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ON BMOA FOR RIEMANN SURFACES

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1. Introduction. Let Δ denote the unit disk in the complex plane $\mathbf{C}$. The space BMO has been extensively studied by many authors (see [**3**] for a good exposition of this topic). Recently, the subspace BMOA(Δ) has become a topic of interest. An analytic function f, in the Hardy class $H^2(\Delta)$, belongs to BMOA(Δ) if

$$(1) \quad \sup_{\omega\in\Delta} \left\| f\left(\frac{z+\omega}{1+z\bar{\omega}}\right) - f(\omega)\right\|_2 < \infty$$

where

$$\|h\|_2 = \lim_{r\to 1} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^2 d\theta\right)^{1/2}.$$

It is known (see [**3**, p. 96]) that (1) is equivalent to

$$(2) \quad \sup_{\omega\in\Delta} \iint_\Delta |f'(z)|^2 \log\left|\frac{1-z\bar{\omega}}{z-\omega}\right| dxdy < \infty.$$

Using (2), we can define BMOA(W) for Riemann surfaces W, when W possesses a Green's function $G(p, q)$. It will be proved that AD(W), the space of Dirichlet finite functions on the surface, is in BMOA(W) and thus as a corollary one gets AD$(W) \subseteq H^p(W)$ for all $p < \infty$. This latter result seems to be new, the fact that it holds for $p = 2$ seems to be the only previous result in this direction (see [**2**]). These results show that the case for an arbitrary Riemann surface is analogous to that of the unit disk.

An important ingredient of the proof is the following striking result due to Hayman and Pommerenke [**1**], which is

THEOREM A. *The domain $G \subseteq \mathbf{C}$ has the property that every function $f(z)$ analytic in Δ with values in G belongs to* BMOA(Δ), *if and only if there exist constants R and $\delta > 0$ such that for all ω_0 in G*

$$(3) \quad \text{cap}\,(E \cap \{\omega: |\omega - \omega_0| \leqq R\}) \geqq \delta,$$

where $E = \mathbf{C} - G$ and cap *denotes logarithmic capacity.*

In the last section we shall note some correlations between the condition (3) and BMOA(G), with G considered as a planar Riemann surface.

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2. The main result. Let W be an arbitrary Riemann surface and assume that W has a Green's function $G(p, q)$. In this case, as is well-known, the universal covering surface of W is Δ and if $\pi: \Delta \to W$ denotes the universal covering map then the group of deck transformations is a Fuchsian group Γ. Let Ω be the Ford fundamental polygon for Γ so that $\pi: \Omega \to W$ is a one to one into map, $\pi: \bar{\Omega} \to W$ is onto and the area measure of $\partial\Omega$ is zero.

In analogy with (2), the space BMOA(W) is defined to be the space of analytic functions on W for which

$$(4) \qquad \sup_{q \in W} \int\int_W |F'(p)|^2 G(p, q) dp d\bar{p}$$

is finite. The space of Dirichlet finite analytic functions, denoted by AD(W), is defined by requiring that

$$(5) \qquad \int\int_W |F'(p)|^2 dp d\bar{p}$$

be finite. We now assert

THEOREM 1. AD$(W) \subseteq$ BMOA(W).

The fact that the containment is strict follows immediately from the fact that $H^\infty(W)$, the space of bounded analytic functions on W, is contained in BMOA(W).

The proof of the inclusion is based on the fact that F can be "pulled back" to a function f acting on Δ by the following method: if F is analytic on W, define $f(z)$ by $f(z) = F(\pi z)$ for all z in Δ. It follows immediately that f is an analytic function on Δ which satisfies

$$(6) \qquad f(\gamma z) = f(z) \text{ for all } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \Delta.$$

Functions satisfying (6) are called *automorphic* functions with respect to Γ. Define the spaces AD(Δ/Γ) and BMOA(Δ/Γ) as the pull backs of the spaces AD(W) and BMOA(W), respectively. It is clear that these spaces are isometrically isomorphic.

If Γ is not the identity and f satisfies (6) then each fundamental polygon Ω contributes the same amount (5) to

$$\int\int_\Delta |f'(z)|^2 dz d\bar{z} = \sum_{\gamma \in \Gamma} \int\int_\Omega |f'(z)|^2 dz d\bar{z}$$

so that f cannot belong to AD(Δ). Nevertheless we have:

PROPOSITION 2. $\mathrm{BMOA}(\Delta/\Gamma) \subseteq \mathrm{BMOA}(\Delta)$.

Proof. If $g(z, \omega; \Gamma) = \sum_{\gamma\in\Gamma} g_\Delta(z, \gamma\omega)$, where g_Δ is the Green's function for Δ, it follows immediately from the result of Myrberg (see [**6**], p. 522) that

$$G(p, q) = g(\pi^{-1}(p), \pi^{-1}(q); \Gamma).$$

Thus f is in $\mathrm{BMOA}(\Delta/\Gamma)$ if and only if

$$(7) \qquad \sup_{\omega\in\Omega} \int\int_\Omega |f'(z)|^2 g(z, \omega; \Gamma) dxdy < \infty.$$

Since $g(z, \gamma\omega; \Gamma) = g(z, \omega; \Gamma)$ for all γ in Γ and $\bigcup_{\gamma\in\Gamma} \gamma\bar{\Omega} = \Delta$, it follows that the sup in (7) can be taken over all ω in Δ. Moreover

$$\int\int_\Omega |f'(z)|^2 \sum_{\gamma\in\Gamma} g_\Delta(z, \gamma\omega) dxdy = \sum_{\gamma\in\Gamma} \int\int_\Omega |f'(z)|^2 g_\Delta(\gamma^{-1}z, \omega) dxdy.$$

However, (6) implies that $|f'(z)|^2\, dxdy$ is invariant under a change of variables by γ in Γ. Hence (4) reduces to

$$(8) \qquad \sup_{\omega\in\Delta} \int\int_\Delta |f'(z)|^2 g_\Delta(z, \omega) dxdy < \infty,$$

and this yields Proposition 2.

Remark. It follows from Proposition 2 that one could have defined $\mathrm{BMOA}(\Delta/\Gamma)$ as those automorphic functions in $\mathrm{BMOA}(\Delta)$ and $\mathrm{BMOA}(W)$ as the projections of functions in $\mathrm{BMOA}(\Delta/\Gamma)$ and then proved that this latter definition was intrinsic to the surface.

Theorem 1 is now proved as follows: if $g(\Delta)$ satisfies (3) then g is in $\mathrm{BMOA}(\Delta)$. Since $f(\Delta) = F(W)$, which has finite area, it follows that the projection of the surface $f(\Delta)$ on the plane has finite area. Thus by taking $\pi R^2 = 2\,\mathrm{area}\,F(W)$, one gets that $f(\Delta)$ satisfies (3) and f is in $\mathrm{BMOA}(\Delta/\Gamma)$ and Proposition 2 completes the proof.

Proposition 2 also implies

COROLLARY 3. *Let F be an analytic function on W with values in a domain G which satisfies* (3): *then* $F \in \mathrm{BMOA}(W)$.

3. Applications. As a first application of Theorem 1 one gets the following, seemingly new, corollary about the Hardy spaces $H^p(W)$ and $\mathrm{AD}(W)$.

COROLLARY 4. $\mathrm{AD}(W) \subset \bigcap_{0<p<\infty} H^p(W)$.

Proof. If F is in $\mathrm{AD}(W)$ then f, its pull back, belongs to $\mathrm{BMOA}(\Delta/\Gamma)$ and therefore f belongs to $\bigcap_{0<p<\infty} H^p(\Delta)$. Moreover, f is an automorphic function so that the least harmonic majorant of $|f|^p$ is also an automorphic function. Thus, by projecting back down to the surface, we see that F is in $\bigcap_{0<p<\infty} H^p(W)$.

Remark. The classical result in this direction is that $\mathrm{AD}(W) \subseteq H^2(W)$ but no results seem to be known for $p > 2$. The result of Corollary 4 is that the case for the disk Δ and arbitrary Riemann surfaces are entirely analogous.

In the language of classification theory for Riemann surfaces one has (see [**2**] for the appropriate definitions)

$$\bar{O}_p \subset O_{\mathrm{BMOA}} \subset O_{\mathrm{AB}} \subset O_{\mathrm{AD}}.$$

Next, an application to automorphic forms is catalogued. If $f(z)$ satisfies (6) then

(9) $\quad f'(\gamma z)\gamma'(z) = f'(z)$ for all γ in Γ and z in Δ.

Such functions are called *automorphic forms of weight* 1. The finiteness of the Dirichlet integral becomes

(10) $$\int\int_\Omega |f'(z)|^2 dxdy < \infty.$$

We shall say f is in $D(\Gamma)$ if f' satisfies (9) and (10). Note that the fact f belongs to $D(\Gamma)$ does not imply that f is an automorphic function. In fact (9) is equivalent to

(11) $\quad f(\gamma z) = f(z) + C(\gamma)$ for all γ in Γ and z in Δ,

where $C(\gamma)$ is an additive constant.

Let $\mathscr{B}$ denote the space of Bloch functions and recall that f belongs to $\mathscr{B}$ if and only if $|f'(z)| = O((1 - |z|^2)^{-1})$. A question which occupied various authors was the Bers space conjecture which in this case becomes: Is $D(\Gamma) \subset \mathscr{B}$? Ch. Pommerenke [**4**] disproved this conjecture by exhibiting a surface W and the associated Fuchsian group Γ such that $D(\Gamma) - \mathscr{B} \neq \emptyset$. In his example the functions in $D(\Gamma) - \mathscr{B}$ had non-zero additive periods $C(\gamma)$, i.e., they were not automorphic functions. In response to a question by A. M. Macbeath we note that $C(\gamma) \neq 0$ is characteristic of functions in $D(\Gamma) - \mathscr{B}$ by asserting

COROLLARY 5. *If f belongs to $D(\Gamma) - \mathscr{B}$ then f is not an automorphic function.*

Proof. If f is an automorphic function in $D(\Gamma)$ then f belongs to $\mathrm{AD}(\Delta/\Gamma)$ and thus to $\mathrm{BMOA}(\Delta/\Gamma)$ by Theorem 1. Proposition 2 implies that f is in $\mathrm{BMOA}(\Delta)$ which is a subspace of $\mathscr{B}$ and the proof is complete.

In view of Theorem 1 it would seem to be of interest to know for which groups Γ, $D(\Gamma) - \text{BMOA}(\Delta/\Gamma) = \emptyset$. This situation can actually occur and the author hopes to give a description of such groups in the near future.

4. On the condition (3). It should be noted that the condition (3) does not characterize the range of functions in BMOA(Δ). To see this one merely constructs a finitely valent Bloch function whose projection covers all of **C**. Since for finitely valent functions $\mathscr{B} = \text{BMOA}(\Delta)$ it follows that (3) does not hold. A. Baernstein II has noted that if one takes $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ with $n_{k+1}/n_k \geqq + > 1$ with $\sum_{k=0}^{\infty} |a_k|^2 < \infty$ and $\sum_{k=0}^{\infty} |a_k| = \infty$ then, for a sufficiently large q, $f(z)$ covers the plane infinitely often. Moreover, $f(z)$ is in $H^1(\Delta)$ and since it has a gap series representation it follows that f must belong to BMOA(Δ).

In contrast to the negative results above we shall give a condition which is equivalent to (3). As before we write $E = \mathbf{C} - G$ and note that if G satisfies (3) then cap $(E) > 0$ and thus G has a Green's function which we denote by $g(p, q)$.

THEOREM 6. *Given a plane domain G,* (3) *holds if and only if*

$$(12) \quad \sup_{q \in G} \int\int_G g(p, q) dp d\overline{p} < \infty .$$

Remark. One can interpret (12) as asserting that the identity function $f(p) = p$ belongs to BMOA(G).

Proof. Let (3) hold and suppose that G^{∞} is the universal (infinite) covering surface of G. If $F: \Delta \rightarrow G^{\infty}$ is the Riemann mapping function then F is in BMOA(Δ) by Theorem A. Consider G as a Riemann surface and let Γ be the associated Fuchsian group and let $\pi: \Delta \rightarrow G$ be the projection map. It follows that $\pi(z) = F(z)$ for all z in Δ and thus π belongs to BMOA(Δ). Thus, by (2),

$$\begin{aligned} \infty &> \sup_{\omega \in \Delta} \int\int_{\Delta} |\pi'(z)|^2 g_{\Delta}(z, \omega) dx dy \\ &= \sup_{\omega \in \Delta} \sum_{\gamma \in \Gamma} \int\int_{\gamma\Delta} |\pi'(z)|^2 g_{\Delta}(z, \omega) dx dy \\ &= \sup_{\omega \in \Delta} \int\int_{\Delta} |\pi'(z)|^2 \sum_{\gamma \in \Gamma} g_{\Delta}(\gamma z, \omega) dx dy. \end{aligned}$$

Since $g_{\Delta}(\gamma z, \omega) = g_{\Delta}(z, \gamma^{-1}\omega)$ one can substitute $\overline{\Omega}$ for Δ in the sup and upon recalling that $g(p, q) = \sum_{\gamma \in \Gamma} g_{\Delta}(z, \gamma\omega)$ whenever $\pi(z) = p$ and $\pi(\omega) = q$, it follows that (12) holds. To see that the converse holds we note that if (12) holds then π belongs to BMOA(Δ) and therefore, F

belongs to BMOA(Δ). However, F in BMOA(Δ) implies that (3) holds as a careful reading of the proof of Theorem A in [**1**] shows and the proof is complete.

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