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# FINITELY DOMINATED COVERING SPACES OF 3- AND 4-MANIFOLDS

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**Abstract**

If $P$ is a closed 3-manifold the covering space associated to a finitely presentable subgroup $\nu$ of infinite index in $\pi_1(P)$ is finitely dominated if and only if $P$ is aspherical or $\widetilde{P} \simeq S^2$. There is a corresponding result in dimension 4, under further hypotheses on $\pi$ and $\nu$. In particular, if $M$ is a closed 4-manifold, $\nu$ is an ascendant, $FP_3$, finitely-ended subgroup of infinite index in $\pi_1(M)$, $\pi$ is virtually torsion free and the associated covering space is finitely dominated then either $M$ is aspherical or $\widetilde{M} \simeq S^2$ or $S^3$. In the aspherical case such an ascendant subgroup is usually $Z$, a surface group or a $PD_3$-group.

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A space $X$ is *finitely dominated* if there is a finite cell complex $Y$ with maps $f : X \to Y$ and $g : Y \to X$ such that $gf \sim id_X$. (Thus, $X$ is homotopically a retract of $Y$.) If the universal covering space $\widetilde{M}$ of a 4-manifold $M$ is finitely dominated then one of the following holds: $M$ is aspherical; $\widetilde{M}$ is homeomorphic to $S^2 \times R^2$ or $S^3 \times R$; or $\pi = \pi_1(M)$ is finite. More generally, if $M$ has a finitely dominated covering space $M_\nu$ such that $\nu = \pi_1(M_\nu)$ is an $FP_3$ normal subgroup of infinite index in $\pi$ there is the additional possibility (when $\nu$ has infinitely many ends) that $M$ might have a finite covering space which is homotopy equivalent to the mapping torus of a self-homotopy equivalence of a $PD_3$-complex. (See [9, Theorems 3.9, 10.1 and 11.1].)

In this paper we relax the hypotheses on $\nu$ further. The arguments we use apply equally well to covering spaces of low-dimensional Poincaré duality complexes. We begin in dimension 3, as the surface case is trivial. In Section 2 we show that if $M$ is a $PD_3$-complex with torsion-free fundamental group $\pi$ and $\nu$ is a subgroup of infinite index in $\pi$ the associated covering space $M_\nu$ is finitely dominated if and only if $\nu$ is finitely presentable and $\pi_2(M) = 0$ or $Z$. In the rest of the paper we consider $PD_4$-complexes. Here we need to assume that the subgroup $\nu$ be $FP_3$ and ascendant in $\pi$. (The notion of ascendant subgroup is recalled in Section 1 below.) If $M$ is aspherical $\pi$ is a $PD_4$-group and finitely dominated covering spaces correspond to $FP_3$ subgroups of $\pi$. In Section 3 we show that an ascendant $FP_3$ subgroup of infinite index in a $PD_4$-group is usually a $PD_r$-group for some $r \leq 3$. However, we have not been able to eliminate other possibilities completely. For instance, it is not known whether a Baumslag–Solitar group may be an ascendant subgroup of a $PD_4$-group. In Section 4 we consider the case of a $PD_4$-complex $M$ with a finitely dominated infinite covering space $M_\nu$ corresponding to an ascendant $FP_3$ subgroup $\nu$, and give homological conditions on $\pi$ and $\nu$ under which either $M$ is aspherical or $\widetilde{M}$ is homotopy equivalent to $S^2$ or $S^3$.

## 1. Notation

The Hirsch–Plotkin radical $\sqrt{\pi}$ of a group $\pi$ is the maximal locally nilpotent, normal subgroup of $\pi$. The Hirsch length $h(\nu)$ of a finitely generated nilpotent group $\nu$ is the number of infinite cyclic factors of a composition series for the group; $h(\sqrt{\pi})$ is the least upper bound of $h(\nu)$ as $\nu$ varies over finitely generated subgroups of $\sqrt{\pi}$. If $G$ is a subgroup of $\pi$ then $C_\pi(G)$ and $N_\pi(G)$ are the centralizer and normalizer of $G$ in $\pi$, respectively. The centre of $G$ is $\zeta G = G \cap C_\pi(G)$.

A subgroup $K$ of a group $G$ is *ascendant* if there is an increasing sequence of subgoups $N_\alpha$, indexed by an ordinal $\beth + 1$, such that $N_0 = K$, $N_\alpha$ is normal in $N_{\alpha+1}$ if $\alpha < \beth$, $N_\beta = \cup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \beth$ and $N_\beth = G$. (If $\beth$ is finite $K$ is *subnormal* in $G$.) Such ascendant series are well suited to arguments by transfinite induction. For instance, it is easily seen that $\sqrt{K} \leq \sqrt{N_\alpha}$, for all $\alpha \leq \beth$. We write $\mathbb{Z}$ for the ring of integers and $Z$ for an abstract infinite cyclic group. If $A$ is an abelian group and $I$ a set let $\oplus^I A$ be the direct sum of copies of $A$ indexed by $I$.

We shall assume that the fundamental group $\pi$ of a space or cell complex $X$ acts on the universal cover $\widetilde{X}$ on the left, and so the (cellular) chain complex $C_*(\widetilde{X})$ is naturally a complex of left $\mathbb{Z}[\pi]$-modules. The equivariant cochain complex $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), \mathbb{Z}[\pi])$ is then a complex of right $\mathbb{Z}[\pi]$-modules. Let $E(\pi) = H^1(\pi; \mathbb{Z}[\pi])$; this is naturally a right $\mathbb{Z}[\pi]$-module.

If $X$ is a Poincaré duality complex with fundamental group $\pi$ and orientation character $w = w_1(X)$ and $R$ is a right $\mathbb{Z}[\pi]$-module we let $\overline{R}$ be the conjugate left module, with module structure given by $g.r = w(g)rg^{-1}$ for all $g \in \pi$ and $r \in R$.

## 2. $PD_3$-complexes

It is easy to see that an infinite covering space of a closed surface is finitely dominated if and only if its fundamental group is finitely generated. Here we show that there is a similar criterion for an infinite covering space of a $PD_3$-complex to be finitely dominated.

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LEMMA 1. *Let $\pi$ be a finitely generated torsion free group which is not free. Then $E(\pi)$ is a free right $\mathbb{Z}[\pi]$-module.*

PROOF. Since $\pi$ is finitely generated it is a free product of finitely many indecomposable groups, and since $\pi$ is torsion-free the latter either have one end or are infinite cyclic. Thus, $\pi$ is an iterated Higman–Neumann–Neumann (HNN) extension with base a nontrivial free product of one-ended groups and trivial associated subgroups. In other words, $\pi$ is the fundamental group of a finite graph of groups $\mathcal{G}$ in which all of the vertex groups have one end and all of the edge groups are trivial. It follows from the Mayer–Vietoris sequences of [1, Theorems 2.10 and 2.11] that $E(\pi)$ is a free right $\mathbb{Z}[\pi]$-module with basis corresponding to the edges of $\mathcal{G}$. □

When $\pi$ is a free group $E(\pi)$ is a finitely presentable $\mathbb{Z}[\pi]$-module of projective dimension 1, and we shall need a different result.

LEMMA 2. *Let $\pi = \nu * \sigma$, where $\nu$ is finitely generated, and let $I = \nu\backslash\pi/\nu$ be the double coset space. Then $\oplus^I E(\nu)$ is a direct summand of the abelian group $E(\pi) \otimes_\nu \mathbb{Z}$.*

PROOF. The group $\nu$ is clearly a retract of $\pi$ and so $H^1(\nu; \mathbb{Z}[\pi])$ is a direct summand of $E(\pi)$ (as a right $\mathbb{Z}[\pi]$-module). Now $H^1(\nu; \mathbb{Z}[\pi]) \cong E(\nu) \otimes_\nu \mathbb{Z}[\pi]$, since $\nu$ is finitely generated. Therefore, $H^1(\nu; \mathbb{Z}[\pi]) \otimes_\nu \mathbb{Z} \cong E(\nu) \otimes_\nu \mathbb{Z}[\pi/\nu] \cong \oplus^I E(\nu)$ is a direct summand of $E(\pi) \otimes_\nu \mathbb{Z}$ (as an abelian group). □

THEOREM 3. *Let $P$ be a $PD_3$-complex with fundamental group $\pi$ and let $\nu$ be a subgroup of infinite index in $\pi$. Then the associated covering space $P_\nu$ is finitely dominated if and only if $\pi$ is virtually $Z$ or $\pi_2(P) = 0$ and $\nu$ is finitely presentable.*

PROOF. The Hurewicz theorem and Poincaré duality give isomorphisms of left $\mathbb{Z}[\pi]$-modules $\Pi = \pi_2(P) \cong H_2(P; \mathbb{Z}[\pi]) \cong \overline{E(\pi)}$. The fundamental group of a $PD_3$-complex is a free product of $PD_3$-groups with a finitely generated, virtually free group, and so is virtually torsion free [4]. Moreover, a complex is finitely dominated if and only if it has a finite covering space which is finitely dominated. Thus we may assume that $\pi$ is torsion free, after passing to a finite covering space, if necessary.

The spectral sequence of the covering $\widetilde{P} \to P_\nu$ gives an exact sequence

$$H_3(\nu; \mathbb{Z}) \to \mathbb{Z} \otimes_\nu \Pi \to H_2(P_\nu; \mathbb{Z}) \to H_2(\nu; \mathbb{Z}) \to 0.$$

We may assume that $\nu$ is finitely presentable. It is then a finite free product of finitely presentable subgroups of $PD_3$-groups with a free group, by the Kurosh subgroup theorem. In particular, $H_s(\nu; \mathbb{Z})$ is finitely generated for all $s \geq 0$, and so $H_2(P_\nu; \mathbb{Z})$ is finitely generated if and only if $\mathbb{Z} \otimes_\nu \Pi$ is finitely generated.

If $\pi$ is free of rank 1 then $\pi \cong Z$ and $E(\pi) \cong \mathbb{Z}$. Hence, $\pi_2(P)$ is infinite cyclic, so $\widetilde{P} \simeq S^2$. In this case every covering space is finitely dominated.

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If $\pi$ is free of rank $r > 1$ then we may assume that $\nu$ is a proper free factor of $\pi$, after passing to a subgroup of finite index, if necessary [3]. We may also assume that $P$ is orientable. It is easy to see that the double set space $I = \nu\backslash\pi/\nu$ is infinite. Since $\mathbb{Z} \otimes_\nu \Pi \cong \overline{E(\pi) \otimes_\nu \mathbb{Z}}$ is not finitely generated, by Lemma 2, $P_\nu$ cannot be finitely dominated.

If $\pi$ is not free $\Pi \cong \mathbb{Z}[\pi]^s$ for some $s \geq 0$, by Lemma 1. Thus, $\mathbb{Z} \otimes_\nu \Pi$ is free of infinite rank as an abelian group, unless $s = 0$. Thus, if $P_\nu$ is finitely dominated $s = 0$ and so $\Pi = 0$.

Suppose conversely that $\Pi = 0$ and that $\nu$ is a finitely presentable subgroup of infinite index in $\pi$. Then the universal covering space $\widetilde{P}$ is contractible, and so $P$ is aspherical. Therefore $c.d.\nu \leq 2$, by [16], and $P_\nu \simeq K(\nu, 1)$. Let $Y$ be the finite 2-complex determined by a finite presentation for $\nu$. The cellular chain complex for $\widetilde{Y}$ gives an exact sequence of $\mathbb{Z}[\nu]$-modules

$$0 \to \pi_2(Y) \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0,$$

where $C_q = C_q(\widetilde{Y})$ is a finitely generated free $\mathbb{Z}[\nu]$-module, for $q \leq 2$. Since $c.d.\nu \leq 2$ the module of 1-cycles $Z_1 = \mathrm{Im}(\partial_2)$ is projective, and so $C_2 \cong Z_1 \oplus \pi_2(Y)$. Thus, $\pi_2(Y)$ is a finitely generated projective module. Let $F$ be a free $\mathbb{Z}[\nu]$-module of countably infinite rank. Then $\pi_2(Y) \oplus F \cong F$, by the 'Eilenberg swindle'. Hence, we may construct a $K(\pi, 1)$ complex $K$ by adding countably many 3-cells to $Y \vee V$, where $V$ is a countable wedge of 2-spheres. Let $c : Y \to K$ be the classifying map and $p : K \to Y$ be the map which collapses $V$ and the adjoined 3-cells. Then $cp \sim id_K$, and so $P_\nu \simeq K$ is finitely dominated. □

In particular, a closed 3-manifold has a finitely dominated infinite covering space if and only if its universal covering space is contractible or homotopy equivalent to $S^2$ or $S^3$.

## 3. Poincaré duality groups

Subgroups of $PD_n$-groups are the algebraic analogues of covering spaces of aspherical $PD_n$-complexes. The analogues of finitely dominated covering spaces are the $FP_{n-1}$ subgroups, for which the trivial module $\mathbb{Z}$ has a projective resolution which is finitely generated in degrees $\leq n-1$. (There is then a finite projective resolution of length at most $n$, since either $\nu$ is a $PD_n$-group or $c.d.\nu < n$ [16].) The algebraic notion is broader in one respect: we do not assume that the $PD_n$-groups or their $FP$ subgroups are finitely presentable.

In [10] it was shown that if $\nu$ is an $FP_2$ ascendant subgroup of infinite index in a $PD_3$-group $\pi$ then either $\nu \cong Z$ and is normal in $\pi$ or $\nu$ is a $PD_2$-group and $[\pi : N_\pi(\nu)] < \infty$ or $\pi$ is a virtually poly-$Z$ group (and every subgroup is $FP_2$).

THEOREM 4. *Let $G$ be a nontrivial $FP_3$ normal subgroup of infinite index in a $PD_4$-group $\pi$. Then either:*

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(1) *$G$ is a $PD_3$-group and $\pi/G$ has two ends;*
(2) *$G$ is a $PD_2$-group and $\pi/G$ is virtually a $PD_2$-group; or*
(3) *$G \cong Z$, $H^s(\pi/G; \mathbb{Z}[\pi/G]) = 0$ for $s \leq 2$ and $H^3(\pi/G; \mathbb{Z}[\pi/G]) \cong Z$.*

PROOF. The subgroup $G$ is $FP$, since $c.d.G < 4$ (see [16]), and hence so is $\pi/G$. The $E_2$ terms of the Lyndon–Hochschild–Serre (LHS) spectral sequence with coefficients $\mathbb{Q}[\pi]$ can then be expressed as tensor products $E_2^{pq} = H^p(\pi/G; \mathbb{Q}[\pi/G]) \otimes H^q(G; \mathbb{Q}[G])$. If $H^j(\pi/G; \mathbb{Q}[\pi/G])$ and $H^k(G; \mathbb{Q}[G])$ are the first nonzero such cohomology groups then $E_2^{jk}$ persists to $E_\infty$. Hence, $j + k = 4$ and $E_2^{jk} \cong \mathbb{Q}$. Therefore, $H^j(\pi/G; \mathbb{Q}[\pi/G])$ and $H^{n-j}(G; \mathbb{Q}[G])$ each have dimension 1 over $\mathbb{Q}$. In particular, $\pi/G$ has one or two ends and $G$ is a $PD_{4-j}$-group over $\mathbb{Q}$ [6]. If $\pi/G$ has two ends then it is virtually $Z$, and so $G$ is a $PD_3$-group (over $\mathbb{Z}$), by [1, Theorem 9.11]. If $H^2(G; \mathbb{Q}[G]) \cong H^2(\pi/G; \mathbb{Q}[\pi/G]) \cong \mathbb{Q}$ then $G$ and $\pi/G$ are virtually $PD_2$-groups [2]. Since $G$ is torsion-free it must be a $PD_2$-group. The only remaining possibility is (3). □

Do the conclusions of this theorem hold if the hypothesis that $G$ be $FP_3$ is relaxed to '$G$ is $FP_2$'? (If $G$ is an $FP_2$ normal subgroup of a $PD_4$-group $\pi$ and $\pi/G$ is virtually a $PD_r$-group then $G$ is a $PD_{4-r}$-group [11].) If $v.c.d.\pi/G < \infty$ then $\pi/G$ is virtually a $PD$-group in case (3) also, by [1, Theorem 9.11].

COROLLARY. *If $G$ is an $FP_3$ normal subgroup of infinite index in $\pi$ and $K$ is an ascendant $FP_2$ subgroup of $G$ then $K$ is a $PD_k$-group for some $k < 4$.*

PROOF. This follows immediately from Theorem 4 together with the [10, corollary of Theorem 11]. □

We shall consider next $FP_3$ ascendant subgroups of $PD_4$-groups.

THEOREM 5. *Let $G$ be a nontrivial $FP_3$ ascendant subgroup of infinite index in a $PD_4$-group $\pi$. If $G$ has finitely many ends then one of the following holds:*

(1) *$G$ is a $PD_3$-group, $[\pi : N_\pi(G)] < \infty$ and $N_\pi(G)/G$ has two ends;*
(2) *$c.d.G = 3$ and $H^2(G; \mathbb{Z}[G])$ is not finitely generated as an abelian group;*
(3) *$G$ is a $PD_2$-group, $[\pi : N_\pi(G)] < \infty$ and $\pi$ is virtually the group of a surface bundle over a surface;*
(4) *$G$ is a $PD_2$-group, $\zeta G = 1$ and $\pi$ is virtually the group of the mapping torus of a self homeomorphism of a surface bundle over the circle;*
(5) *$c.d.G = 2$, $\chi(G) = 0$, $H^2(G; \mathbb{Z}[G])$ is not finitely generated as an abelian group and $[\pi : N_\pi(G)] = \infty$; or*
(6) *$G \cong Z$ and either $G < \sqrt{\pi} < \pi$ is a subnormal chain or $\pi$ is virtually nilpotent of Hirsch length 4.*

PROOF. Let $G = N_0 < N_1 < \cdots < N_\beth = \pi$ be an ascendant sequence, and let $\phi$ be the union of the finite ordinals $\leq \beth$. If $G$ is normal in $\pi$ then the theorem follows from Theorem 4. Otherwise, replacing $N_1$ by the union of the terms $N_\alpha$ which normalize $G$ and reindexing, if necessary, we may assume that $G$ is not normal in $N_2$.

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Since $[\pi : G] = \infty$ we have $c.d.G < 4$, by [16]. Suppose first that $c.d.G = 3$ and that $H^2(G; \mathbb{Z}[G])$ is finitely generated as an abelian group. Then $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by [5] or [2]. If $\phi$ is infinite then $N_\phi$ is not finitely generated, and so $c.d.N_\phi = 4$, by [8, Theorem 3.3]. However, then $[\pi : N_\phi] < \infty$ [16] and so $N_\phi$ is finitely generated. Therefore, $\phi$ is finite, so $N_\phi$ is one-ended, $FP$ and ascendant in $\pi$, and it is easily seen that the theorem holds for $G$ if it holds for $N_\phi$. Thus, we may assume that $[N_1 : G] = \infty$. It follows immediately from the LHS spectral sequence that $H^s(N_1; W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[N_1]$-module $W$. Hence, $c.d.N_1 = 4$ and so $[\pi : N_1] < \infty$, by [16]. Hence, $N_1$ is a $PD_4$-group and (1) follows from Theorem 4. If $c.d.G = 3$ and $H^2(G; \mathbb{Z}[G])$ is not finitely generated as an abelian group (2) holds.

Suppose now that $c.d.G = 2$ and that $\chi(G) \neq 0$. If $[N_i : G]$ is finite, then $\chi(G) = [N_i : G]\chi(N_i)$. Hence, we again find that $\phi$ is finite. If $G_1 < G_2$ are two such groups with $G_1$ normal in $G_2$, then $[G_2 : G_1]$ is finite, by [1, Theorem 8.2]. Moreover, if $G_2$ is normal in $J$ then $[J : N_J(G_1)] < \infty$, since $G_2$ has only finitely many subgroups of index $[G_2 : G_1]$. Therefore, we may assume that $G$ is maximal among normal subgroups of $N_1$ with cohomological dimension 2 and that $[N_1 : G] = \infty$. If $N_1 = \pi$, then (3) holds, by Theorem 4. Otherwise, we may assume that $G$ is not normal in $N_2$, as observed earlier, and so there is an $n$ in $N_2$ such that $nGn^{-1} \neq G$. Let $H = G.nGn^{-1}$. Then $G < H$ and $H$ is normal in $N_1$, so $[H : G] = \infty$ and $c.d._Q H = 3$. Moreover, $H$ is $FP$ and $H^s(H; \mathbb{Z}[H]) = 0$ for $s \leq 2$, so either $N_1/H$ is locally finite or $c.d_{\mathbb{Q}} N_1 > c.d._{\mathbb{Q}} H$, by [1, Theorem 8.2]. If $N_1/H$ is locally finite but not finite, then we again have $c.d_{\mathbb{Q}} N_1 > c.d._{\mathbb{Q}} H$, by [8, Theorem 3.3]. If $c.d._{\mathbb{Q}} N_1 = 4$, then $[\pi : N_1] < \infty$, so $N_1$ is a $PD_4$-group and (3) holds, by Theorem 4. Otherwise $[N_1 : H] < \infty$ and then $c.d.N_1 = 3$, $N_1$ is $FP$ and $H^s(N_1; \mathbb{Z}[N_1]) = 0$ for $s \leq 2$. Hence, $N_1$ is a $PD_3$-group by (1), and so (4) holds.

Suppose that $\chi(G) = 0$ and that $G$ is a $PD_2$-group. Then $G \cong Z^2$ or $Z \times_{-1} Z$, so $h(\sqrt{\pi}) \geq 2$ and $\chi(\pi) = 0$. We may assume that $\pi$ is orientable, so $\mathrm{Hom}(\pi, Z) \neq 0$. If $h(\sqrt{\pi}) > 2$ then $\pi$ is virtually poly-$Z$, by [9, Theorem 8.1]. Therefore, we may also assume that $h(\sqrt{\pi}) = 2$. In this case $\sqrt{\pi} \cong Z^2$ and $\pi$ is virtually the group of a torus bundle over a surface, by [9, Theorem 9.2]. Since $[\sqrt{\pi} : G] < \infty$ it follows also that $[\pi : N_\pi(G)] < \infty$ and so (3) holds. If $c.d.G = 2$ but $G$ is not a $PD_2$-group then $H^2(G; \mathbb{Z}[G])$ is not finitely generated as an abelian group [6] and $[\pi : N_\pi(G)] = \infty$, and so (5) covers the remaining possibilities with one end.

If $G$ has two ends, then $G \cong Z$, so $G \leq \sqrt{\pi}$. If $h = h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi}$ is abelian of rank $h$, by [9, Theorem 9.2]. If $h > 2$ then $\pi$ is virtually poly-$Z$ of Hirsch length 4, by [9, Theorem 8.1]. If $\sqrt{\pi}$ is abelian or nilpotent of class 2 then $G$ is a normal subgroup of $\sqrt{\pi}$; otherwise $\pi$ is virtually nilpotent of type $\mathbb{N}il^4$, by [9, Theorem 1.5]. □

To what extent can the hypotheses be relaxed? Are all ascendant $FP$ subgroups $PD$-groups? If so then cases (2) and (5) cannot arise. (This is certainly so if there is a subnormal sequence consisting of $FP$ subgroups.) Can a finitely generated noncyclic free group be an ascendant subgroup of a $PD_4$-group?

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EXAMPLE. Let $G$ be a $PD_2$-group such that $\zeta G = 1$. Let $\theta : G \to G$ have infinite order in $Out(G)$, and let $\lambda : G \to Z$ be an epimorphism. Let $\pi = (G \times Z) \times_\phi Z$ where $\phi(g, n) = (\theta(g), \lambda(g) + n)$ for all $g \in G$ and $n \in Z$. Then $G$ is subnormal in $\pi$ but this group is not virtually the group of a surface bundle over a surface.

Any group with a finite two-dimensional Eilenberg–Mac Lane complex is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to $D^4$. On applying the reflection group trick of Davis to the boundary we see that each such group embeds in a $PD_4$-group (see [12]). The simplest such groups $G$ with $\chi(G) = 0$ which are not $PD_2$-groups are the Baumslag–Solitar 1-relator groups $G_{p,q} = \langle a, t \mid ta^p t^{-1} = a^q \rangle$ with $|pq| > 1$. Can they be realized as *ascendant* subgroups of $PD_4$-groups?

## 4. $PD_4$-complexes

In this section we consider $PD_4$-complexes $M$ with a finitely dominated covering space associated to an ascendant $FP_3$ subgroup of $\pi_1(M)$.

THEOREM 6. *Let $M$ be a $PD_4$-complex with fundamental group $\pi$ and let $\nu$ be an ascendant $FP_3$ subgroup of infinite index in $\pi$. Suppose that the associated covering space $M_\nu$ is finitely dominated. Then:*

(1) *if $\nu$ is finite then the universal covering space $\widetilde{M}$ is contractible or homotopy equivalent to $S^2$ or to $S^3$, and $[\pi : N_\pi(\nu)]$ is finite;*

(2) *if $\nu$ has one end then $M$ is aspherical;*

(3) *if $\nu$ has two ends then either $M$ is aspherical or it is finitely covered by $S^2 \times S^1 \times S^1$ or $h(\sqrt{\pi}) = 1$ and $H^2(\pi; \mathbb{Z}[\pi])$ is not finitely generated as an abelian group;*

(4) *if $\nu$ has infinitely many ends and $\nu \leq N$ where $N$ is an $FP_2$ normal subgroup of infinite index in $\pi$ then either $M$ has a finite covering space which is homotopy equivalent to the mapping torus of a self-homotopy equivalence of a $PD_3$-complex and $[\pi : N_\pi(\nu)]$ is finite or $M$ is aspherical and $N$ is not $FP_3$.*

PROOF. Let $\nu = N_0 < N_1 < \cdots < N_{\beth} = \pi$ be an ascendant sequence. Suppose first that $\nu$ is finite. Then $\widetilde{M}$ is also finitely dominated, and so is contractible (in which case $\nu = 1$) or is homotopy equivalent to $S^2$ or $S^3$, by [9, Theorem 3.9]. If $\widetilde{M} \simeq S^2$ the kernel of the natural homomorphism from $\pi$ to $Aut(\pi_2(M))$ is torsion free. Hence, $\nu = Z/2Z$ and so $\nu$ is central in $N_1$. Moreover as it is the torsion subgroup of $\zeta N_1$ it is characteristic in $N_1$, and hence normal in $N_2$. Transfinite induction now shows that $\nu$ is normal in $\pi$. If $\widetilde{M} \simeq S^3$ then $\pi$ has two ends, and so $[\pi : N_\pi(\nu)]$ is finite.

If $\nu$ is infinite then transfinite induction using the LHS spectral sequence, [8, Theorem 3.3] and [15, Lemma 4.1] shows that $\pi$ has one end, and that if $\nu$ has one end $H^2(\pi; \mathbb{Z}[\pi]) = 0$. Since $\nu$ is $FP_3$ and $M_\nu$ is finitely dominated $\pi_2(M) = \pi_2(M_\nu)$ is finitely generated as a $\mathbb{Z}[\nu]$-module, and so $\mathrm{Hom}_\pi(\pi_2(M), \mathbb{Z}[\pi]) = 0$.

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Therefore, $\pi_2(M) \cong \overline{H^2(\pi; \mathbb{Z}[\pi])}$, by [9, Lemma 3.3]. In particular, if $\nu$ has one end then $\pi_2(M) = 0$ and so $M$ is aspherical.

If $\nu$ has two ends then it has an infinite cyclic normal subgroup of finite index, and so we may assume without loss of generality that $\nu \cong Z$. Hence $\nu \leq \sqrt{\pi}$. If $h(\sqrt{\pi}) > 2$ then $H^2(\pi; \mathbb{Z}[\pi]) = 0$, by [9, Theorem 1.16], and so $M$ is aspherical. (In fact $M$ is then homeomorphic to an infrasolvmanifold, by [9, Theorem 8.1].) If $h(\sqrt{\pi}) = 2$ and $\sqrt{\pi}$ has infinite index in $\pi$, then we again have $H^2(\pi; \mathbb{Z}[\pi]) = 0$ and so $M$ is aspherical. (If $\sqrt{\pi}$ is finitely generated it is nilpotent, hence $FP$, and the vanishing of $H^2(\pi; \mathbb{Z}[\pi])$ follows immediately from an LHS spectral sequence argument. If $\sqrt{\pi}$ is not finitely generated then it is the increasing union of finitely generated subgroups of Hirsch rank 2, and we may apply [8, Theorem 3.3] to conclude that $H^s(\sqrt{\pi}; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$.) If $h(\sqrt{\pi}) = 2$ and $\sqrt{\pi}$ has finite index in $\pi$ then $\pi$ is virtually $Z^2$. We may then assume that $\pi \cong Z^2$ and $\pi/\nu \cong Z$. Since $H_*(M_\nu; \mathbb{Q})$ is finitely generated it follows from the Wang sequence for the projection of $M_\nu$ onto $M$ that $\chi(M) = 0$. Hence, $M$ is finitely covered by $S^2 \times S^1 \times S^1$, by [9, Theorem 10.10].

Suppose that $h(\sqrt{\pi}) = 1$ and let $\sqrt{M}$ be the associated covering space. Since $h(\nu) = h(\sqrt{\pi})$ the stages of a subnormal chain between $\nu$ and $\sqrt{\pi}$ are locally finite, and so the rational homology spectral sequences between the corresponding covering spaces collapse, to show that $H_*(\sqrt{M}; \mathbb{Q})$ is finitely generated and $\chi(\sqrt{M}) = \chi(M_\nu)$. In particular, $\pi/\sqrt{\pi}$ has finitely many ends, since $H_3(\sqrt{M}; \mathbb{Q})$ is finite dimensional.

If $[\pi : \sqrt{\pi}]$ is finite then $\sqrt{\pi}$ is finitely generated. However, then $[\sqrt{\pi} : \nu] < \infty$ and so $[\pi : \nu] < \infty$, contrary to hypothesis.

If $\pi/\sqrt{\pi}$ has two ends then we may assume that $\pi/\sqrt{\pi} \cong Z$. However, then $\pi$ is an ascending HNN construction over a finitely generated base, and so the torsion subgroup $T$ of $\sqrt{\pi}$ is finite, while $\sqrt{\pi}/T$ is abelian. Therefore, $\sqrt{\pi}$ has a finitely generated infinite normal subgroup and so $H^2(\pi; \mathbb{Z}[\pi])$ is free abelian [13]. Since $H_*(\sqrt{M}; \mathbb{Q})$ is finitely generated $\sqrt{M}$ satisfies Poincaré duality with simple coefficients $\mathbb{Q}$ and formal dimension 3 [14] and so $\chi(\sqrt{M}) = 0$. Hence $\chi(M_\nu) = 0$. This in turn implies that $\pi_2(M_\nu)$ is a torsion $\mathbb{Z}[\nu]$-module. Now $\pi_2(M_\nu)$ is finitely generated as a $\mathbb{Z}[\nu]$-module, and is $\mathbb{Z}$-torsion-free, since $\pi_2(M_\nu) = \pi_2(M) \cong H^2(\pi; \mathbb{Z}[\pi])$. Therefore, $\pi_2(M_\nu)$ is finitely generated as an abelian group, since $\mathbb{Z}[\nu] \cong \mathbb{Z}[t, t^{-1}]$. Since $\pi$ has elements of infinite order $H^2(\pi; \mathbb{Z}[\pi])$ must therefore be 0 or $Z$, by [5, Corollary 5.2]. But $M$ cannot be aspherical as $c.d._{\mathbb{Q}}(\pi) \leq c.d._{\mathbb{Q}}\sqrt{\pi} + c.d._{\mathbb{Q}}Z = 2$. Therefore, $\widetilde{M} \simeq S^2$. As $\pi$ is elementary amenable it must be virtually $Z^2$, by [9, Theorem 10.10]. However, this contradicts the assumption that $h(\sqrt{\pi}) = 1$. Therefore, $\pi/\sqrt{\pi}$ has one end. As we may again exclude the possibility that $H^2(\pi; \mathbb{Z}[\pi]) \cong Z$, either $M$ is aspherical or $H^2(\pi; \mathbb{Z}[\pi])$ is not finitely generated as an abelian group.

Suppose that $\nu$ has infinitely many ends and $\nu \leq N$ where $N$ is an $FP_2$ normal subgroup of infinite index in $\pi$. If $[N : \nu]$ is finite then $N$ has infinitely many ends and $M_N$ is finitely dominated, so $\pi/N$ has two ends and the covering space associated to $N$ is a $PD_3$-complex, by [9, Theorem 3.9]. If $[N : \nu] = \infty$ then $N$ has one end (as above).

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Hence $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$ and so $M$ is aspherical, as before. This cannot happen if $N$ is $FP_3$, by the corollary to Theorem 4. □

The hypothesis that $\nu$ be $FP_3$ is used to ensure that $\mathrm{Hom}_\pi(\pi_2(M),\mathbb{Z}[\pi])=0$, and is automatic if $\pi$ is finite or has two ends. Does the theorem hold without this hypothesis?

Products $M=S^1\times N$ where $N=S^3$, $S^2\times S^1$, $(S^1)^3$ or $(S^2\times S^1)\#(S^2\times S^1)$ give examples realizing most of the possibilities allowed by the theorem. The main exception is the final alternative in case (3); the following corollary suggests that this is rather unlikely.

COROLLARY. *If $\nu$ has finitely many ends and either $\sqrt{\pi}$ is abelian or $h(\sqrt{\pi})\neq 1$ then $M$ is aspherical or $\widetilde{M}$ is homotopy equivalent to $S^2$ or $S^3$.*

PROOF. We may assume that $\sqrt{\pi}$ is abelian of rank 1 and $\pi/\sqrt{\pi}$ has one end. However, then $H^2(\pi;\mathbb{Z}[\pi])=0$, by [7] and [13], and so $M$ is aspherical. □

In case (4) the question raised after Theorem 4 also remains: is every $FP_2$ normal subgroup of a $PD_4$-group $FP_3$?

What happens if we drop the hypothesis on ascendancy? If a $PD_4$-complex $M$ has a finitely dominated infinite covering space must $\pi_1(M)$ have one or two ends?

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