LINEAR NORMED SPACES WITH EXTENSION PROPERTY

George Elliott[1] and Israel Halperin

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1. Introduction. In this paper we shall say "E has the (F, G) (extension) property" to mean the following: F is a subspace of the real normed linear space G, E is a real normed linear space, and any bounded linear mapping $F \to E$ has a linear extension $G \to E$ with the same bound (equivalently, every linear mapping $F \to E$ of bound 1 has a linear extension $G \to E$ with bound 1).

The Hahn-Banach theorem asserts that the real field R has the unrestricted (F, G) property (that is, for all F and G with $F \subset G$).

If X is a topological space, C(X) will denote the normed linear lattice of all continuous, bounded functions $f : X \to R$ with supremum norm: $\|f\| = \sup_{t \in X} |f(t)|$ (if X is compact every continuous f is necessarily bounded). We recall that X is called extremally disconnected if the closure of every open set is again open. Now M.H. Stone has proved [5, Theorem 14] that if X is extremally disconnected then C(X) is boundedly complete. Using Stone's result, it is easy to verify that Banach's proof [1, page 28] that R has the unrestricted (F, G) property remains valid when R is replaced by C(X) provided that X is extremally disconnected.

Nachbin [4, page 42], Goodner [2, page 103] and Kelley [3] have shown a converse: if E has the unrestricted (F, G) property then E must be isometric to C(X) for some extremally disconnected compact Hausdorff space X.

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The authors are indebted to the referee for pointing out that the results of this paper are closely related to some results in the paper by Lindenstrauss [6].

2. Main theorem of this paper. The question arises: is there a single pair (F_o, G_o) such that whenever E has the (F_o, G_o) property then E must have the unrestricted (F, G) property?

We shall show that the answer is <<yes>> if E is restricted to have finite dimension. More precisely, let G_o be the real normed space C(3) (the integer n will be used to denote the discrete topological space of n elements) and let F_o be the subspace of G_o generated by (0, 1, 1) and (1,0, 1). We shall prove the theorem: if $\dim E = n < \infty$ and E has the (F_o, G_o) property then E is isometric to the space C(n).

Our proof is elementary, is independent of the literature referred to above, and gives a new result for the finite dimensional case. But in most of our arguments it is not assumed that dim E is finite.

We shall give examples (see Theorem 5) of spaces which have the (F_o, G_o) property and yet fail to have the unrestricted (F, G) property (necessarily they are infinite dimensional).

3. Some definitions and notations.

(i) B = B(E), S = S(E) will denote respectively the closed unit ball and the unit sphere of the real normed linear space E.

(ii) J = J(a, b, . . .) will denote the convex hull of a, b, ... J(a, b) will be called a segment if $a \neq b$.

L = L(a, b, . . .) will denote the linear set of a, b,. . . (of course, L is a subspace if and only if $0 \in L$).

(iii) We shall use the following notation:

$\|x - J(a, b)\| = k$ means $\|x - y\| = k$ for all $y \in J(a, b)$;

$\|x - J(a, b)\| < k$ means $\|x - y\| < k$ for all $y \in J(a, b)$ with $a \neq y \neq b$;

$\|x - L(a, b)\| \geq k$ means $\|x - y\| \geq k$ for all $y \in L(a, b)$.

We note that in any normed linear space: if $c \in J(a, b)$ and $a \neq c \neq b$ and $\|x - c\| \geq \max(\|x - a\|, \|x - b\|)$ then $\|x - L(a, b)\| \geq \|x - c\|$.

(iv) Suppose that V is a linear space, K a convex subset. Then a point $x \in K$ will be called an <u>extreme point</u> of K if

$$a, b \in K \text{ and } x \in J(a, b) \Longrightarrow x = a \text{ or } x = b;$$

a segment $J(a, b) \subset K$ will be called an <u>edge</u> of K if

$$J(u, v) \subset K \text{ and } p \in J(u, v) \cap J(a, b) \text{ with } u \neq p \neq v$$
$$\Longrightarrow J(u, v) \subset J(a, b).$$

We note that if $J(a, b)$ is an edge then a, b are different extreme points but the converse may be false.

4. LEMMA. E has the (F_o, G_o) property if and only if: $x, y, x-y \in B(E)$ implies that there is a point $z \in B(E)$ such that the points $2x - z$, $2y - z$, $2x + 2y - 3z$ (obtained by reflecting z in $x, y, x+y-z$, respectively) are also in $B(E)$.

<u>Proof.</u> (i) $B(F_o)$ is the convex hull of $\pm(0, 1, 1)$, $\pm(1, 0, 1)$ and $\pm(-1, 1, 0)$. Every linear mapping $f: F_o \to E$ is determined by (arbitrary) values of $f(0, 1, 1) = x$ (say) and $f(1, 0, 1) = y$ (say). Then necessarily $f(-1, 1, 0) = x-y$ and hence $f(B(F_o)) \subset B(E)$ if and only if $x, y, x-y$ are all in $B(E)$.

(ii) Clearly, E has the (F_o, G_o) property if and only if for every linear mapping $f: F_o \to E$ for which $f(B(F_o)) \subset B(E)$ there is a point $z \in B(E)$ such that the linear extension $\hat{f}$ determined by $\hat{f}(1, 1, 1) = z$ maps $B(G_o)$ into $B(E)$.

(iii) Next, $B(G_o)$ is the convex hull of $\pm(1, 1, 1)$, $\pm(-1, 1, 1)$, $\pm(1, -1, 1)$ and $\pm(-1, -1, 1)$. Suppose that $\hat{f}: G_o \to E$ is a linear extension of $f: F_o \to E$. Let $f(0, 1, 1) = x$ and $f(1, 0, 1) = y$. Then $\hat{f}$ is determined by $\hat{f}(1, 1, 1) = z$ (say) and then $\hat{f}(-1, 1, 1) = 2x-z$, $\hat{f}(1, -1, 1) = 2y-z$, $\hat{f}(-1, -1, 1) = 2x + 2y - 3z$. Hence $\hat{f}(B(G_o)) \subset B(E)$ if and only if z, $2x - z$, $2x + 2y - 3z$ are all in $B(E)$.

The Lemma follows easily from (i), (ii), (iii).

5. THEOREM. Let X denote any topological space,

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let Y denote any closed subset of X and let E denote the subspace of C(X) consisting of those functions in C(X) which vanish on Y. Then E has the (F_o, G_o) property.

Proof. By Lemma 4 we need only show that if $x, y, x-y \in B = B(E)$ then there exists $z \in B$ such that $2x - z$, $2y - z$, $2x + 2y - 3z$ are also in B.

Set $z = \frac{x+y}{2-|x-y|}$. We note that $2 - |x-y| \geq 1$. Hence z is defined and is in E.

To prove z, $2x - z$, $2y - z$, $2x + 2y - 3z$ are all in B, we fix $t \in X$, write x, y, z for x(t), y(t), z(t), and show $-1 \leq 2x - z$, $2y - z$, $2x + 2y - 3z \leq 1$.

We may suppose $y \geq x$. Then $0 \leq y - x \leq 1$. Now $z = \frac{x+y}{2-(y-x)}$.

Since $-1 \leq x, y \leq 1$ we have $(-2-x) + y \leq x+y \leq x+(2-y)$, hence $-1 \leq z \leq 1$.

Next, $z = \frac{x+y}{2-(y-x)} = x + \frac{(y-x)(x+1)}{(x+1)+(1-y)} \geq x$, hence $2x - z = x - \frac{(y-x)(x+1)}{(x+1)+(1-y)} \leq x \leq 1$ and $2x - z \geq x - \frac{x+1}{1} = -1$.

Thus $-1 \leq 2x - z \leq 1$.

Again, $z = y - \frac{(y-x)(1-y)}{2-(y-x)} \leq y$, hence $2y - z = y + \frac{(y-x)(1-y)}{2-(y-x)} \geq y \geq -1$ and $2y - z \leq y + \frac{(1-y)}{1} = 1$.

Thus $-1 \leq 2y - z \leq 1$.

Now $x \leq z \leq y$, hence $2x - z \leq 2x + 2y - 3z \leq 2y - z$.

This completes the proof.

6. Remark. Theorem 5 shows that $C_{\{0\}}([0, 1])$ has the (F_o, G_o) property. If $C_{\{0\}}([0, 1])$ had the unrestricted (F, G) property, then the results of Nachbin, Goodner and Kelley would imply that it was isometric to some C(X) with X compact and extremally disconnected. However $C_{\{0\}}([0, 1])$ is not isometric to C(X) for any X. (Indeed, the unit ball of $C_{\{0\}}[0, 1]$ has

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no extreme points but in every $C(X)$ the function $f(t) = 1$ for all $t \in X$ is an extreme point of the unit ball of $C(X)$.)

7. LEMMA. (Corollary of Lemma 4). Suppose that E has the (F_o, G_o) property and x is an extreme point of B and $y_1 \in B$, $y_1 \neq x$. Then the segment $J(x, y_1)$ is part of a chord of S of length 2. In particular, if e, e_1 are different extreme points of B then $\|e - e_1\| = 2$.

Proof. We may obviously pass to the case that $J(x, y_1)$ is an inextensible chord of S, and we need only show that $\|x - y_1\| = 2$. Clearly, $\|x - y_1\| \leq \|x\| + \|y_1\| \leq 2$. Suppose if possible that $\|x - y_1\| < 2$. Then choose $y \in J(x, y_1)$ so that $\|x - y_1\| / 2 < \|x - y\| \leq 1$. Then Lemma 4 applies; the resulting z must coincide with x, since x is an extreme point. Hence the segment $J(x, 2y-x)$ is in B, $J(x, y_1) \underset{\neq}{\subset} J(x, 2y-x)$, contradicting the fact that $J(x, y_1)$ was chosen to be inextensible.

This contradiction shows that $\|x - y_1\| = 2$ holds.

8. LEMMA. (Corollary of Lemma 4). Suppose that E has the (F_o, G_o) property and $J(a, b)$ is an edge of B, $c \in B$, $c \notin J(a, b)$, and $\|b - J(a, c)\| = 2$. Then $J(a, c) \subset S$, $J(b, b+c-a) \subset S$, and $\|a - J(b, b+c-a)\| = 2$.

Proof. Apply Lemma 4 with $x = \frac{a+b}{2}$, $y = \frac{a+c}{2}$. The resulting z must be in $J(a, b)$, since this is an edge.

We shall now show that $z = a$. We have: $2y - z \in B$, hence $\|b - (2y - z)\| \leq 2$; $z \neq a$ would imply $\|b - z\| < 2$ and hence $\|b - y\| = \left\| \frac{(b-z)+(b-2y+z)}{2} \right\| < 2$, contradicting $\|b - J(a, c)\| = 2$, since $y \in J(a, c)$.

Consequently $z = a$, $b + c - a = 2x + 2y - 3z \in B$, and $J(b, b+c-a) \subset B$.

If $m \in J(b, b+c-a)$ then $a - m = a - b - \theta(c-a)$ for some $0 \leq \theta \leq 1$, hence $\|a-m\| = \|u-b\|$ with $u = a - \theta(c-a) \in L(a, c)$ so $\|a - m\| \geq 2$. Since $a, m \in B$ it follows that $\|a - m\| \leq 2$,

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hence $\|a - m\| = 2$. Thus $\|a - J(b, b+c-a)\| = 2$. It follows that $J(b, b+c-a) \subset S$, for if $u, v \in B$ and $\|u - v\| = 2$ then necessarily $u, v \in S$.

9. LEMMA. (Corollary of Lemmas 7, 8). Suppose that E has the (F_o, G_o) property and $J(a, b)$, $J(a, c)$ are different edges of B. Then $J(b, b+c-a)$ is an edge.

<u>Proof.</u> $J(a, b)$ is an edge, $c \notin J(a, b)$ and by Lemma 7, since b is an extreme point and $J(a, c)$ is an edge, $\|b - J(a, c)\| = 2$; hence by Lemma 8, $\|a - J(b, b+c-a)\| = 2$.

Now suppose if possible that $J(b, b+c-a)$ is not an edge. Then there exist $u, v \in B$ such that $J(u, v) \cap J(b, b+c-a)$ is a single point x, $u \neq x \neq v$.

We shall now show that $\|a - J(b, u)\| = 2$. Suppose that $\theta < \theta < 1$ and set $u' = b + \theta(u-b)$, $v' = b + \theta(v-b)$, $x' = b + \theta(x-b)$. Then $\|a - u'\| \leq 2$, $\|a - v'\| \leq 2$, $\|a - x'\| = 2$, and $x' \in J(u', v')$. Hence $\|a - u'\| = 2$. Thus $\|a - J(b, u)\| = 2$.

Now $J(b, a)$ is an edge, $u \in B$, $u \notin J(b, a)$, and $\|a - J(b, u)\| = 2$, hence by Lemma 8, $J(a, a+u-b) \subset S$. Similarly $J(a, a+v-b) \subset S$.

Then $J(a+u-b, a+v-b) \subset B$ and $a + x-b \in J(a+u-b, a+v-b) \cap J(a, c)$ with $a+u-b \neq a+x-b \neq a+v-b$, but $J(a+u-b, a+v-b) \not\subset J(a, c)$. This contradicts the fact that $J(a, c)$ is an edge and shows that Lemma 9 must hold.

10. LEMMA. Suppose that E is an n-dimensional $(n < \infty)$ real normed linear space which has the (F_o, G_o) property. Then

(i) the set W of extreme points of the unit ball B is finite;

(ii) there exist in B different extreme points $e_o, e_1, \ldots, e_n$ such that the vectors $v_i = e_i - e_o$, $i = 1, \ldots, n$ are linearly independent and each $J(e_o, e_i)$, $i = 1, \ldots, n$ is an edge of B.

(iii) if $e_o, e_1, \ldots, e_n$ are chosen as in (ii) then the unit ball B coincides with the parallelepiped

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$$\{e_o + \sum_{i=1}^{n} t^i v_i \mid 0 \leq t^i \leq 1, \quad i = 1, \ldots, n\} ;$$

hence $e_o + \sum_{i=1}^{n} \frac{1}{2} v_i = 0$, and B coincides with the parallelepiped

$$\{\sum_{i=1}^{n} t^i (\frac{v_i}{2}) \mid -1 \leq t^i \leq 1, \ i=1, \ldots, n\} ;$$

(iv) E is isometric with $C(n)$.

Proof of (i). Since the dimension of E is finite, B is compact. Since $\|e - e_1\| = 2$ whenever e, e_1 are different extreme points of B it follows that the number of extreme points of B is finite.

Proof of (ii). We now introduce a euclidean metric into E. With respect to this metric B is a bounded, convex, closed subset of the finite dimensional space E and $L(B) = E$. Hence W is not empty and $L(W) = E$. Suppose that $e_o, e_1, \ldots, e_m$ are the different extreme points of B and let $\mathrm{Conv}(e_1, \ldots, e_m)$ consist of all $\sum_{i=1}^{m} t^i e_i$ with all $t^i \geq 0$ and $\sum_{i=1}^{m} t^i = 1$. Let d be the point in $\mathrm{Conv}(e_1, \ldots, e_m)$ which is closest to e_o in the euclidean metric. Then $d \neq e_o$.

Let $y = \frac{d+e_o}{2}$ and let H be the $n-1$ dimensional hyperplane which contains y and is orthogonal to $e_o - d$ with respect to the euclidean metric. Then $H \cap B$ is a closed, bounded convex subset of H.

If x is an extreme point of $H \cap B$ it follows that $x \in J(e_o, e_x)$ for some unique extreme point e_x of B. Hence $H \cap B$ has a finite number of extreme points $x_1, \ldots, x_r$, and it is easily seen that $L(x_1, \ldots, x_r) = H$ and each $J(e_o, e_{x_i})$, $i = 1, \ldots, r$, is an edge of B, and $L(e_o, e_{x_1}, \ldots, e_{x_r}) = E$. It

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follows that $r \geq n$ and it is possible to choose $x_1, \ldots, x_n$ so that the vectors $v_i = e_{x_i} - e_o$, $i = 1, \ldots, n$ are linearly independent.

Proof of (iii). By repeated application of Lemma 9 it follows that

$$J(p, p+v_i) \text{ is an edge of } B$$

whenever $p = e_o + \Sigma_{j \in J} v_j$ with $J \subset \{1, 2, \ldots, n\}$, $i \notin J$.

Hence $\{e_o + \sum_{i=1}^{n} t^i v_i \mid 0 \leq t^i \leq 1, \; i = 1, \ldots, n\}$ is part of B. Moreover, if $x = e_o + \sum_{i=1}^{n} t^i v_i$ with $0 \leq t^i \leq 1$ for all i but $t^j = 0$ for some j, then x and $x + v_j$ are both in B and $\|x - (x+v_j)\| = 2$ which implies that $x \in S$. It follows that the parallelepiped

$$\{ e_o + \sum_{i=1}^{n} t^i v_i \mid 0 \leq t^i \leq 1, \; i = 1, \ldots, n\}$$

coincides with B.

Since $x \in B$ implies that $-x \in B$, the rest of (iii) and then (iv), the main result of this note, follow at once.

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Queen's University, Kingston, Ontario