A|$. On the other hand, it will be shown that $|A^*|>|A|$. Indeed, since $((B\cap C_A(B))\Phi(B))/\Phi(B)$ is an A-invariant subgroup of V, it follows by an argument used in (b) that $B\cap C_A(B)\subseteq\Phi(B)$ and finally we get:

$$|A^*|=|C_A(B)B|=|B:B\cap C_A(B)||C_A(B)|\geqq|B/\Phi(B)||C_A(B)|$$
$$=|V||C_A(B)|>|\bar{A}||C_A(B)|=|A|.$$

Thus the proof of Theorem C.1 is complete. □

Proof of the main theorem. The five cases will be proved simultaneously. We use k to denote m, 2, 15, $p-1$ and ∞ in case (a), (b), (c), (d), (e), respectively. If $O_p(G)=1$, then $F(G)=O_q(G)\neq 1$. Let A be a nilpotent subgroup of G of maximal order among all nilpotent subgroups C of G satisfying class $(C_p)\leqq k$ and class$(C_q)\leqq 2$. By Lemma B.4 $d(k,S_p(GL(n,q)))<q^n$ and therefore we can apply Theorem C.1 with $F(G)$ being B, the subgroup of G normalized by A. We obtain that $AO_q(G)$ is nilpotent. Since $d(k,P)>d(2,Q)$, the definition of A implies that A is not a q-group. Hence, there is a non-q-element in A which centralizes $O_q(G)=F(G)$ contradicting that $C(F(G))\subseteq F(G)$ for a solvable group G. □

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