J. Austral. Math. Soc. (Series A) 51 (1991), 95–105

STAR-CENTER POINTS OF UNIVALENT FUNCTIONS

NICOLAS K. ARTÉMIADIS

(Received 19 February 1990; revised 13 July 1990)

Abstract

Let $\mathscr{S}$ be the class of normalized univalent functions in the unit disk. For $f \in \mathscr{S}$ let S_f be the set of all star center points of f. Let $\mathscr{S}_0 = \{f \in \mathscr{S}: 0 \in S_f^0\}$ where S_f^0 is the interior of S_f. The influence that the size of the set S_f^0 has on the Taylor coefficients of a function $f \in \mathscr{S}_0$ is examined, and estimates of these coefficients depending only on S_f^0, as well as other results, are obtained.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 30 C 50.

1. Introduction

Let $\mathscr{S}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$. For $f \in \mathscr{S}$ the set $f(D)$ is a nonempty open connected proper subset of the complex plane $\mathbb{C}$. A point $w \in f(D)$ is called a star center point (s.c.p) of $f(D)$ if and only if

$$tf(z) + (1-t)w \in f(D), \qquad z \in D, \ 0 \le t \le 1.$$

For $f \in \mathscr{S}$, let S_f be the set of all s.c.p of $f(D)$. Also let $\mathscr{S}_0$ be the subclass of $\mathscr{S}$ having the property that if $f \in \mathscr{S}_0$ then $0 \in S_f^0$, where S_f^0 is the interior of S_f.

In this paper we examine the influence that roughly the size of S_f^0 has on the Taylor coefficients, a_n, of a function in $\mathscr{S}_0$.

In Theorem 1, we obtain estimates of $|a_n|$, depending on the size of S_f^0 for $f \in \mathscr{S}_0$.

Theorem 2 provides additional information concerning the coefficient estimates obtained in Theorem 1. More precisely it is shown that if $f_1, f_2 \in \mathscr{S}_0$ and $S^0_{f_1} \subset S^0_{f_2}$ then $B(f_2, n) \leq B(f_1, n)$, $n = 1, 2, \dots$, where $B(f_1, n)$, $B(f_2, n)$ are the estimates for the nth coefficients of f_1 and f_2 respectively. Finally we give examples of functions in $\mathscr{S}_0$ and compare our results with those obtained in [1].

I would like to thank the referees for their helpful comments on the subject.

2. Preliminaries

In this section we prove three lemmas which will be used later.

LEMMA 1. *The set of all star center points of a function in* $\mathscr{S}$ *is convex.*

PROOF. Let $g \in \mathscr{S}$, $z_1, z_2 \in D$ such that $g(z_1), g(z_2)$ belong to S_g. We show that the segment $[g(z_1), g(z_2)]$ is contained in S_g. Suppose $[g(z_1), g(z_2)] \not\subset S_g$ and let $w \in (g(z_1), g(z_2))$ be such that $w \notin S_g$. Since $g(z_1), g(z_2)$ are s.c.p of $g(D)$ we have $w \in g(D)$.

By the hypothesis on w there is $z_0 \in D$ such that $[g(z_0), w] \not\subset g(D)$. Observe that if the points $g(z_0), g(z_1), g(z_2)$ are collinear then there is nothing to prove. Otherwise there is $w_1 \in (g(z_0), w)$ such that $w_1 \notin g(D)$. We have $[g(z_1), g(z_0)] \subset g(D)$ because $g(z_1) \in S_g$ and $g(z_0) \in g(D)$. Let w_2 be the intersection of the segment $[g(z_1), g(z_0)]$ and the straight line determined by the points $g(z_2)$ and w_1. These two sets intersect because w_1 is an interior point of the triangle $\{g(z_0), g(z_1), g(z_2)\}$. We have $w_2 \in g(D)$. Since $g(z_2) \in S_g$ it follows that $w_1 \in g(D)$ which contradicts $w_1 \notin g(D)$. Hence S_g is convex.

LEMMA 2. *Let* $f \in \mathscr{S}_0$, $\xi: D \to S^0_f$ *be a univalent analytic function such that* $\xi(0) = 0$, $\xi(D) = S^0_f$, *and let* z_0, z_1 *be complex numbers such that* $|z_0| < |z_1| = r < 1$. *Then the segment* $[f(z_1), \xi(z_0)]$ *is contained in* $f(\overline{D}_r)$, *where* $\overline{D}_r = \{z: |z| \leq r\}$.

PROOF. For $\xi(z_0) = 0$ the lemma is known [2, page 220]. Let ρ and θ be two real numbers such that $0 < \rho < 1$, $-\pi \leq \theta \leq \pi$, $\rho e^{i\theta} z_1 = z_0$. Put $\Phi(z) = tf(z) + (1-t)\xi(\rho e^{i\theta} z)$, $z \in D$, $0 \leq t \leq 1$. Clearly Φ is analytic in D, $\Phi(0) = f(0) = 0$, and for each z the point $\xi(\rho e^{i\theta} z)$ is a s.c.p of $f(D)$. Hence Φ is subordinate to f, so $\Phi(z) = f(\varphi(z))$, where φ is analytic in

D, and $|\varphi(z)| \le |z|$. We have

$$\Phi(z_1) = tf(z_1) + (1-t)\xi(\rho e^{i\theta} z_1) = tf(z_1) + (1-t)\xi(z_0) = f(\varphi(z_1))$$

and $|\varphi(z_1) \le |z_1|$. Hence $\Phi(z_1) \in f(\overline{D}_r)$.

LEMMA 3. *Let* $n > 2$ *be an integer. Given* $1/2 \le x \le 1$ *and integers* $1 \le p \le q$, *define*

$$F_{q,p} = (p-x)(p+1-x)\cdots(q-x).$$

Then

$$(1)\quad -n!n + nF_{n,2}(x) + 2x[F_{n,3}(x) + 2!2F_{n,4}(x) + \cdots + (n-2)!(n-2)F_{n,n}(x) + (n-1)!(n-1)] \le 0.$$

PROOF. We proceed by induction on n. Observe that (1) holds for $n = 3$. We assume that it holds for n and we prove that it holds for $n+1$. It suffices to show that the left-hand side of (1) is nonincreasing in n, for each fixed $x \in [1/2, 1]$, or equivalently

$$\begin{aligned}(n+1)F_{n+1,2}(x) - nF_{n,2}(x) + 2xn!n + 2x(n-x)\\ \cdot [F_{n,3}(x) + 2!2F_{n,4}(x) + \cdots + (n-1)!(n-1)]\\ \le (n+1)!(n+1) - n!n.\end{aligned}$$

Now by the induction hypothesis we have

$$2x[F_{n,3}(x) + 2!2F_{n,4}(x) + \cdots + (n-1)!(n-1)] \le n!n - nF_{n,2}(x).$$

Hence (2) will hold if the following holds:

$$(3)\quad \begin{aligned}(n+1)F_{n+1,2}(x) - nF_{n,2}(x) + 2xn!n + (n-x)(n!n - nF_{n,2}(x))\\ \le (n+1)!(n+1) - n!n.\end{aligned}$$

This is equivalent to

$$(4)\qquad F_{n+1,2}(x) + n!nx - (n+1)! \le 0.$$

To prove (4) we proceed as follows. We put $\Phi(x) = F_{n+1,2}(x) + n!nx - (n+1)!$ and we claim that the derivative $\Phi'(x)$ is nonnegative for $1/2 \le x \le 1$. If this is proven it will mean that $\Phi(x)$ is nondecreasing so that its maximum value will be taken for $x = 1$. But since $\Phi(1) = 0$ (4) will be proven.

We show that

$$(5)\qquad \Phi'(x) = n!n + F'_{n+1,2}(x) \ge 0, \qquad 1/2 \le x \le 1.$$

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Observe that from the definition of $F_{q,p}(x)$ it follows that

$$F'_{n+1,2}(x) = -F_{n+1,2}(x) \cdot \sum_{k=2}^{n+1} \frac{1}{k-x}, \tag{6}$$

so (5) can be written

$$n!n - F_{n+1,2}(x) \cdot \sum_{k=2}^{n+1} \frac{1}{k-x} \geq 0. \tag{7}$$

Since $1/2 \leq x \leq 1$, to prove (7) it suffices to show

$$n!n - F_{n+1,2}\left(\frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k-\frac{1}{2}} \geq 0. \tag{8}$$

We again proceed by induction on n. It is easily seen that (8) holds for $n = 3$. Assume that it holds for n. To show that (8) holds for $n+1$ we prove that the left-hand side of (8) is nondecreasing in n, or that

$$(n+1)!(n+1) - n!n \geq F_{n+1,2}\left(\frac{1}{2}\right)\left[\left(n+\frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k-\frac{1}{2}} + 1\right]. \tag{9}$$

If in (9) the expression $F_{n+1,2}(\frac{1}{2})$ is replaced by

$$n!n \Big/ \sum_{k=2}^{n+1} \frac{1}{k-\frac{1}{2}}$$

we get

$$\begin{aligned}(n+1)!(n+1) - n!n \geq & \left[\frac{n!n}{\sum_{k=2}^{n+1} \frac{1}{k-\frac{1}{2}}}\right] \\ & \cdot \left[\left(n+\frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k-\frac{1}{2}} + 1\right].\end{aligned} \tag{10}$$

Since (8) holds, it follows that (9) is true if (10) holds. But (10) is equivalent to

$$\frac{n}{n+2} \leq \sum_{k=1}^{n} \frac{1}{2k+1}, \qquad n \geq 3, \tag{11}$$

which is easily seen to be true by induction. It follows that (9) holds, and this proves the lemma.

3. The main results

We wish to give coefficient estimates for the Taylor expansion of a function in $\mathscr{S}_0$.

Let $f \in \mathcal{S}_0$. From Lemma 1 it follows that S_f^0 is convex. Also $S_f^0 \neq \mathbb{C}$ since $f(C) \neq \mathbb{C}$.

Let α be any point of S_f^0. Riemann's Mapping Theorem asserts that there is a unique analytic function

$$g_\alpha : S_f^0 \to D \tag{12}$$

having the properties

(a) $g_\alpha(\alpha) = 0$ and $g'_\alpha(\alpha) > 0$,

(b) g_α is one-to-one,

(c) $g_\alpha(S_f^0) = D$.

Put

$$\mu(f, \alpha) = [1 - |g_\alpha(0)|^2]/g'_\alpha(0).$$

THEOREM 1. *Let* $f(z) = z + \sum_{n=2}^\infty a_n z^n$ *be a function in* $\mathcal{S}_0$ *and let* α *be a point of* S_f^0. *Then*

(i) $0 < \mu(f, \alpha) \leq 1$.

(ii) *If* $\mu(f, \alpha) = 1$ *then* $|a_n| \leq 1$, $n = 1, 2, \ldots$.

(iii) $\mu(f, \alpha) = 1$ *if and only if* $S_f^0 = f(D)$.

(iv) *If* $\mu(f, \alpha) < 1$ *then* $|a_n| \leq A_n(f, \alpha) + R_{n-1}(\sigma) = M_n(f, \alpha)$, $n \geq 2$, *where* $A_n(f, \alpha) = 1 + (n-1)\prod_{k=2}^n (k-1)/(k-\sigma)$, $\sigma = 1/(1+\mu(f, \alpha))$, *and*

$$R_n(\sigma) = \frac{-n!n}{\prod_{k=2}^{n+1}(k-\sigma)} + \frac{n}{n+1-\sigma} + \frac{2\sigma}{n+1-\sigma} \cdot \left[\frac{1}{2-\sigma} + \frac{2!2}{(2-\sigma)(3-\sigma)} + \cdots + \frac{(n-2)!(n-2)}{(2-\sigma)\cdots(n-1-\sigma)} + \frac{(n-1)!(n-1)}{(2-\sigma)\cdots(n-\sigma)}\right].$$

(v) $|a_n| \leq B(f, n)$, $n \geq 2$, *where* $B(f, n) = \inf_{\alpha \in S_f^0}(M_n(f, \alpha))$.

PROOF. Put $g = g_\alpha^{-1}$ where g_α is the function defined in (12). Then $g : D \to S_f^0$ is analytic in D and has the following properties:

(a′) $g(0) = \alpha$, $g'(0) = 1/g'_\alpha(\alpha) > 0$;

(b′) g is one-to-one;

(c′) $g(D) = S_f^0$.

Let $g_\alpha(0) = \beta$. Then $\beta \in D$ and $g(\beta) = 0$.

Put

$$G(z) = g\left(\frac{z+\beta}{1+\overline{\beta}z}\right), \qquad z \in D. \tag{13}$$

The function $G : D \to S_f^0$ is analytic in D and has the following properties:

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(a″) $G(0) = g(\beta) = 0$; $G'(0) = g'(\beta)(1 - |\beta|^2) = (1 - |\beta|^2)/g'_\alpha(0) = (1 - |g_\alpha(0)|^2)/g'_\alpha(0)$;

(b″) G is one-to-one;

(c″) $G(D) = S_f^0$.

Clearly G is subordinate to f. It follows that

$$G(z) = f(\omega(z)) \tag{14}$$

where ω is analytic on D and $|\omega(z)| \le |z|$.

Put $G(z) = \sum_{n=1}^{\infty} b_n z^n$, $z \in D$. We have since, $G'(0)$ does not vanish, that

$$0 < b_1 = G'(0) = \omega'(0) = [1 - |g_\alpha(0)|^2]/g'_\alpha(0) = \mu(f, \alpha) \le 1. \tag{15}$$

This proves assertion (1) of Theorem 1.

The function $G(z)/b_1 = \sum_{n=1}^{\infty}(b_n/b_1)z^n$ belongs to the class $\mathscr{S}_0$ and maps D onto the region $(1/b_1)S_f^0 = \{w/b_1 : w \in S_f^0\}$ which is convex since S_f^0 is convex. It follows that

$$|b_n/b_1| \le 1, \qquad n = 1, 2, \ldots. \tag{16}$$

Observe that ω is univalent in D because the composition of two univalent functions is univalent.

Summarizing the properties of ω, we have:

(i) ω is univalent in D;

(ii) $\omega(D) \subset D$;

(iii) $\omega(0) = 0$;

(iv) $0 < \omega'(0) = b_1 \le 1$.

If in addition we had $\omega(D) = D$ then we would have $\omega(z) = z$, $\omega'(0) = b_1 = 1$ and it would follow from (14) and (16) that $G(z) = f(z)$ so that $a_n = b_n$, $|a_n| \le 1$. This proves assertion (ii) of Theorem 1.

Next assume that $\omega(D)$ is a proper subset of D. Then it follows from the condition for equality in Schwarz's lemma that $\omega'(0) < 1$.

The above imply

$$\begin{aligned} &\text{(i) } \omega(D) = D \quad \text{if and only if } \omega'(0) = 1, \\ &\text{(ii) if } \omega'(0) < 1 \quad \text{then } 0 < b_1 < 1, \\ &\text{(iii) } |b_n| \le |b_1| \le 1, \end{aligned} \tag{17}$$

and assertion (iii) of Theorem 1 follows from (17)(i).

Let $z, z_0 \in D$ such that $|z_0| < |z| = r < 1$. Put $G(z_0) = w \in S_f^0$, $f(z) - w = Re^{i\tau}$, $z = re^{i\theta}$. It follows from Lemma 2 that w is a s.c.p of $f(\overline{D}_r)$. Therefore

$$\frac{\partial}{\partial\theta}\arg[f(z) - w] = \frac{\partial\tau}{\partial\theta} \ge 0.$$

We have

$$\log[f(z) - w] = \log R + i\tau$$

so

$$\operatorname{Im}\left[\frac{\partial}{\partial\theta}\log(f(z) - w)\right] \geq 0.$$

In view of

$$\frac{\partial}{\partial\theta} = ire^{i\theta}\frac{d}{dz} = iz\frac{d}{dz}$$

we get

$$\operatorname{Re}[zf'(z)/(f(z) - G(z_0))] \geq 0.$$

The last inequality holds for all z, z_0 in D for which $|z| > |z_0|$. Therefore if λ is a real number such that $0 \leq \lambda < 1$, we have

$$\operatorname{Re}[zf'(z)/(f(z) - G(-\lambda z))] \geq 0, \qquad z \in D.$$

Put

$$(18) \qquad F(z) = [zf'(z)/(f(z) - G(-\lambda z))] = \sum_{n=0}^{\infty} c_n z^n, \qquad z \in D.$$

It is easily seen that F is analytic in D and that $c_0 = 1/(1 + b_1\lambda)$.

Due to the inequality

$$\operatorname{Re} F(z) \geq 0, \qquad z \in D$$

we have

$$(19) \qquad |c_n| \leq 2c_0 = \frac{2}{1 + b_1\lambda}.$$

From (18) we get

$$zf'(z) = \sum_{n=1}^{\infty} na_n z^n = \sum_{n=1}^{\infty}[a_n - b_n(-\lambda)^n]z^n \cdot \sum_{n=0}^{\infty} c_n z^n.$$

The last equation gives

$$na_n = \sum_{k=1}^{n}[a_k - (-\lambda)^k b_k]c_{n-k}, \qquad n = 1, 2, \ldots,$$

or

$$(20) \qquad (n - c_0)a_n = \sum_{k=1}^{n-1} a_k c_{n-k} - \sum_{k=1}^{n}(-\lambda)^k b_k c_{n-k}.$$

If we set $\lambda = 0$ then (20) and (19) provide the well known inequality $|a_n| \leq n$, $n = 2, 3, \ldots$.

From (20) we obtain, on account of (16) and (19),

$$|a_n| \le \frac{2c_0}{n-c_0}\sum_{k=1}^{n-1}|a_k| + \frac{1}{n-c_0}\sum_{k=1}^{n}\lambda^k|c_{n-k}||b_k|$$
$$\le \frac{2c_0}{n-c_0}\sum_{k=1}^{n-1}|a_k| + \frac{\lambda^n b_1 c_0}{n-c_0} + \frac{1}{n-c_0}\sum_{k=1}^{n-1}2b_1c_0\lambda^k.$$

Now if we let $\lambda \to 1$ we get, since $b_1\sigma = 1-\sigma$, that

$$|a_n| \le \frac{2\sigma}{n-\sigma}\sum_{k=1}^{n-1}|a_k| + \frac{(1-\sigma)(2n-1)}{n-\sigma}, \qquad n \ge 2. \tag{21}$$

From (21) we deduce that for $n \ge 2$,

$$|a_n| \le A_n(f,\alpha) + R_{n-1}(\sigma) = M_n(f,\alpha) \le A_n(f,\alpha). \tag{22}$$

The last part of (22) follows immediately from Lemma 3, since $R_n(\sigma)$ is nonpositive for $n \ge 1$ and $1/2 \le \sigma \le 1$.

To prove the first part of (22) we proceed by induction on n. It is easily seen that for $n = 2, 3$, (21) provides

$$|a_2| \le 1 + \frac{1}{2-\sigma} = A_2(f,\alpha) + R_1(\sigma) = A_2(f,\alpha),$$
$$|a_3| \le 1 + \frac{2!2}{(2-\sigma)(3-\sigma)} = A_3(f,\alpha) + R_2(\sigma) = A_3(f,\alpha),$$

because $R_1(\sigma) = R_2(\sigma) = 0$, which proves that (22) holds for $n = 2, 3$. Assume that (22) holds for n. We get from (21), after some calculations, that

$$|a_{n+1}| \le \frac{2\sigma}{n+1-\sigma}\sum_{k=1}^{n}|a_k| + \frac{(1-\sigma)(2n+1)}{n+1-\sigma}$$
$$\le A_{n+1}(f,\alpha) + R_n(\sigma) = M_{n+1}(f,\alpha).$$

It follows that (22) holds for $n+1$. This proves assertion (iv) of Theorem 1, while assertion (v) is obvious. The theorem is proved.

Remark. If in (19) and (16) equality holds for $n = 2, 3, 4$ then for $c_1 = c_2 = c_3 = 2\sigma$, $b_2 = b_4 = -b_1$, $b_3 = b_1$, $\lambda = 1$, it is easily checked that (22) is sharp for $n \le 4$. Indeed we find

$$a_2 = 1 + \frac{1}{2-\sigma}, \qquad a_3 = 1 + \frac{4}{(2-\sigma)(3-\sigma)},$$
$$a_4 = 1 + \frac{18}{(2-\sigma)(3-\sigma)(4-\sigma)} + \frac{\sigma^2-\sigma}{(2-\sigma)(3-\sigma)(4-\sigma)}.$$

However the sharpness of (22) for all n remains open.

We make the following conjecture.

CONJECTURE. Let $f \in \mathscr{S}_0$, $\alpha \in S_f^0$. Then

$$|a_n| \le A_n(f, \alpha) + R_{n-1}(\sigma) + H_n(\sigma), \qquad n \ge 2, \tag{$*$}$$

where

$$H_n(\sigma) = \sum_{k=3}^{n-2} \left[R_k(\sigma)(2\sigma)^{n-k-1} \Big/ \prod_{p=5}^{n+3-k} (p - \sigma) \right],$$

for $n \ge 5$ and $H_n(\sigma) = 0$ for $n < 5$.

Furthermore, if equality holds in (16) and (19) and if

$$c_n = 2\sigma, \qquad b_{2q} = -b_1, \qquad b_{2q-1} = b_1, \qquad n = 1, 2, \dots, q = 1, 2, \dots,$$

then for the a_n obtained from (20), $(*)$ is sharp.

THEOREM 2. *Let f_1, f_2 be functions in $\mathscr{S}_0$. Let $B(f_1, n)$, $B(f_2, n)$ be the corresponding bounds on the Taylor coefficients of f_1 and f_2 respectively, as these are defined in Theorem* 1(v). *Suppose $S_{f_1}^0 \subset S_{f_2}^0$. Then*

$$B(f_2, n) \le B(f_1, n). \tag{23}$$

PROOF. Let $\alpha \in S_{f_1}^0$. Let G_1 be the function obtained from f_1 exactly the same way as G was obtained from f in (13). Similarly, since α also belongs to $S_{f_2}^0$, let G_2 be the function obtained from f_2. We have

$$G_1(D) = S_{f_1}^0 \subset S_{f_2}^0 = G_2(D), \qquad G_1(0) = G_2(0) = 0,$$

and both G_1 and G_2 are regular and univalent in D. It follows that G_1 is subordinate to G_2, so $G_1(z) = G_2(\varphi(z))$, where φ is analytic in D and $|\varphi(z)| \le |z|$. We have $G_1'(z) = G_2'(\varphi(z))\varphi'(z)$, or

$$G_1'(0) = \mu(f_1, \alpha) = G_2'(0)\varphi'(0) = \mu(f_2, \alpha)\varphi'(0).$$

Since $|\varphi'(0)| \le 1$ we have

$$\mu(f_1, \alpha) \le \mu(f_2, \alpha). \tag{24}$$

Put

$$\sigma_1 = \frac{1}{1 + \mu(f_1, \alpha)}, \qquad \sigma_2 = \frac{1}{1 + \mu(f_2, \alpha)}.$$

We have from (24) that

$$\sigma_1 \ge \sigma_2. \tag{25}$$

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Now the function $M_n(f, \alpha) = A_n(f, \alpha) + R_{n-1}(\sigma)$, defined in the statement of Theorem 1, can be written as follows

$$M_n(f, \alpha) = 1 + \frac{n-1}{n-\sigma} + \frac{2\sigma}{n-\sigma} \cdot \left[\frac{1}{2-\sigma} + \frac{2!2}{(2-\sigma)(3-\sigma)} + \cdots + \frac{(n-2)!(n-2)}{(2-\sigma)\cdots(n-1-\sigma)}\right].$$

It is easily seen that the derivative of $M_n(f, \alpha)$ with respect to σ is non-negative, which implies that $M_n(f, \alpha)$ is an increasing function of σ. It follows that

$$M_n(f_1, \alpha) \geq M_n(f_2, \alpha). \tag{26}$$

By taking the infinum of the left side of (26) for $\alpha \in S^0_{f_1}$ and of the right side for $\alpha \in S^0_{f_2}$, we get (23) because $S^0_{f_1} \subset S^0_{f_2}$. This proves the theorem.

4. Examples and comments

EXAMPLE (from [1]). The function

$$f(z) = \frac{1}{2\varepsilon}\left[\left(\frac{1+z}{1-z}\right)^{\varepsilon} - 1\right], \qquad z \in D, \ 1 < \varepsilon < 2,$$

belongs to the class $\mathscr{S}_0$. This is easily seen if we sketch $f(D)$. More precisely let L_1, L_2 be the rays which start from the point $(-1/2\varepsilon, 0)$ and make with the positive x-axis the angles $(2-\varepsilon)\frac{\pi}{2}$, $(\varepsilon-2)\frac{\pi}{2}$ respectively. Then S^0_f is the open set which contains the origin and is bounded by the rays L_1, L_2. Let T be the symmetric set of S^0_f with respect to the line $x = -1/2\varepsilon$. Then $f(D) = \mathbb{C} - \overline{T}$.

If we choose $\alpha = 0 \in S^0_f$ then the function G considered in (13), which maps D onto S^0_f, is

$$G(z) = \frac{1}{2\varepsilon}\left[\left(\frac{1+z}{1-z}\right)^{2-\varepsilon} - 1\right], \qquad z \in D,$$

and we have $\mu(f, 0) = G'(0) = (2-\varepsilon)/\varepsilon$ and $\sigma = \varepsilon/2$.

Other examples can be found in [2, pages 196, 197].

We close with the following comment.

In [1] the authors present a different approach to the subject. Given $f \in \mathscr{S}$ the index δ of starlikeness of f is defined to be

$$\delta = \sup\{r : f(z) \text{ is a s.c.p of } f(D), \ \text{whenever } |z| < r\}.$$

Let Δ_δ be the class of all starlike functions whose index is equal to δ, $0 \leq \delta \leq 1$. For $f \in \Delta_\delta$ the following inequality holds:

$$|a_n| \leq \prod_{k=1}^{n-1} \frac{k(1+\delta) + 1 - (-\delta)^k}{k(1+\delta) + \delta + (-\delta)^k}. \tag{27}$$

The estimates given by (27) depend on δ, or equivalently on the size of $f(D_\delta)$ which (in the cases of interest, that is, when $0 < \delta < 1$) is *always* a bounded subset of S_f^0.

On the other hand the estimates, given in Theorem 1 above, depend on the entire set S_f^0. If S_f^0 is unbounded (see example given above) then $f(D_\delta)$ is a proper subset of S_f^0. Now it is possible in this case (that is, when S_f^0 is unbounded) that the "unused" part of S_f^0 "hides" some additional information on the a_n, including some concerning the sharpness of (27).

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169, Megalou Alexandrou Street
136 71 Thrakomakedones
Athens
Greece

https://doi.org/10.1017/S1446788700033334 Published online by Cambridge University Press