ON THE DERIVATIVES AT THE ORIGIN OF ENTIRE HARMONIC FUNCTIONS

by D. H. ARMITAGE

(Received 11 January, 1978)

1. Introduction. If f is an entire function in the complex plane such that

$$\max_{|z|=r} |f(z)| = O(e^{\alpha r}) \qquad (r \to \infty),$$

where $0 \leq \alpha < 1$, and all the derivatives of f at 0 are integers, then it is easy to show that f is a polynomial (see e.g. Straus [**10**]). The best possible result of this type was proved by Pólya [**9**]. The main aim of this paper is to prove two analogous results for harmonic functions defined in the whole of the Euclidean space $\mathbf{R}^n$, where $n \geq 2$ (i.e. entire harmonic functions).

Before stating the main results, we give some notations. A point of $\mathbf{R}^n$ is denoted by $X = (x_1, \ldots, x_n)$. Throughout the paper a denotes an n-tuple $(a_1, \ldots, a_n)$ of non-negative integers, and we put

$$|a| = a_1 + \ldots + a_n, \qquad a! = a_1! \ldots a_n!$$

and

$$D^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{a_n}.$$

We shall use m consistently to denote a non-negative integer. If f is an infinitely differentiable function in an open subset of $\mathbf{R}^n$, the norm of the gradient of order m of f is defined by

$$|\nabla_m f| = \left\{ m! \sum_{|a|=m} (D^a f)^2 (a!)^{-1} \right\}^{1/2}.$$

Thus $|\nabla_0 f| = |f|$ and $|\nabla_1 f|$ is the usual norm of the gradient (of order 1) of f. Also, it is easy to show that

$$|\nabla_m f| = \left\{ \sum_{b_1=1}^{n} \ldots \sum_{b_m=1}^{n} (\partial^m f / \partial x_{b_1} \ldots \partial x_{b_m})^2 \right\}^{1/2} \tag{1}$$

(see Calderón and Zygmund [**3**]), whence it follows that if h is harmonic in $\mathbf{R}^n$, then

$$|\nabla_m h|^2 = 2^{-m} \Delta^m (h^2), \tag{2}$$

Glasgow Math. J. **20** (1979) 147–154.

where Δ^m is the mth iterated Laplacian operator in $\mathbf{R}^n$. In particular, we note that by (1) our definition of $|\nabla_m f|$ agrees with that given by Kuran [**6**]. We denote the origin of $\mathbf{R}^n$ by O. If r is a positive number, the open ball and the sphere of centre O and radius r in $\mathbf{R}^n$ are denoted by $B(r)$ and $S(r)$ respectively. If a function g is defined and continuous at least on $S(r)$, then the mean of g over $S(r)$ is given by the equation

$$\mathcal{M}(g, r) = (s_n r^{n-1})^{-1} \int_{S(r)} g \, d\sigma,$$

where σ is the surface area measure on $S(r)$ and s_n is the surface area of $S(1)$.

THEOREM 1. *Let h be harmonic in $\mathbf{R}^n$ and suppose that*

$$\mathcal{M}(|h|, r) = O(e^{\alpha r}) \qquad (r \to \infty), \tag{3}$$

where $0 \leq \alpha < 1$. If $D^a h(O)$ is an integer for each n-tuple a, then h is a polynomial. The result is false with $\alpha = 1$.

THEOREM 2. *Let h be harmonic in $\mathbf{R}^n$ and suppose that* (3) *holds for some α such that $0 \leq \alpha < 1/\sqrt{2}$. If $|\nabla_m h(O)|$ is an integer for all m, then h is a polynomial. The result is false with $\alpha = 1/\sqrt{2}$.*

It will become obvious that, in proving Theorem 1, we need only suppose that $D^a h(O)$ is an integer when a is sufficiently large. Similarly, in Theorem 2 we need only suppose that $|\nabla_m h(O)|$ is an integer for all sufficiently large m. In fact, in Theorem 1, we require only that there is a positive integer p such that $D^a h(O)$ is an integer whenever $a_2 + \ldots + a_n \geq p$ and $a_1 = 0$ or 1, for the identity $\Delta^1 D^a h \equiv 0$, which holds for each a, will then imply that $D^a h(O) = 0$ for any a such that $|a| > p$.

Theorems 1 and 2 will follow easily from the following lemmas respectively.

LEMMA 1. *If h is harmonic in $\mathbf{R}^n$ and* (3) *holds for some non-negative number α, then*

$$D^a h(O) = O(|a|^{n-3/2} \alpha^{|a|}) \qquad (|a| \to \infty).$$

LEMMA 2. *If h is harmonic in $\mathbf{R}^n$ and* (3) *holds for some non-negative number α, then*

$$|\nabla_m h(O)| = O(m^{3n/4-1} (\alpha\sqrt{2})^m) \qquad (m \to \infty).$$

The special case of Lemma 1 in which $a_2 = \ldots = a_m = 0$ (so that $D^a h$ is an x_1-derivative) was proved in [**1**].

2. Preliminary results. In this section we reduce the proofs of Lemmas 1 and 2 to problems about harmonic polynomials.

The Poisson kernel of $B(r)$ is the function K_r, defined in $B(r) \times S(r)$ by the equation

$$K_r(X, Y) = (s_n r)^{-1} (r^2 - |X|^2) \, |X - Y|^{-n}, \tag{4}$$

where

$$|X| = (x_1^2 + \ldots + x_n^2)^{1/2}.$$

It is well known that if h is harmonic in an open set containing the closure $\bar{B}(r)$ of $B(r)$, then

$$h(X) = \int_{S(r)} K_r(X, Y)h(Y)\, d\sigma(Y) \qquad (X \in B(r))$$

(see e.g. Helms [5, p. 16]). Since K_r and all its partial derivatives with respect to $x_1, \ldots, x_n$ are continuous in $B(r) \times S(r)$, we have

$$D^a h(O) = \int_{S(r)} D^a K_r(O, Y)h(Y)\, d\sigma(Y) \tag{5}$$

for any a. The main problem thus becomes that of estimating $D^a K_r(O, Y)$, and this will be solved by expressing $K_r(\cdot, Y)$ as a series of harmonic polynomials and studying the terms of this series.

The vector space of all homogeneous harmonic polynomials of degree m in $\mathbf{R}^n$ is denoted by $\mathcal{H}_m$. (Note that $0 \in \mathcal{H}_m$). Brelot and Choquet [2] introduced the norm $\|\ \|$ on $\mathcal{H}_m$, defined by the equation

$$\|H\| = \left\{(s_n)^{-1} \int_{S(1)} H^2\, d\sigma\right\}^{1/2}.$$

We shall need the following results.

Theorem A. *If $Y \in \mathbf{R}^n \backslash \{O\}$, then there exists a unique element $I_{m,Y}$ (a Brelot-Choquet axial polynomial) of $\mathcal{H}_m$ such that $I_{m,Y}$ is invariant under rotation about the line OY (i.e. for each orthonormal transformation T of $\mathbf{R}^n$ for which $T(Y) = Y$, we have $I_{m,Y} \circ T = I_{m,Y}$) and*

$$I_{m,Y}(Y) = |Y|^m.$$

The polynomial $I_{m,Y}$ is given in $\mathbf{R}^n \backslash \{O\}$ by the equation

$$I_{m,Y}(X) = |X|^m P_m(t), \tag{6}$$

where

$$t = (x_1 y_1 + \ldots + x_n y_n)(|X|\,|Y|)^{-1} \tag{7}$$

and P_m is the n-dimensional Legendre polynomial of degree m. Further,

$$\|I_{m,Y}\|^2 = (\dim \mathcal{H}_m)^{-1} = (N(m, n))^{-1}, \quad \text{say}. \tag{8}$$

Most of this theorem can be found in [**2**]. The relation (6) is well known (see e.g. Müller [**8**]).

Theorem B. *The Poisson kernel K_r is given in $B(r)\times S(r)$ by the equation*

$$K_r(X, Y)=(s_n r^{n-1})^{-1}\sum_{k=0}^{\infty} N(k, n)r^{-k}I_{k,Y}(X). \tag{9}$$

When $X=O$ this equation is trivial. When $X\neq O$, we deduce it from (4), (6) and the equation

$$\sum_{k=0}^{\infty} N(k, n)u^k P_k(t)=(1-u^2)(1+u^2-2ut)^{-n/2} \qquad (0\leq u<1, -1\leq t\leq 1)$$

(see e.g. [**8**, p. 30]) by taking $u=|X|/r$ and t to be given by (7).

Lemma 3. *If h is harmonic in an open set containing $\bar{B}(r)$, then*

$$D^a h(O)=(s_n r^{n-1})^{-1}N(|a|, n)r^{-|a|}\int_{S(r)} D^a I_{|a|,Y}h(Y)\,d\sigma(Y).$$

From (4) and (8), we easily obtain

$$D^a h(O)=(s_n r^{n-1})^{-1}\int_{S(r)} D^a\left\{\sum_{k=0}^{\infty} N(k, n)r^{-k}I_{k,Y}(O)\right\}h(Y)\,d\sigma(Y). \tag{10}$$

Clearly

$$D^a I_{k,Y}(O)=0 \qquad (k\neq|a|), \quad D^a I_{|a|,Y}\equiv D^a I_{|a|,Y}(O).$$

Hence, to prove the lemma, it is enough to show that the operator D^a can be taken inside the summation in (10). Now, for each fixed Y on $S(r)$, the function $K_r(\cdot, Y)$ is harmonic in $B(r)$ and therefore real-analytic in $B(r)$. Hence $K_r(\cdot, Y)$ is equal to its multiple Taylor series about O in some neighbourhood of O. Bracketing together terms of equal degree in this Taylor series, we obtain a series of homogeneous polynomials, convergent to $K_r(\cdot, Y)$ in some neighbourhood of O. Since such a series is unique, it is equal term-by-term to the right-hand side of (9). Since the Taylor series can be differentiated term-by-term arbitrarily often, so also can the series in (10).

3. Harmonic polynomials. In view of Lemma 1, our interest now turns to the estimation of the partial derivatives of $I_{m,Y}$ at O.

Lemma 4. *If $H\in\mathscr{H}_m$ and $|a|=m$, then*

$$|D^a H|\leq m!(N(m, n))^{1/2}\,\|H\|,$$

and in particular

$$|D^a I_{m,Y}| \leq m!$$

for each Y *in* $\mathbf{R}^n \backslash \{O\}$.

When $m = 0$ the lemma is trivial. For positive values of m, we appeal to the inequality

$$N(m-1, n) \left\| \frac{\partial H}{\partial x_i} \right\|^2 \leq m^2 N(m, n) \|H\|^2 \qquad (i = 1, \ldots, n). \tag{11}$$

This inequality is implicit in the work of Calderón and Zygmund [**3**, Chapter 1]. (To deduce (11) from their work one needs an explicit formula for $N(m, n)$, for which see e.g. [**8**].) Kuran [**7**, p. 17] gives (11) explicitly together with the cases of equality. Observing that each of the operators $\partial/\partial x_i$ $(i = 1, \ldots, n)$ maps $\mathcal{H}_m$ into $\mathcal{H}_{m-1}$ and using (11) repeatedly, we find that

$$N(0, n) \|D^a H\|^2 \leq (m!)^2 N(m, n) \|H\|^2.$$

Since $N(0, n) = 1$ and $D^a H \equiv \|D^a H\|$, the main result of the lemma now follows. The special case where $H = I_{m,Y}$ comes from the main result and (8).

LEMMA 5. *If* $H \in \mathcal{H}_m$, *then*

$$|\nabla_m H| = \{m! n(n+2) \ldots (n+2m-2)\}^{1/2} \|H\|.$$

In particular,

$$|\nabla_m I_{m,Y}| = \{m! n(n+2) \ldots (n+2m-2)(N(m, n))^{-1}\}^{1/2}$$

for each Y *in* $\mathbf{R}^n \backslash \{O\}$.

When $m = 0$ the lemma is trivial. For positive values of m, we use a result of Kuran [**7**; Lemma 2] which states that if Q is a homogeneous polynomial of degree $2m$ in $\mathbf{R}^n$, then

$$\Delta^m Q = 2^m m! \{n(n+2) \ldots (n+2m-2)\}(s_n)^{-1} \int_{S(1)} Q \, d\sigma.$$

Applying this equation with $Q = H^2$ and using (2), we obtain the main result of the lemma, from which by using (8) we obtain the particular result for $H = I_{m,Y}$.

4. Proof of Lemmas 1 and 2. To prove Lemma 1, we have, by Lemmas 3 and 4, for each positive number r

$$\begin{aligned} |D^a h(O)| &\leq (s_n)^{-1} r^{-|a|-n+1} N(|a|, n) \int_{S(r)} |D^a I_{|a|,Y} \, h(Y)| \, d\sigma(Y) \\ &\leq r^{-|a|} N(|a|, n) \, |a|! M(|h|, r) \\ &\leq A r^{-|a|} N(|a|, n) \, |a|! e^{\alpha r}, \end{aligned}$$

https://doi.org/10.1017/S0017089500003864 Published online by Cambridge University Press

where A is the constant implied by the O-notation in (3). Now, there is a constant C, depending only on n, such that

$$N(m, n) \leq Cm^{n-2} \qquad (m \geq 1).$$

Hence

$$|D^a h(O)| \leq ACr^{-|a|}\,|a|^{n-2}\,|a|!\,e^{\alpha r} \qquad (|a| \geq 1, r > 0).$$

In particular, taking $r = |a|/\alpha$, we obtain

$$|D^a h(O)| \leq AC\,|a|^{n-2}\,|a|!(\alpha e)^{|a|}\,|a|^{-|a|} \qquad (|a| \geq 1),$$

and the theorem now follows by an application of Stirling's formula.

To prove Lemma 2, we have, by Lemma 1 and the Cauchy–Schwarz inequality, for each positive number r

$$\begin{aligned}|\nabla_m h(O)| &= (s_n r^{n-1})^{-1} N(m, n) m! r^{-m} \left\{ \sum_{|a|=m} (a!)^{-1} \left(\int_{S(r)} D^a I_{m,Y} h(Y)\, d\sigma(Y) \right)^2 \right\}^{1/2} \\ &\leq (s_n r^{n-1})^{-1} N(m, n) m! r^{-m} \left\{ \sum_{|a|=m} (a!)^{-1} \int_{S(r)} (D^a I_{m,Y})^2\, |h(Y)|\, d\sigma(Y) \right. \\ &\quad \left. \times \int_{S(r)} |h(Y)|\, d\sigma(Y) \right\}^{1/2} \\ &= (s_n r^{n-1})^{-1} N(m, n) r^{-m} \left\{ \int_{S(r)} |\nabla_m I_{m,Y}|^2\, |h(Y)|\, d\sigma(Y) \right. \\ &\quad \left. \times \int_{S(r)} |h(Y)|\, d\sigma(Y) \right\}^{1/2}.\end{aligned}$$

By Lemma 5, we now have

$$\begin{aligned}|\nabla_m h(O)| &\leq \{m!\,n(n+2)\ldots(n+2m-2) N(m, n)\}^{1/2} r^{-m} M(|h|, r) \\ &\leq A\{Cm!\,n(n+2)\ldots(n+2m-2) m^{n-2}\}^{1/2} r^{-m} e^{\alpha r},\end{aligned}$$

where A and C are as before. Hence, taking $r = m/\alpha$, we obtain

$$\begin{aligned}|\nabla_m h(O)| &\leq A\{Cm!\,n(n+2)\ldots(n+2m-2) m^{n-2}\}^{1/2} (\alpha e)^m m^{-m} \\ &= O(\{(m!)^{-1} n(n+2)\ldots(n+2m-2) m^{n-1}\}^{1/2} \alpha^m) \qquad (m \to \infty),\end{aligned}$$

by Stirling's formula. When $m \geq 1$,

$$(m!)^{-1} n(n+2)\ldots(n+2m-2) = 2^m \left(1 + \frac{\frac{1}{2}n - 1}{m}\right)\left(1 + \frac{\frac{1}{2}n - 1}{m-1}\right)\ldots\left(1 + \frac{\frac{1}{2}n - 1}{1}\right)$$

and

$$\log\left\{\left(1+\frac{\frac{1}{2}n-1}{m}\right)\left(1+\frac{\frac{1}{2}n-1}{m-1}\right)\cdots\left(1+\frac{\frac{1}{2}n-1}{1}\right)\right\}\leq(\tfrac{1}{2}n-1)\sum_{j=1}^{m}j^{-1}\leq(\tfrac{1}{2}n-1)(\log m+1).$$

Hence

$$(m!)^{-1}n(n+2)\ldots(n+2m-2)m^{n-1}=O(2^m m^{3n/2-2})\qquad(m\to\infty),$$

and the lemma follows.

5. Proofs of Theorems 1 and 2. If h satisfies the hypotheses of Theorem 1, then, by Lemma 1,

$$D^a h(O)\to 0\qquad(|a|\to\infty).$$

Hence there exists a non-negative integer q such that $D^a h(O)=0$ whenever $|a|\geq q$. It follows that the multiple Taylor series of h about O has only finitely many non-zero terms and hence that h, being equal in $\mathbf{R}^n$ to the sum of this series (see e.g. [**4**]), is a polynomial.

If h satisfies the hypotheses of Theorem 2, then by Lemma 2,

$$|\nabla_m h(O)|\to 0\qquad(m\to\infty).$$

Hence there exists a non-negative integer q such that $|\nabla_m h(O)|=0$ when $m\geq q$. This implies that $D^a h(O)=0$ when $|a|\geq q$ and hence that h is a polynomial.

Consideration of the functions h_1 and h_2, defined in $\mathbf{R}^n$ by the equations

$$h_1(X)=e^{x_1}\cos x_2$$

and

$$h_2(X)=e^{x_1/\sqrt{2}}\{\cos(x_2/\sqrt{2})+\sin(x_2/\sqrt{2})\},$$

shows that Theorems 1 and 2 fail with $\alpha=1$ and $\alpha=1/\sqrt{2}$, respectively. The verifications are left to the reader.

REFERENCES

1. D. H. Armitage, Uniqueness theorems for harmonic functions which vanish at lattice points, *J. Approximation Theory* (to appear).

2. M. Brelot and G. Choquet, Polynômes harmoniques et polyharmoniques, *Deuxième colloque sur les équations aux derivées partielles, Bruxelles* (1954), 45–66.

3. A. P. Calderón and A. Zygmund, On higher gradients of harmonic functions, *Studia Math.* **24** (1964), 211–226.

https://doi.org/10.1017/S0017089500003864 Published online by Cambridge University Press

4. W. K. Hayman, Power series expansions for harmonic functions, *Bull. London Math. Soc.* **2** (1970), 152–158.

5. L. L. Helms, *Introduction to potential theory* (New York, 1969).

6. Ü. Kuran, On norms of higher gradients of harmonic functions, *J. London Math. Soc.* (2) **3** (1971), 761–766.

7. Ü. Kuran, On Brelot-Choquet axial polynomials, *ibid.* (2) **4** (1971), 15–26.

8. C. Müller, *Spherical harmonics*, Lecture Notes in Mathematics, No. 17, (Springer-Verlag, 1966).

9. G. Pólya, Über die kleinsten ganzen Funktionen, deren sämtliche Derivierten im Punkte $z=0$ ganzzahlig sind, *Tôhoku Math. J.* **19** (1921), 65–68.

10. E. G. Straus, On entire functions with algebraic derivatives at certain algebraic points, *Ann. of Math.* **52** (1950), 188–198.

THE QUEEN'S UNIVERSITY,
BELFAST BT7 1NN.

https://doi.org/10.1017/S0017089500003864 Published online by Cambridge University Press