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Realizable Oriented Matroids

To adapt a phrase from Whitney (1935), the study of oriented matroids is the study of "the abstract properties of linear dependence over ordered fields." (For our purposes, the field in question may be assumed to be $\mathbb{R}$.) Briefly put, from a finite subset S of $\mathbb{R}^n$, one can extract a certain combinatorial structure, and from that structure one can reconstruct various dependency relationships in S. From properties of $\mathbb{R}^n$ we find properties satisfied by these structures, and we call any structure with these properties – whether it arises from a subset of $\mathbb{R}^n$ or not – an oriented matroid. Thus, oriented matroids are "combinatorial abstractions of finite vector arrangements in $\mathbb{R}^n$."

As we will see in this chapter, oriented matroids could as well be described as combinatorial abstractions of linear subspaces of $\mathbb{R}^n$, or of hyperplane arrangements in $\mathbb{R}^n$. Each of these interpretations has led to beautiful interplay between combinatorics, geometry, and topology. There are various other well-known mathematical objects that can be abstracted to oriented matroids (most notably, directed graphs), but we'll focus on direct connections from oriented matroids to linear algebra.

The first steps in learning about oriented matroids can be annoying, because every really honest introduction puts off the definition of oriented matroid until at least the second chapter. To justify this annoyance, here's a short preview.

In Chapter 2 we will introduce several different axiom systems, writing preliminary definitions of the form:

Definition 1.1 An object $\mathcal{A}$ is a **Type 1 expression of an oriented matroid** if it satisfies the following axioms ...

Definition 1.2 An object $\mathcal{B}$ is a **Type 2 expression of an oriented matroid** if it satisfies the following axioms ...

Et cetera. We will then show that these different types of expressions are *cryptomorphic*, that is, there are nice bijections between the expressions of different types, so that each expression of one type determines a unique expression of each other type. Thus, each type encodes the same data in a different form. We will then define an oriented matroid to be this data, however expressed.

Why not stick with one type of expression? One answer is pragmatic – different types of expressions are easier to work with in different settings. A more interesting answer is that the different expressions reflect different aspects of the relationship between oriented matroids and linear algebra. Radon partitions, Grassmann–Plücker relations, orthogonality of vector spaces, and the combinatorics of hyperplane arrangements are some of the geometric notions that will have elegant combinatorial analogs in one or the other of the oriented matroid axiom systems. This wealth of different routes from $\mathbb{R}^n$ to oriented matroids is one indication that oriented matroids are the "right" combinatorial model for this kind of geometry. (Two other strong indications are the Topological Representation Theorem (Chapter 4) and the results in Section 10.5 on the *MacPhersonian*).

In this chapter we'll introduce our "Type X expressions" for oriented matroids via their concrete manifestations in $\mathbb{R}^n$. We'll look at different ways to extract combinatorial structures from a finite subset S of $\mathbb{R}^n$, and we'll see that these structures encode geometric data interesting in several different contexts. Finally, we'll see that these different structures each encode the same data about S. We call that data, however encoded, the *oriented matroid corresponding to S*.

The combinatorial structures arising from finite subsets S of $\mathbb{R}^n$ are called **realizable oriented matroids**. (S is then called a **realization** of that oriented matroid.) They're the examples that motivate the theory, but they're not the whole picture. In general, oriented matroids are defined in purely combinatorial terms, with no reference to $\mathbb{R}^n$, and not all oriented matroids are realizable. The relationship between oriented matroids and their realizations is a fraught topic, as Chapter 7 will show.

Despite the existence of nonrealizable oriented matroids, and the scandalously non-realizable behavior in which oriented matroids occasionally indulge, for the most part oriented matroids model real vector sets admirably. In working with the abstract combinatorics of general oriented matroids, it's almost always a good idea to be guided by one's intuition from $\mathbb{R}^n$. The point of this introductory chapter is to develop that intuition. The concepts and proofs of this chapter are mostly rather simple. In Chapter 2 they'll all be combinatorialized into more abstract notions, which will be less daunting if you keep the geometric inspirations in mind.

1.1 Some Notation

- $\mathrm{Mat}(r,n)$ will denote the set of all $r \times n$ real matrices of rank r.
- $\mathrm{row}(M)$ denotes the row space of the matrix M, and $\mathrm{null}(M)$ denotes the nullspace.
- A matrix will be viewed as a list of column vectors. Thus an element of $\mathrm{Mat}(r,n)$ will often be written as $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.
- If $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathrm{Mat}(r,n)$ then $M_{i_1,\ldots,i_r}$ denotes the square matrix $(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r})$.
- For sets P and E, P^E denotes the set of all functions $E \to P$. An element of P^E will sometimes be written as $(p_e : e \in E)$.
- For a positive integer n, $[n]$ denotes $\{1, 2, \ldots, n\}$. For a set P and natural number n, $P^{[n]}$ will be abbreviated P^n.
- Depending on what's convenient, we will write elements of P^n as functions or as n-component vectors with entries in P. In particular, we will refer to the support of $X \in \{0, +, -\}^n$, which will mean the support as a function (that is, $\{i \in [n] : X(i) \neq 0\}$).
- For a sign vector X, we will sometimes denote $X^{-1}(+)$ by X^+, $X^{-1}(-)$ by X^-, and $X^{-1}(0)$ by X^0. If $A = X^+$, $B = X^-$, and $C = X^0$, we will sometimes denote X by A^+B^- or $A^+B^-C^0$. If $A = \emptyset$, then we may denote X by B^-, and if $B = \emptyset$, then we may denote X by A^+. (Aside: Sign vectors written in the A^+B^- notation are elsewhere sometimes called *signed sets*.)
- When the support of a sign vector has just a few elements, we may write it as a string of symbols $e^{X(e)}$, with e in the support. For instance, the sign vector $\{a,c\}^+\{b\}^-$ may be denoted $a^+b^-c^+$.
- For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathrm{sign}(\mathbf{x})$ denotes the sign vector

$$(\mathrm{sign}(x_1), \ldots, \mathrm{sign}(x_n)) \in \{0, +, -\}^n.$$

- $\mathbf{0}$ denotes a vector all of whose components are 0. Context will tell us the number of components and whether the vector is a row or column vector.
- $\mathcal{P}(S)$ denotes the power set of S.

The set $\{0, +, -\}$ will be partially ordered by $+ > 0, - > 0, + \not> -, - \not> +$. The set $\{0, +, -\}^E$ will be ordered componentwise: If $X, Y \in \{0, +, -\}^E$, then $X \geq Y$ if and only if $X(e) \geq Y(e)$ for every $e \in E$.

For $X \in \{0, +, -\}^E$, the **orthant** in $\mathbb{R}^E$ corresponding to X is $\{\mathbf{x} \in \mathbb{R}^E : \mathrm{sign}(\mathbf{x}) = X\}$, and the **closed orthant** corresponding to X is $\{\mathbf{x} \in \mathbb{R}^E : \mathrm{sign}(\mathbf{x}) \leq X\}$. Thus our partial order on sign vectors corresponds to the partial order on closed orthants by inclusion.

1.2 Discrete Models for Matrices

It will be convenient to consider an ordered list of vectors spanning $\mathbb{R}^r$ as the set of columns of a matrix $M \in \mathrm{Mat}(r,n)$. Such a list is called a *vector arrangement*.

We begin by considering some "discrete models for matrices." That is, for every $r,n \in \mathbb{N}$, we will consider some finite set $\mathcal{O}$ and function

$$\mathrm{Mat}(r,n) \to \mathcal{O}$$

that seem somehow natural from the point of view of linear algebra.

The five models we'll look at are:

1. $\mathcal{V}$: $\mathrm{Mat}(r,n) \to \mathcal{P}(\{0,+,-\}^n)$ defined by
$$\mathcal{V}(M) = \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in \mathrm{null}(M)\}.$$
$\mathcal{V}(M)$ is called the **set of vectors**[1] corresponding to M. It has a partial order as a subposet of $\{0,+,-\}^n$.
2. $\mathcal{V}^*$: $\mathrm{Mat}(r,n) \to \mathcal{P}(\{0,+,-\}^n)$ defined by
$$\mathcal{V}^*(M) = \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in \mathrm{row}(M)\} = \{\mathrm{sign}(\mathbf{y}M) : \mathbf{y} \in \mathbb{R}^r\}.$$
$\mathcal{V}^*(M)$ is called the **set of covectors** corresponding to M. Again, this is partially ordered as a subposet of $\{0,+,-\}^n$.
3. $\mathcal{C}$: $\mathrm{Mat}(r,n) \to \mathcal{P}(\{0,+,-\}^n)$ takes each M to the set of minimal elements of $\mathcal{V}(M)\backslash\{\mathbf{0}\}$. Here, as always, minimality is with respect to the partial order on sign vectors described in Section 1.1. $\mathcal{C}(M)$ is called the **set of signed circuits** corresponding to M.
4. $\mathcal{C}^*$: $\mathrm{Mat}(r,n) \to \mathcal{P}(\{0,+,-\}^n)$ takes each M to the set of minimal elements of $\mathcal{V}^*(M)\backslash\{\mathbf{0}\}$. $\mathcal{C}^*(M)$ is called the **set of signed cocircuits** corresponding to M.
5. χ: $\mathrm{Mat}(r,n) \to \{0,+,-\}^{[n]^r}$ defined by: If $M \in \mathrm{Mat}(r,n)$, then $\chi(M)$: $[n]^r \to \{0,+,-\}$ is the function taking each $(i_1,i_2,\ldots,i_r)$ to the sign of the determinant $|M_{i_1,\ldots,i_r}|$. $\chi(M)$ is called the **chirotope** corresponding to M.

Problem 1.3 Let

$$M = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Determine $\mathcal{V}(M)$, $\mathcal{V}^*(M)$, $\mathcal{C}(M)$, $\mathcal{C}^*(M)$, and $\chi(M)$.

[1] The terminology is terrible but firmly established.

Problem 1.4 For each of the sets $\mathcal{C}(N)$, $\mathcal{C}^*(N)$, and $\chi(N)$ associated to a matrix $N = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, describe how to determine each of the following:

1. whether $\mathbf{v}_i = 0$,
2. whether $\mathbf{v}_i$ is a positive multiple of $\mathbf{v}_j$,
3. whether $\mathbf{v}_i$ is a negative multiple of $\mathbf{v}_j$,
4. whether a set of columns $\{\mathbf{v}_i : i \in I\}$ is independent, and
5. whether $\mathbf{v}_i$ is in the span of the remaining columns.

1.2.1 Invariance under Change of Coordinates

Problem 1.5 1. Prove that the set of vectors, set of covectors, set of signed circuits, and set of signed cocircuits of a vector arrangement in $\mathbb{R}^r$ are invariant under change of coordinates. That is, prove for each $A \in GL_r$ that $\mathcal{V}(AM) = \mathcal{V}(M)$, et cetera.
2. Prove that χ is invariant under orientation-preserving change of coordinates (that is, under left multiplication by $A \in GL_r^+$).

This shows one reason these models might be more interesting models for vector arrangements than, say, $\{\text{sign}(\mathbf{x}) : \vec{x} \text{ a column of } M\}$.

Problem 1.6 Prove that the set of vectors, set of covectors, set of signed circuits, set of signed cocircuits, and chirotope of a vector arrangement in $\mathbb{R}^r$ are invariant under scaling of columns by positive scalars. That is, prove for each diagonal $n \times n$ matrix D with all diagonal entries positive that $\mathcal{V}(MD) = \mathcal{V}(M)$, et cetera.

So far we have only considered vector arrangements that are expressed as the columns of $r \times n$ matrices of rank r. Thus we have assumed that our vector space is $\mathbb{R}^r$ and that our arrangement spans the space. But Exercise 1.5 points out one way to associate vector sets, covector sets, and so on, to a finite arrangement $(\mathbf{v}_i : i \in S)$ in any real vector space. One can simply fix a vector space isomorphism from the span $\langle \mathbf{v}_i : i \in S \rangle$ to $\mathbb{R}^r$ and then define $\mathcal{V}$, $\mathcal{V}^*$, and so on in terms of the corresponding arrangement in $\mathbb{R}^r$. This is actually the cheesy way to do it: A better way is to keep reading this chapter and see coordinate-free descriptions of each of our combinatorial structures. Either way it makes sense to talk about the oriented matroid of a finite arrangement of vectors in an arbitrary vector space over $\mathbb{R}$.

We will see in Section 1.5 that $\mathcal{V}(M)$, $\mathcal{V}^*(M)$, $\mathcal{C}(M)$, and $\mathcal{C}^*(M)$ encode the same data about M. ($\chi(M)$ encodes a bit more.) We will call this data, however encoded, the *oriented matroid* corresponding to M. The oriented matroids arising in this way are called *realizable*. The definition of general oriented matroids will come in Chapter 2.

1.2.2 Support-Minimality and Reduced Row-Echelon Form

Definition 1.7 For every $X, Y \in \{0, +, -\}^E$, their **separation set** is

$$S(X,Y) = \{e : \{X(e), Y(e)\} = \{+,-\}\}.$$

Remark 1.8 Observe that $X \geq Y$ if and only if $\mathrm{supp}(X) \supseteq \mathrm{supp}(Y)$ and $S(X,Y) = \emptyset$.

Proposition 1.9 *Let V be a linear subspace of $\mathbb{R}^n$, and let $\mathcal{F} = \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in V\backslash\{\mathbf{0}\}\}$. Then $\min(\mathcal{F})$ is exactly the set of elements of $\mathcal{F}$ of minimal support.*

By applying Proposition 1.9 to the null space and to the row space of a matrix, we get the following.

Corollary 1.10 *$\mathcal{C}(M)$ consists exactly of the elements of $\mathcal{V}(M)\backslash\{\mathbf{0}\}$ of minimal support, and $\mathcal{C}^*(M)$ consists exactly of the elements of $\mathcal{V}^*(M)\backslash\{\mathbf{0}\}$ of minimal support.*

Proof of Proposition 1.9: Let $X \in \mathcal{F}$. Clearly if $\mathrm{supp}(X)$ is minimal then $X \in \min(\mathcal{F})$.

If $\mathrm{supp}(X)$ is not minimal then let $Y \in \mathcal{F}$ such that $\mathrm{supp}(Y) \subset \mathrm{supp}(X)$. We show that X is not minimal in $\mathcal{F}$ by induction on $|S(X,Y)|$. If $S(X,Y) = \emptyset$ then X is not minimal because $Y < X$. Otherwise, we will find a $Y' \in \mathcal{F}$ such that $\mathrm{supp}(Y') \subset \mathrm{supp}(X)$ and $S(X,Y') \subset S(X,Y)$.

Let $\mathbf{x}, \mathbf{y} \in V$ such that $X = \mathrm{sign}(\mathbf{x})$ and $Y = \mathrm{sign}(\mathbf{y})$, and consider the set $C = \{a\mathbf{x} + b\mathbf{y} : a, b > 0\}$ of all positive linear combinations of $\{\mathbf{x}, \mathbf{y}\}$. Notice that $C \subset V$, that $\mathrm{supp}(\mathbf{v}) \subseteq \mathrm{supp}(X)$ for each $\mathbf{v} \in C$, and that $S(X, \mathrm{sign}(\mathbf{v})) \subseteq S(X,Y)$ for each $\mathbf{v} \in C$. Let $e \in S(X,Y)$. Then $\mathrm{sign}(x_e) = -\mathrm{sign}(y_e)$, and so $|y_e|x_e + |x_e|y_e = 0$. Thus we see an element $\mathbf{z} = |y_e|\mathbf{x} + |x_e|\mathbf{y}$ of C with $z_e = 0$. Let $Y' = \mathrm{sign}(\mathbf{z})$. Then $\mathrm{supp}(Y') \subseteq \mathrm{supp}(X)\backslash\{e\}$ and $S(X,Y') \subseteq S(X,Y)\backslash\{e\}$. □

We can see the idea of the preceding proof by way of an example we can draw: This is also an opportunity to introduce a type of picture we'll be revisiting often. Consider a two-dimensional subspace V of $\mathbb{R}^4$. While we can't draw $\mathbb{R}^4$, we can draw V by itself, and in our drawing we can include the intersection of V with each coordinate hyperplane. Unless V is very special, each such intersection will be a line $L_i = \{\mathbf{v} \in V : v_i = 0\}$, and the half-space on one side of this line will be $\{\mathbf{v} \in V : v_i > 0\}$. In our picture we'll label L_i by the number i, and we'll draw a small arrow starting at L_i and pointing into the half-space $\{\mathbf{v} \in V : v_i > 0\}$. Figure 1.1 depicts such a V. This drawing is enough to see the poset $\mathcal{F}$ arising from V. For each W in the associated set

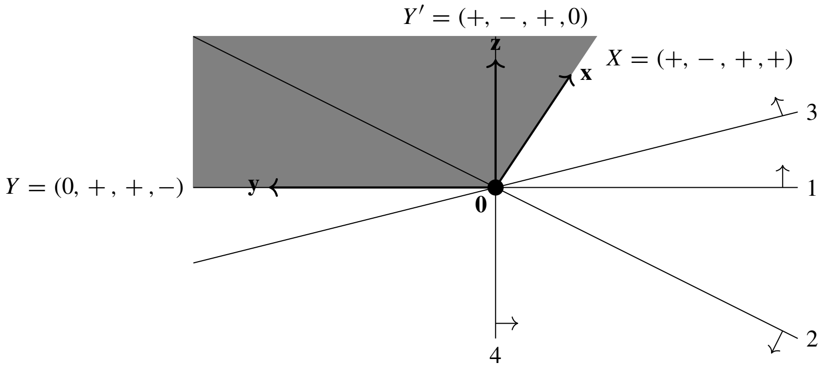


Figure 1.1 Illustration for the proof of Proposition 1.9.

$\mathcal{F}$, $\{\mathbf{v} \in V : \text{sign}(\mathbf{v}) = W\}$ is a cone: In Figure 1.1 some of these cones are labeled with the corresponding elements of $\mathcal{F}$. The figure also shows the cone C and the vector $\mathbf{z}$ from the proof of Proposition 1.9, for a particular choice of X, Y, $\mathbf{x}$, and $\mathbf{y}$, and with $e = 4 \in S(X, Y)$.

Lemma 1.11 *Let* $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \text{Mat}(r, n)$ *and* $S \subseteq [n]$. *The space* $V_S := \{\mathbf{x} \in \text{row}(M) : \text{supp}(\mathbf{x}) \cap S = \emptyset\}$ *has dimension* $r - \text{rank}\{\mathbf{v}_i : i \in S\}$.

Proof: Since the rows of M are linearly independent, the map from $\mathbb{R}^r$ to $\text{row}(M)$ taking each $\mathbf{w}$ to $\mathbf{w}M = (\mathbf{w}\mathbf{v}_1, \ldots, \mathbf{w}\mathbf{v}_n)$ has kernel $\{\mathbf{0}\}$. The preimage of V_S is $\{\mathbf{v}_i : i \in S\}^{\perp}$, and so the dimension of V_S is $r - \text{rank}\{\mathbf{v}_i : i \in S\}$. □

Remark 1.12 Lemma 1.11 is our first example of a useful, quirky aspect of oriented matroid theory: using information about the columns of M to derive information about the row space, or vice versa. We'll see the vice versa aspect in Section 1.3.3, when we give an interpretation of covectors for vector arrangements.

Definition 1.13 Let $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \text{Mat}(r, n)$ and $B = \{i_1 < \cdots < i_r\} \subseteq [n]$. If $M_{i_1,\ldots,i_r}$ is rank r, then the **reduced row-echelon form of** M **with respect to** B is $(M_{i_1,\ldots,i_r})^{-1}M$.

This is familiar from linear algebra as the unique matrix N obtainable from M by elementary row operations such that $N_{i_1,\ldots,i_r} = I$.

Proposition 1.14 *Let* $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \text{Mat}(r, n)$. $\mathcal{C}^*(M)$ *consists of all* $\pm\text{sign}(\mathbf{x})$ *such that* $\mathbf{x}$ *is a row in some reduced row-echelon form for* M.

Equivalently, $\mathcal{C}^*(M)$ *consists of all* $X \in \mathcal{V}^*(M)$ *such that* $\{\mathbf{v}_i : i \in X^0\}$ *has rank* $r - 1$.

Proof: By Corollary 1.10 $\mathcal{C}^*(M)$ consists of the sign vectors of elements $\mathbf{x}$ of $\text{row}(M)\backslash\{\mathbf{0}\}$ of minimal support. So consider such an $\mathbf{x}$. Let $S = [n]\backslash\text{supp}(\mathbf{x})$, and consider the space V_S of Lemma 1.11. Notice that V_S is the set of elements of $\text{row}(M)$ whose support is contained in $\text{supp}(x)$, and so by our minimality assumption, $V_S\backslash\{\mathbf{0}\}$ is the set of elements of $\text{row}(M)$ with support equal to the support of $\mathbf{x}$.

Assume by way of contradiction that $\dim(V_S) > 1$. Let $\{\mathbf{y}, \mathbf{z}\}$ be a linearly independent subset of V_S. As already observed, $\mathbf{y}$ and $\mathbf{z}$ have the same support as $\mathbf{x}$. Let j be an element of this support. Then $z_j\mathbf{y} - y_j\mathbf{z}$ is a nonzero element of $\text{row}(M)$ with support contained in $\text{supp}(\mathbf{x})\backslash\{j\}$, contradicting our minimality assumption.

Since $\dim(V_S) = 1$, by Lemma 1.11, $\{\mathbf{v}_i : i \in S\}$ has rank $r - 1$. Take a maximal independent subset $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{r-1}}\}$ of this set, and choose $a \in [n]$ so that $B = \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{r-1}}, \mathbf{v}_a\}$ is a basis of $\mathbb{R}^r$. Let M' be the reduced row-echelon form of M with respect to B. Since $x_{i_1} = \cdots = x_{i_{r-1}} = 0$ and $\mathbf{x}$ is a linear combination of the rows of M', we see that $\mathbf{x}$ is a scalar multiple of the row corresponding to the column indexed by a. Thus $\text{sign}(\mathbf{x})$ is indeed a multiple of the sign of this row. □

1.3 Arrangements

In this section we'll look at two types of geometric objects:

- *vector arrangements* – finite ordered lists of vectors in $\mathbb{R}^r$. These vectors will always be viewed as columns of a matrix.
- *signed hyperplane arrangements* – finite ordered lists of signed hyperplanes in $\mathbb{R}^r$.

Actually an arrangement is not necessarily a list (a collection indexed by $[n]$) – see the definition of arrangement in Section 1.3.1.

We'll note how the discrete models of Section 1.2 describe fundamental notions associated to each of these objects.

1.3.1 Geometry Glossary

1. For us, a **hyperplane** is always a linear hyperplane in a vector space V over $\mathbb{R}^r$. We allow the **degenerate hyperplane** consisting of V itself. (In terms of an inner product on V, a hyperplane is the normal $\{\mathbf{x} : \mathbf{x} \cdot \mathbf{v} = 0\}$ to some vector $\mathbf{v}$, and the degenerate hyperplane is the normal to $\mathbf{0}$.)

2. A **signed hyperplane** in V is a triple $\mathcal{H} = (H^0, H^+, H^-)$, where H^0 is a hyperplane in V and H^+ and H^- are the two open half-spaces bounded by H^0. By convention, we call $(V, \emptyset, \emptyset)$ the **degenerate signed hyperplane**. H^+ is the **positive open half-space** of $\mathcal{H}$, and $H^+ \cup H^0$ is the **positive closed half-space** of $\mathcal{H}$. Likewise, H^- and $H^- \cup H^0$ are the **negative open half-space** and **negative closed half-space** of $\mathcal{H}$.
3. For a vector $\mathbf{v}$ in $\mathbb{R}^r$, we'll use $\mathbf{v}^\perp$ to denote the signed hyperplane with

$$(\mathbf{v}^\perp)^0 = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{x} \cdot \mathbf{v} = 0\},$$

$$(\mathbf{v}^\perp)^+ = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{x} \cdot \mathbf{v} > 0\},$$

$$(\mathbf{v}^\perp)^- = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{x} \cdot \mathbf{v} < 0\}.$$

The degenerate signed hyperplane is $\mathbf{0}^\perp$.
4. For a signed hyperplane $\mathcal{H} = (H^0, H^+, H^-)$ in V and a subset W of V, let $\mathcal{H} \cap W$ denote the triple $(H^0 \cap W, H^+ \cap W, H^- \cap W)$. This will most commonly arise with W a linear subspace of V, in which case $\mathcal{H} \cap W$ is a signed hyperplane in W.
5. An **affine hyperplane in** V is a set $\mathbf{w} + W$, where W is a hyperplane in V and $\mathbf{w} \neq \mathbf{0}$. An **affine space in** $\mathbb{R}^r$ is an intersection of affine hyperplanes. We will denote a d-dimensional affine space by $\mathbb{A}^d$, or simply $\mathbb{A}$. When the particular ambient space V is not important, we will call elements of $\mathbb{A}$ **points**.
6. The **affine span** of $S \subseteq \mathbb{A}$ is the intersection of $\mathbb{A}$ with the linear span of S. The **relative interior** of $S \subseteq \mathbb{A}$ is the topological interior of S as a subset of its span.
7. An **affine subspace** of an affine space $\mathbb{A}^r$ is a nonempty intersection of a linear subspace with $\mathbb{A}^r$. A **signed affine hyperplane in** $\mathbb{A}^r$ is a triple $\mathcal{H} = (H^0 \cap \mathbb{A}^r, H^+ \cap \mathbb{A}^r, H^- \cap \mathbb{A}^r)$, where H^0 is a hyperplane, $H^0 \cap \mathbb{A}^r \neq \emptyset$, and H^+ and H^- are the two open half-spaces bounded by H^0.
8. S^d denotes the unit sphere in $\mathbb{R}^{d+1}$.
9. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset V$.

    1. The **open cone** on S is $\{\sum_{i=1}^k a_i \mathbf{v}_i : \forall i \ a_i > 0\}$.
    2. The **closed cone** on S is $\{\sum_{i=1}^k a_i \mathbf{v}_i : \forall i \ a_i \geq 0\}$.
    3. The **convex hull** of S is $\{\sum_{i=1}^k a_i \mathbf{v}_i : \forall i \ a_i \geq 0, \sum_{i=1}^k a_i = 1\}$.

    In particular, if S is contained in an affine space $\mathbb{A}^r$, then the convex hull of S is contained in $\mathbb{A}^r$.

    We also declare that $\mathbf{0} = \sum_{\mathbf{v} \in \emptyset} \mathbf{v}$, so $\{\mathbf{0}\}$ is the open cone on $\emptyset$, the closed cone on $\emptyset$, and the convex hull of $\emptyset$.

10. A **convex polytope** is the convex hull of a finite set of elements of $\mathbb{R}^r$. A subset Q of a convex polytope P is a **face** of P if there is a signed hyperplane $\mathcal{H}$ such that $Q = H^0 \cap P$ and $P \subset H^0 \cup H^+$. A **vertex** of a convex polytope P is a 0-dimensional face.

    A nontrivial theorem (cf. chapter 1 of Ziegler 1995) says that a set is a convex polytope if and only if it is a bounded intersection of closed half-spaces. A convex polytope P is the convex hull of its vertex set, and each face Q is the convex hull of the vertices of P in Q.
11. Let $\mathcal{O}$ be a set of geometric objects and E a finite set. An **arrangement of elements of $\mathcal{O}$ indexed by E** is an element of $\mathcal{O}^E$. An arrangement indexed by E is typically written as $\mathcal{A} = (A_e : e \in E)$. If $E = [n]$ then we may write an arrangement indexed by E as a list $(A_1, \ldots, A_n)$, as we have been doing in the case $\mathcal{O} = \mathbb{R}^r$. We will continue to denote a rank r arrangement of vectors in $\mathbb{R}^r$ indexed by $[n]$ by $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathrm{Mat}(r, n)$.

    This definition has one exception that will become prominent near the end of Chapter 2. We'll introduce the set $\mathcal{O}$ of *pseudospheres* in a fixed sphere and then define an *arrangement of pseudospheres* indexed by E to be an element of $\mathcal{O}^E$ satisfying certain additional properties.

1.3.2 Representing Arrangements

A signed hyperplane $\mathcal{H}$ in $\mathbb{R}^2$ will always be drawn with an arrow pointing from H^0 into H^+, as in Figure 1.1.

We'll represent a signed hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$ in two ways.

- Let $\mathbf{v}_\infty$ be a nonzero vector such that neither $\mathbf{v}_\infty^\perp$ nor $-\mathbf{v}_\infty^\perp$ is in $\mathcal{A}$. We will consider the affine plane $\mathbb{A} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{v}_\infty = 1\}$. The set $\{\mathcal{H} \cap \mathbb{A} : \mathcal{H} \in \mathcal{A}\}$ is a signed affine hyperplane arrangement. Again, we draw an arrow from H^0 into H^+ to indicate the orientation. (See Figure 1.2 for an example.)

  Besides being easier to draw than an arrangement in $\mathbb{R}^3$, this kind of representation is useful for applying results in affine planar geometry to oriented matroids.
- We will draw the arrangement of equators $\{H^0 \cap S^2 : \mathcal{H} \in \mathcal{A}\}$ in S^2. We draw arrows from each equator $H^0 \cap S^2$ into $H^+ \cap S^2$. See Figure 1.3.

  This representation is preferred for two reasons: It saves us from worrying about the special plane $\mathbf{v}_\infty^\perp$, and, as we'll see in Chapter 4, it's the best route to representing arbitrary oriented matroids (not just realizable ones).

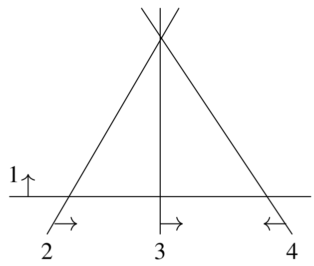


Figure 1.2 An affine representation of a signed hyperplane arrangement in $\mathbb{R}^3$.

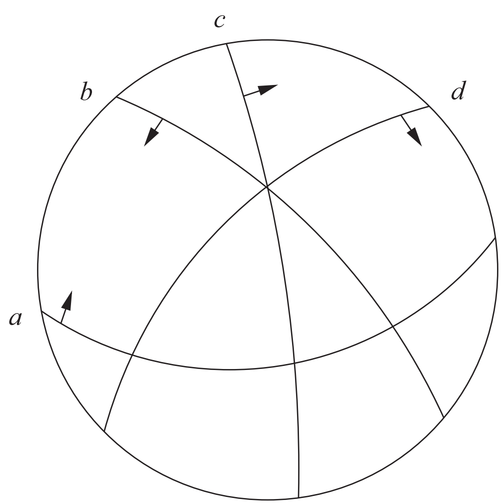


Figure 1.3 A spherical representation of a signed hyperplane arrangement in $\mathbb{R}^3$.

Consider a vector arrangement $M = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ in $\mathbb{R}^r$. If, for some $\mathbf{w} \in \mathbb{R}^r$, we have that $\mathbf{v}_i \cdot \mathbf{w} = 1$ for all i, then M can be viewed as a point arrangement in the affine space $\mathbb{A} := \{\mathbf{x} : \mathbf{x} \cdot \mathbf{w} = 1\}$. We will use this both to represent rank 3 vector arrangements in the plane and to imagine vector arrangements in higher dimensions.

Recall from Problem 1.6 that $\mathcal{V}(M)$, $\mathcal{V}^*(M)$, $\mathcal{C}(M)$, $\mathcal{C}^*(M)$, and $\chi(M)$ are all invariant under scaling the columns by positive scalars. Thus, if we have an arrangement $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ with all $\mathbf{v}_i$ in an open half-space $\{\mathbf{x} : \mathbf{x} \cdot \mathbf{w} > 0\}$, then we can scale each $\mathbf{v}_i$ to get an arrangement in the affine space $\{\mathbf{x} : \mathbf{x} \cdot \mathbf{w} = 1\}$, and this new arrangement can replace the original arrangement for our purposes. We call such an affine point arrangement an **affine representation** of M.

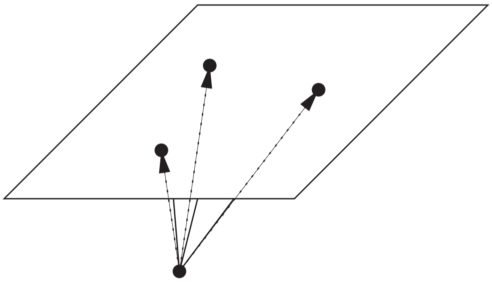


Figure 1.4 A point arrangement in an affine plane.

1.3.3 Vector Arrangements

Each of our models $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{C}$, $\mathcal{C}^*$ has a nice geometric interpretation for vector arrangements. We'll discuss each in turn.

Vectors and Circuits

Let $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be a matrix. The set $\mathcal{V}(M)$ of vectors corresponding to M is useful in studying convexity relationships between $\mathbf{v}_i$. For instance, a sign vector A^+B^- is in $\mathcal{V}(M)$ if and only if there are positive constants a_i, b_j for all $i \in A$, $j \in B$ such that

$$\sum_{i\in A} a_i \mathbf{v}_i + \sum_{j\in B} (-b_j)\mathbf{v}_j = \mathbf{0},$$
$$\sum_{i\in A} a_i \mathbf{v}_i = \sum_{j\in B} b_j \mathbf{v}_j.$$

Thus, $A^+B^- \in \mathcal{V}(M)$ if and only if the open cones spanned by $\{\mathbf{v}_i : i \in A\}$ and by $\{\mathbf{v}_i : i \in B\}$ intersect. (Recall that $\{\mathbf{0}\}$ is the open cone spanned by the empty set.)

If the columns of V lie in an affine space $\mathbb{A}$, then for nonempty sets A and B, $A^+B^- \in \mathcal{V}(M)$ if and only if the convex hulls of the point sets corresponding to A and B intersect in their relative interiors. The following proposition tells us a bit more about $\mathcal{C}(M)$.

Proposition 1.15 *Let $\{\mathbf{v}_i : i \in A\}$ and $\{\mathbf{v}_i : i \in B\}$ be the vectors corresponding to independent nonempty subsets of an affine space $\mathbb{A}$. If $A^+B^- \in \mathcal{C}(M)$ then the convex hulls of these two point sets intersect in a single point.*

(The converse is false: Consider the affine arrangement in Figure 1.5.)

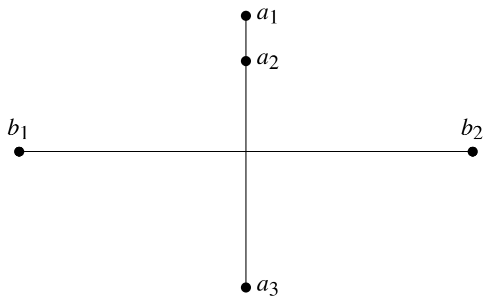


Figure 1.5 The convex hulls intersect in a single point, but this is not a minimal dependency.

Proof: By our preceding observations the intersection of the two convex hulls is nonempty. Assume by way of contradiction that it has two different elements $\sum_{i\in A} a_i \mathbf{v}_i = \sum_{i\in B} b_i \mathbf{v}_i$ and $\sum_{i\in A} \hat{a}_i \mathbf{v}_i = \sum_{i\in B} \hat{b}_i \mathbf{v}_i$. (Here all coefficients are nonnegative, and the sum of coefficients in each sum is 1.) Then for every ϵ

$$\sum_{i\in A}(a_i - \epsilon\hat{a}_i)\mathbf{v}_i = \sum_{i\in B}(b_i - \epsilon\hat{b}_i)\mathbf{v}_i.$$

Take $\epsilon > 0$ minimal such that some coefficient in this equation is 0. Then the other coefficients are all nonnegative, so this linear combination gives a smaller element of $\mathcal{V}(M)$. □

Here are further geometric insights about M that can be gleaned from $\mathcal{V}(M)$ and $\mathcal{C}(M)$:

- A subset S of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent if and only if there is no $X \in \mathcal{V}(M)$ whose support is a subset of $\{i : \mathbf{v}_i \in S\}$. In particular, from $\mathcal{V}(M)$ (or just $\mathcal{C}(M)$) we see the rank of the arrangement. The **rank** of $\mathcal{V}(M)$ is defined to be the rank of M.
- From $\mathcal{V}(M)$, we can tell if M has an affine representation, as follows. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is contained in some open half-space $\{\mathbf{x} : \mathbf{x} \cdot \mathbf{w} > 0\}$ of $\mathbb{R}^n$ if and only if $\mathbf{0}$ is not in any of the relatively open cones spanned by nonempty subsets of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.[2] $\mathbf{0}$ is in the open cone spanned by a subset S of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if and only if $\{i : i \in S\}^+ \in \mathcal{V}(M)$. Thus, M has an affine representation if and only if there is no nonempty S with $S^+ \in \mathcal{V}(M)$. In this case we say $\mathcal{V}(M)$ is *acyclic*. (A formal definition of acyclic is coming in Definition 3.7.)

[2] One direction of this assertion depends on the Farkas Lemma. See Section 1.5.2.

- If M does have an affine representation $(p_1, \ldots, p_n)$ with all p_i distinct, then $\{p_1, \ldots, p_n\}$ is the vertex set of a convex polytope if and only if no element of $\mathcal{V}(M)$ has the form $A^+\{b\}^-$. In this case, a subset $\{p_i : i \in S\}$ of the vertex set is the set of vertices of a face of the polytope if and only if no element of $\mathcal{V}(M)$ has the form A^+B^- with $B \subseteq S$. (Proving this requires some work.)

Covectors and Cocircuits

$\mathcal{V}^*(M)$ and $\mathcal{C}^*(M)$ also have good geometric interpretations in terms of the columns of M. This is another example of the oriented matroid trick brought up in Remark 1.12: $\mathcal{V}^*(M)$ is about the row space of M, but we can use it to talk about geometric properties of the set of columns.

Let $\mathbf{x} \in \text{row}(M)$, and so $\mathbf{x} = \mathbf{y}M$ for some $\mathbf{y} \in \mathbb{R}^r$. Thus $\text{sign}(\mathbf{x}) = (\text{sign}(\mathbf{y} \cdot \mathbf{v}_i) : i \in [n])$. Writing this in terms of the signed hyperplane $\mathcal{H} = \mathbf{y}^\perp$ and the sign vector $X = \text{sign}(\mathbf{x})$,

$$X(i) = \begin{cases} + & \text{if } \mathbf{v}_i \in H^+, \\ - & \text{if } \mathbf{v}_i \in H^-, \\ 0 & \text{if } \mathbf{v}_i \in H^0. \end{cases}$$

This proves the following.

Proposition 1.16 $A^+B^-C^0 \in \mathcal{V}^*(M)$ *if and only if there is a signed hyperplane* $\mathcal{H}$ *with* $\{\mathbf{v}_i : i \in A\} \subset H^+, \{\mathbf{v}_i : i \in B\} \subset H^-$, *and* $\{\mathbf{v}_i : i \in C\} \subset H^0$.

Proposition 1.17 $A^+B^-C^0 \in \mathcal{C}^*(M)$ *if and only if there is an oriented hyperplane* $\mathcal{H}$ *with* $A = \{i : \mathbf{v}_i \in H^+\}$, $B = \{i : \mathbf{v}_i \in H^-\}$, *and* H^0 *the span of* $\{\mathbf{v}_i : i \in C\}$.

Proof: Let $A^+B^-C^0 \in \mathcal{V}^*(\mathcal{M}) - \{\mathbf{0}\}$, and let $\mathcal{H}$ be a signed hyperplane with $\{\mathbf{v}_i : i \in A\} \subset H^+, \{\mathbf{v}_i : i \in B\} \subset H^-$, and $\{\mathbf{v}_i : i \in C\} \subset H^0$. Then certainly the span of $\{\mathbf{v}_i : i \in C\}$ is contained in H^0. Further, this span is exactly H^0 if and only if $\text{rank}(\mathbf{v}_i : i \in C) = r - 1$. Proposition 1.14 says this holds if and only if $A^+B^-C^0 \in \mathcal{C}^*(\mathcal{M})$. □

Let's go back to some of the geometric ideas previously discussed for $\mathcal{V}$ and $\mathcal{C}$:

- A set $\{\mathbf{v}_i : i \in S\}$ contains a maximal independent set if and only if it is not contained in any hyperplane in $\mathbb{R}^r$, hence if and only if there is no element of $V^*(M)$ whose support is contained in the complement of S.

- The columns of M all lie in one open half-space (and hence M has an affine representation) if and only if $[n]^+ \in V^*(M)$.
- If M does have an affine representation, then the convex hull of the elements of an affine representation is a convex polytope P. A set $\{\mathbf{v}_i : i \in A\}$ is the set of elements in a face of P if and only if there is a signed hyperplane $\mathcal{H}$ with $\{\mathbf{v}_i : i \in A\} \subset H^0$ and $P \cap H^- = \emptyset$, hence if and only if $([n] \setminus A)^+ \in V^*(M)$.

Chirotopes

Definition 1.18 An arrangement $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ in $\mathbb{R}^r$ is a **positively oriented basis** of $\mathbb{R}^r$ if $\text{sign}(\det(\mathbf{v}_1, \ldots, \mathbf{v}_r)) = +$ and is a **negatively oriented basis** of $\mathbb{R}^r$ if $\text{sign}(\det(\mathbf{v}_1, \ldots, \mathbf{v}_r)) = -$.

For $M \in \text{Mat}(r,n)$, the chirotope $\chi(M)$ encodes for each ordered r-tuple of columns of M, whether that r-tuple is a positively oriented basis, a negatively oriented basis, or not an ordered basis at all.

To review the geometric meaning of this: The determinant function from GL_r to $\mathbb{R} - \{0\}$ is continuous and surjective, so GL_r has at least two connected components. With a little work one can see that GL_r has exactly two path components, namely $GL_r^+ = \{A \in GL_r : \det(A) > 0\}$ and $GL_r^- = \{A \in GL_r : \det(A) < 0\}$, and these components are homeomorphic. Thus an ordered basis $(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r})$ is positively oriented if and only if we can continuously deform the matrix $(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r})$ to the identity matrix while maintaining linear independence of columns, and similarly we can describe negatively oriented bases.

Another geometric way to think of this: Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_{r-1}\}$ be a linearly independent set in $\mathbb{R}^r$, and let H be the span $\langle \mathbf{w}_1, \ldots, \mathbf{w}_{r-1} \rangle$. Then H is the zero locus of the continuous function $\mathbf{v} \to \det(\mathbf{w}_1, \ldots, \mathbf{w}_{r-1}, \mathbf{v})$ from $\mathbb{R}^r$ to $\mathbb{R}$, and so the two connected components of $\mathbb{R}^r - H$ are the sets

$$\{\mathbf{v} : \text{sign}(\det(\mathbf{w}_1, \ldots, \mathbf{w}_{r-1}, \mathbf{v})) = +\}$$

and

$$\{\mathbf{v} : \text{sign}(\det(\mathbf{w}_1, \ldots, \mathbf{w}_{r-1}, \mathbf{v})) = -\}.$$

Now consider a vector arrangement $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, the associated chirotope χ, and $i_1, \ldots, i_{r-1}$ such that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{r-1}}\}$ is linearly independent. Let $H = \langle \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{r-1}} \rangle$. For each $j \in [n]$, we have that $\chi(i_1, \ldots, i_{nr1}, j) = 0$ if and only if $\mathbf{v}_j \in H$, and for each $j, k \in [n]$, we have that $\chi(i_1, \ldots, i_{r-1}, j) = -\chi(i_1, \ldots, i_{r-1}, k) \neq 0$ if and only if $\mathbf{v}_j$ and $\mathbf{v}_k$ are on opposite sides of H.

Notice a difference between our geometric interpretation of chirotopes for vector arrangements and our geometric interpretations of $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{C}$, and $\mathcal{C}^*$.

Each of our earlier interpretations was coordinate-independent – given a vector arrangement in an arbitrary finite-dimensional real vector space (not necessarily $\mathbb{R}^r$), we could associate sets $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{C}$, and $\mathcal{C}^*$ by way of the interpretations we've described. By contrast, our interpretation of the chirotope references a preferred ordered basis for $\mathbb{R}^r$ (the list of columns of the identity matrix).

Given a general rank r vector space W over $\mathbb{R}$, we get an equivalence relation on the set of ordered bases of W, by saying $(\mathbf{v}_1, \ldots, \mathbf{v}_r) \sim (\mathbf{w}_1, \ldots, \mathbf{w}_r)$ if, for some (and therefore all) isomorphisms $f: W \to \mathbb{R}^r$, the matrices $(f(\mathbf{v}_1), \ldots, f(\mathbf{v}_r))$ and $(f(\mathbf{w}_1), \ldots, f(\mathbf{w}_r))$ are in the same connected component of GL_r. An *orientation* of W is a choice of one of these equivalence classes to be considered the set of positively oriented bases. So, given a rank r vector arrangement $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ in W and a choice of orientation of W, we get an associated chirotope $\chi: [n]^r \to \{0, +, -\}$. The opposite choice of orientation for W gives the chirotope $-\chi$.

We'll see in Section 1.5 that the same information about a vector arrangement M is encoded by each of $\mathcal{V}(M)$, $\mathcal{V}^*(M)$, $\mathcal{C}(M)$, $\mathcal{C}^*(M)$, or $\{\chi(M), -\chi(M)\}$.

1.3.4 Signed Hyperplane Arrangements

For a matrix $M = (\mathbf{v}_1 \cdots \mathbf{v}_n)$, consider the associated signed hyperplane arrangement $\mathcal{A} = (\mathbf{v}_1^\perp, \ldots, \mathbf{v}_n^\perp)$. Each triple $((\mathbf{v}_i^\perp)^0, (\mathbf{v}_i^\perp)^+, (\mathbf{v}_i^\perp)^-)$ with $\mathbf{v}_i \neq \mathbf{0}$ is a partition of $\mathbb{R}^r$ into three parts. The common refinement of these partitions is a decomposition of $\mathbb{R}^r$ into convex cones. Each cone can be specified by its relationship to each hyperplane: $A^+B^-C^0$ represents the cone $\{\mathbf{y} \in \mathbb{R}^r : \mathbf{y} \cdot \mathbf{v}_i > 0 \text{ for all } i \in A;\ \mathbf{y} \cdot \mathbf{v}_i < 0 \text{ for all } i \in B;\ \mathbf{y} \cdot \mathbf{v}_i = 0 \text{ for all } i \in C\}$. (See Figure 1.6 for an example.) Thus we see a bijection from $\mathcal{V}^*(M)$ to cones of this partition, taking each X to $\{\mathbf{y} \in \mathbb{R}^r : \operatorname{sign}(\mathbf{y}^\top M) = X\}$. The cones that are rays correspond to $\mathcal{C}^*(M)$ under this bijection.

Often it's more convenient to work with the arrangement of oriented equators $\mathbf{v}_i^\perp \cap S^{r-1}$ in S^{r-1}. This arrangement defines a decomposition of S^{r-1} into spherically convex cells that are in bijection with $\mathcal{V}^*(M)\backslash\{\mathbf{0}\}$.

The interpretations of $\mathcal{V}$, $\mathcal{C}$, and χ for signed hyperplane arrangements are not as illuminating and won't be discussed here. See Section 4.8 for some development of these interpretations.

1.4 Subspaces

Every rank r subspace W of $\mathbb{R}^n$ is the row space of some $M \in \operatorname{Mat}(r, n)$, and by definition $\mathcal{V}(M) = \{\operatorname{sign}(x) : x \in W^\perp\}$ and $\mathcal{V}^*(M) = \{\operatorname{sign}(x) : x \in W\}$.

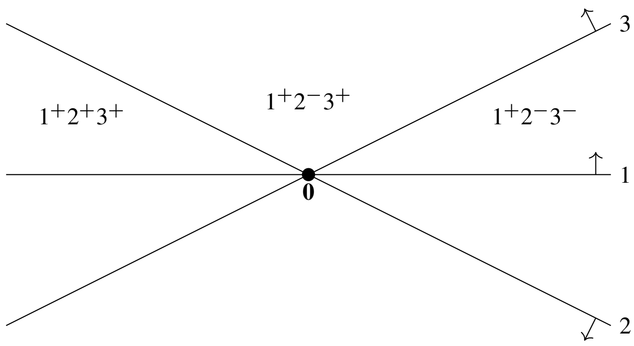


Figure 1.6 An arrangement of signed hyperplanes in $\mathbb{R}^2$ and some of the resulting sign vectors.

So our "discrete models for real matrices" could as well be thought of as "discrete models for real subspaces of $\mathbb{R}^n$," and these models nicely reflect orthogonality of vector spaces.

To recast things in these terms: The **real Grassmannian** $G(r, \mathbb{R}^n)$ is the set of all rank r subspaces of $\mathbb{R}^n$. Thus each element of $G(r, \mathbb{R}^n)$ is the row space of a matrix $M \in \mathrm{Mat}(r, n)$. Notice that two matrices $M, M' \in \mathrm{Mat}(r, n)$ have the same row space if and only if each row of M is a linear combination of rows of M', that is, $M = AM'$ for some invertible matrix A. Thus $G(r, \mathbb{R}^n)$ can be identified with the quotient of $\mathrm{Mat}(r, n)$ by the left action of GL_r. Our maps $\mathcal{V}\colon \mathrm{Mat}(r, n) \to \mathcal{P}(\{0, +, -\}^n)$ and $\mathcal{V}^*\colon \mathrm{Mat}(r, n) \to \mathcal{P}(\{0, +, -\}^n)$ quotient to maps

$$\begin{array}{rccc} \mathcal{V}\colon & G(r, \mathbb{R}^n) & \to & \mathcal{P}(\{0, +, -\}^n) \\ & W & \to & \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in W^{\perp}\} \\ \mathcal{V}^*\colon & G(r, \mathbb{R}^n) & \to & \mathcal{P}(\{0, +, -\}^n) \\ & W & \to & \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in W\}, \end{array}$$

and we can define functions $\mathcal{C}$ and $\mathcal{C}^*$ on $G(r, \mathbb{R}^n)$ either as quotients of our maps $\mathcal{C}, \mathcal{C}^*$ on $\mathrm{Mat}(r, n)$, or in terms of $\mathcal{V}$ and $\mathcal{V}^*$ as before.

The view of vector and covector sets as simply the sets of sign vectors arising from subspaces of $\mathbb{R}^n$ is fundamental. We'll frequently bring up the idea of oriented matroids as combinatorial analogs to subspaces of $\mathbb{R}^n$. One useful point to notice is that the set of nonzero elements of $\{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in W\}$ is just the set $\{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in W, \|x\| = 1\}$ of sign vectors arising from the unit sphere in W. Loosely put, *an oriented matroid $\mathcal{M}$ is a combinatorial analog to a vector space, and $\mathcal{V}^*(\mathcal{M})\backslash\{0\}$ is a combinatorial analog to the unit sphere in that vector space.* This idea is central to Chapter 4.

Note that orthogonality of vector spaces is reflected nicely in oriented matroids: $\mathcal{V}(W) = \mathcal{V}^*(W^\perp)$ and $\mathcal{V}^*(W) = \mathcal{V}(W^\perp)$. In general, we will say two oriented matroids are *dual* if the vector set of one is the covector set of the other. As we shall see in Chapter 2, *every* oriented matroid has a dual, and these dual pairs behave like pairs of orthogonally complementary vector spaces. Duality is an idea of central importance in oriented matroid theory and applications. For instance, in convex polytopes it arises in the form of *Gale diagrams* – see Section 1.6 for details.

1.4.1 Representing Subspaces by Vector Arrangements

Let's tie the current discussion of subspaces together with our earlier discussion on vector arrangements. Given a subspace W of $\mathbb{R}^n$, choose a basis of W and make it the set of *rows* of a matrix M. Then the set of *columns* of M is a vector arrangement, and the vectors, covectors, and so on associated to W coincide with the vectors, covectors, and so on of this vector arrangement. Thus studying oriented matroid properties associated to subspaces of $\mathbb{R}^n$ is the same as studying oriented matroid properties of vector arrangements. This is often a useful way to address subspace issues – see, for instance, Exercise 1.6.

This transition from rows to columns may seem mysterious, but the following proposition shows that the arrangement of column vectors is essentially just the vector arrangement in W obtained by projecting the unit coordinate vectors in $\mathbb{R}^n$ onto W.

Proposition 1.19 *Let* $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathrm{Mat}(r,n)$ *and let* $W = \mathrm{row}(M)$. *Let* $\pi_W : \mathbb{R}^n \to \mathbb{R}^n$ *denote the orthogonal projection onto* W. *Then there is an isomorphism* $\mathbb{R}^r \to W$ *sending each* $\mathbf{v}_i$ *to* $\pi_W(\mathbf{e}_i)$.

Proof: There is an automorphism of W sending the rows of M to an orthonormal basis for W, that is, there is an $A \in GL_r$ such that AM has orthonormal rows. The vector arrangement given by the columns of AM is the image under an isomorphism of the vector arrangement given by the columns of M. So without loss of generality we assume the rows of M to be orthonormal.

Let $\mathcal{B}$ be the vector arrangement in W consisting of the orthogonal projections $\pi_W(\mathbf{e}_i)$ of the unit coordinate vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ onto W. We can think of $\mathcal{B}$ as the columns of the $n \times n$ matrix C_W representing orthogonal projection of $\mathbb{R}^n$ onto W, but in contrast to our usual identification of vector arrangements with matrices, the rank of C_W is typically less than the number of rows. Since the set of rows of M is an orthonormal basis for W, the columns of MC_W represent $\mathcal{B}$ in terms of this basis.

But, letting $C_{W^\perp}$ be the $n \times n$ matrix representing orthogonal projection of $\mathbb{R}^n$ onto $W^\perp$, we have

$$\begin{aligned} MC_W &= M(C_W + C_{W^\perp}) \qquad \text{since } MC_{W^\perp} = 0 \\ &= MI \\ &= M, \end{aligned}$$

and so the map $\mathbb{R}^r \to W$ sending the standard basis to the rows of M sends each $\mathbf{v}_i$ to $\pi_W(\mathbf{e}_i)$. □

1.4.2 Representing Subspaces by Signed Hyperplane Arrangements

Given a subspace W of $\mathbb{R}^n$, let $\mathcal{A}$ be the signed hyperplane arrangement $\{\mathbf{e}_i^\perp \cap W : i \in [n]\}$ in W. Then from the definition of $\mathcal{V}^*(W)$ and our earlier discussion of hyperplane arrangements, we see that the oriented matroid of $\mathcal{A}$ is the oriented matroid of W.

In fact, since $\mathbf{w} \cdot \mathbf{e}_i = \mathbf{w} \cdot \pi_W(\mathbf{e}_i)$ for each $\mathbf{w} \in W$, we see that $\mathcal{A}$ is the signed hyperplane arrangement corresponding to the vector arrangement $\mathcal{B} = \{\pi_W(\mathbf{e}_i) : i \in [n]\}$ of Section 1.4.1.

Problem 1.20 Consider the map $G(1,\mathbb{R}^n) \to \mathcal{P}(\{0,+,-\}^n)$ sending each W to $\mathcal{V}^*(W)$. Describe the partition of $\mathbb{RP}^{n-1} = G(1,\mathbb{R}^n)$ by preimages under this map.

1.4.3 Representing Subspaces by Points in Projective Space: The Plücker Embedding

Our discussion so far has put chirotopes in a different realm from the other oriented matroid characterizations. Each of those was given as the sign of a collection of vectors in $\mathbb{R}^n$, with each of these vectors having a nice geometric interpretation. In this section we'll see how to view the pair $\pm\chi(M)$ as giving the sign pair of a point in projective space, again with useful geometric meaning.

The key result is the following:

Proposition 1.21 *Let* $M, M' \in \mathrm{Mat}(r,n)$. *Then* $\mathrm{row}(M) = \mathrm{row}(M')$ *if and only if there is a nonzero scalar* c *such that* $|M_{i_1,\ldots,i_r}| = c|M'_{i_1,\ldots,i_r}|$ *for each* $i_1,\ldots,i_r \in [n]$.

We can prove Proposition 1.21 in a naive way, or we can use the machinery of *exterior algebras*. The exterior algebra perspective has found powerful application (cf. Bokowski and Sturmfels 1989), but little of that has made it into this book. Here we'll give both the naive and fancier approaches: The reader may choose which to follow.

Naive proof: ($\Rightarrow$) If $\text{row}(M) = \text{row}(M')$ then $M = AM'$ for some $A \in GL_r$. For each $i_1 < \cdots < i_r$ in $[n]$, the submatrix $(AM')_{i_1,\ldots,i_r}$ is just $A(M'_{i_1,\ldots,i_r})$. Thus our scalar c is $|A|$.

($\Leftarrow$) Choose a $\{a_1, \ldots, a_r\} \subseteq [n]$ with $a_1 < \cdots < a_r$ such that $|M_{a_1,\ldots,a_r}| \neq 0$. Thus $\hat{M} := (M_{a_1,\ldots,a_r})^{-1}M$ is in reduced row-echelon form with respect to $\{a_1, \ldots, a_r\}$, and we can read off the entries of $\hat{M}$ as follows.

Let $\mathbf{v}_j$ denote the jth column of M, $\hat{\mathbf{v}}_j$ the jth column of $\hat{M}$, and $\hat{m}_{ij}$ denote individual entries of $\hat{M}$. Then

$$\begin{aligned}
\hat{m}_{ij} &= |\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \hat{\mathbf{v}}_j, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_r| \\
&= |\hat{\mathbf{v}}_{a_1}, \ldots, \hat{\mathbf{v}}_{a_{i-1}}, \hat{\mathbf{v}}_j, \hat{\mathbf{v}}_{a_{i+1}}, \ldots, \hat{\mathbf{v}}_{a_r}| \\
&= |\hat{M}_{a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r}| \\
&= |(M_{a_1,\ldots,a_r})^{-1}||M_{a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r}|.
\end{aligned}$$

Our hypothesis tells us that

$$|(M_{a_1,\ldots,a_r})^{-1}||M_{a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r}| = |(M'_{a_1,\ldots,a_r})^{-1}||M'_{a_1,\ldots,a_{i-1},j,a_{i+1},\ldots,a_r}|.$$

Thus $\hat{M} = (M'_{a_1,\ldots,a_r})^{-1}M'$ as well, and so $M = M_{a_1,\ldots,a_r}(M'_{a_1,\ldots,a_r})^{-1}M'$. Since $M_{a_1,\ldots,a_r}(M'_{a_1,\ldots,a_r})^{-1} \in GL_r$, we have $\text{row}(M) = \text{row}(M')$. □

To give the exterior algebra perspective, we start with a quick review of tensor products.

Definition 1.22 Let $A_1, \ldots, A_r$, and V be vector spaces over a field $\mathbb{K}$. A function $f\colon A_1 \times \cdots \times A_r \to V$ is **multilinear** if it is linear in each coordinate.

One example of a multilinear map is the determinant, viewed as a function taking a sequence of row vectors to a field element. A generalization of this example: Let $r \leq n$, and for every $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{K}^n$, let $R(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ be the matrix with rows $\mathbf{v}_1, \ldots, \mathbf{v}_r$. We get a multilinear map $(\mathbb{K}^n)^r \to (\mathbb{K})^{\binom{n}{r}}$ by sending each $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ to the vector $(|R(\mathbf{v}_1, \ldots, \mathbf{v}_r)_{i_1,\ldots,i_r}| : 1 \leq i_1 < \cdots < i_r \leq n)$.

The **tensor product** $A \otimes B$ of vector spaces A and B over $\mathbb{K}$ is the essentially unique vector space with the following universal property: There is a map $A \times B \to A \otimes B$ such that for each vector space V, each multilinear map $m\colon A \times B \to V$ factors uniquely as $A \times B \to A \otimes B \xrightarrow{l} V$, where l is linear. Thus the tensor product is a gadget we use to move from multilinear maps to linear maps. The tensor product is constructed as a quotient of the free vector space on $A \times B$: We simply quotient out by what we need to in order to get our universal property. That is, we identify formal sums $\sum_{\substack{\mathbf{a}\in A\\ \mathbf{b}\in B}} c_{\mathbf{a},\mathbf{b}}(\mathbf{a},\mathbf{b})$ and $\sum_{\substack{\mathbf{a}\in A\\ \mathbf{b}\in B}} d_{\mathbf{a},\mathbf{b}}(\mathbf{a},\mathbf{b})$ if, for every vector space V and multilinear map $f\colon A\times B\to V$, we have $\sum_{\substack{\mathbf{a}\in A\\ \mathbf{b}\in B}} c_{\mathbf{a},\mathbf{b}} f(\mathbf{a},\mathbf{b}) = \sum_{\substack{\mathbf{a}\in A\\ \mathbf{b}\in B}} d_{\mathbf{a},\mathbf{b}} f(\mathbf{a},\mathbf{b})$.

For each $\mathbf{a}\in A$ and $\mathbf{b}\in B$, let $\mathbf{a}\otimes\mathbf{b}$ denote the equivalence class of $(\mathbf{a},\mathbf{b})$ under this equivalence. Hence every element of $A\otimes B$ can be expressed (usually not uniquely) as the equivalence class of a sum of terms of the form $k(\mathbf{a}\otimes\mathbf{b})$ with $k\in\mathbb{K}$, $\mathbf{a}\in A$, and $\mathbf{b}\in B$. The map $A\times B\to A\otimes B$ sends $(\mathbf{a},\mathbf{b})$ to $\mathbf{a}\otimes\mathbf{b}$. It is easy to check that if $\{\mathbf{a}_1,\dots,\mathbf{a}_\alpha\}$ is a basis for A and $\{\mathbf{b}_1,\dots,\mathbf{b}_\beta\}$ is a basis for B then $A\otimes B$ has basis $\{\mathbf{a}_i\otimes\mathbf{b}_j : i\in[\alpha], j\in[\beta]\}$.

For a vector space A, the **exterior product** $\bigwedge^r A$ is the quotient of the tensor product $\otimes^r A$ by $\langle \mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_r - \operatorname{sign}(\sigma)\mathbf{v}_{\sigma(1)}\otimes\cdots\otimes\mathbf{v}_{\sigma(r)} : \mathbf{v}_i\in A, \sigma\in S_r\rangle$. It follows from the universal property for tensor products that the composition $A^r\to\otimes^r A\to\bigwedge^r A$ is universal for alternating multilinear maps. That is, each alternating multilinear map $A^r\to V$ factors through this map.

The image of $\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_r$ under the quotient map $\otimes^r A\to\bigwedge^r A$ is denoted $\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_r$. Since $\bigwedge^r A$ is the quotient of a vector space by a subspace, it inherits a vector space structure. If $\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$ is a basis for A then $\{\mathbf{a}_{i_1}\wedge\cdots\wedge\mathbf{a}_{i_r} : i_i<\cdots<i_r\}$ is a basis for $\bigwedge^r A$. Additionally, $\bigoplus_r\bigwedge^r A$ has a product operation, sending a pair $(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_r, \mathbf{x}_{r+1}\wedge\cdots\wedge\mathbf{x}_{r+s})$ to $\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_{r+s}$.

Now let's focus on the vector space $\mathbb{R}^n$. The vector space $\bigwedge^r\mathbb{R}^n$ has basis $\{\mathbf{e}_{i_1}\wedge\mathbf{e}_{i_2}\wedge\cdots\wedge\mathbf{e}_{i_r} : 1\le i_1<i_2<\cdots<i_r\le n\}$, hence is isomorphic as a vector space to $\mathbb{R}^{\binom{n}{r}}$.

Proposition 1.23 *Let $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_r$ be the rows of a $r\times n$ matrix M. Then*

$$\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_r = \sum_{i_1<\cdots<i_r} |M_{i_1,\dots,i_r}|\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_r}.$$

To see this, write each $\mathbf{v}_i$ as $\sum_j m_{i,j}\mathbf{e}_j$ and expand out $\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_r$, remembering the formula for the determinant $|A| = \sum_{\sigma\in S_r}\operatorname{sign}(\sigma)a_{1,\sigma_1}\cdots a_{r,\sigma_r}$.

Corollary 1.24 *Let $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_r\in\mathbb{R}^n$. Then $\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_r=0$ if and only if $\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_r\}$ has rank less than r.*

This happens when either the sequence $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_r$ has repeated elements or the set $\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_r\}$ is dependent.

Proof: Let M be a matrix with rows $\mathbf{v}_1,\dots,\mathbf{v}_r$. Then

$$\begin{aligned}\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_r=0 &\Leftrightarrow |M_{i_1,\dots,i_r}|=0 \text{ for all } i_1,\dots,i_r\\ &\Leftrightarrow \operatorname{rank}(M_{i_1,\dots,i_r})<r \text{ for all } i_1,\dots,i_r\\ &\Leftrightarrow \operatorname{rank}(M)<r.\end{aligned}$$

□

Exterior Algebra Proof of Proposition 1.21: The proof of ($\Rightarrow$) is the same as that for the naive proof. To see ($\Leftarrow$), let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the rows of M and $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be the rows of M'. Thus

$$\begin{aligned}\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r &= \sum\nolimits_{i_1 < \cdots < i_r} |M_{i_1,\ldots,i_r}| \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r} \\ &= \sum\nolimits_{i_1 < \cdots < i_r} |cM'_{i_1,\ldots,i_r}| \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r} \\ &= c\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r.\end{aligned}$$

Then for each i we have $(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r) \wedge \mathbf{w}_i = c(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r) \wedge \mathbf{w}_i$, which is 0 by Corollary 1.24. Thus each $\mathbf{w}_i$ is in $\langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$. Similarly, for each i we have $\mathbf{v}_i \in \langle \mathbf{w}_1, \ldots, \mathbf{w}_r \rangle$. □

Notation 1.25 For a set S and an $r < n$, let $S^{\binom{n}{r}}$ denote the set of all vectors $(s_{i_1,\ldots,i_r} : 1 \leq i_1 < \cdots < i_r \leq n)$ with each component in S.

$\mathbb{R}P^{\binom{n}{r}-1}$ denotes the real projective space consisting of all one-dimensional subspaces of $\mathbb{R}^{\binom{n}{r}}$.

Proposition 1.21 gives us the **Plücker embedding**

$$P : G(r, \mathbb{R}^n) \to \mathbb{R}P^{\binom{n}{r}-1}$$

$$P(\mathrm{row}(M)) = \mathbb{R}\left(\sum_{i_1 < \cdots < i_r} |M_{i_1,\ldots,i_r}| : 1 \leq i_1 < \cdots < i_r \leq n\right),$$

or, in terms of the exterior algebra,

$$P : G(r, \mathbb{R}^n) \to \mathbb{P}\left(\bigwedge^r \mathbb{R}^n\right) \cong \mathbb{R}P^{\binom{n}{r}-1}$$

$$\langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle \to \mathbb{R}\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r = \mathbb{R}\left(\sum_{i_1 < \cdots < i_r} |M_{i_1,\ldots,i_r}| \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}\right).$$

This gives us a geometric interpretation of the chirotope as the sign of a vector: If χ is one of the two chirotopes arising from a space W and $i_1 < \cdots < i_r$, then $\chi(i_1, \ldots, i_r)$ is just the sign of the $(i_1, \ldots, i_r)$ coordinate of the Plücker embedding of W.

Remark 1.26 The naive proof of Proposition 1.21 pointed out how to find the entries of a matrix in reduced row-echelon form given just the maximal minors. Using this, we see how to recover the signs of entries of the matrix given only the chirotope. Let χ be the chirotope of a matrix $M = (m_{ij}) \in \mathrm{Mat}(r, n)$

that is in reduced row-echelon form with respect to $\{a_1, \ldots, a_r\}$. Then for each i, j,

$$\operatorname{sign}(m_{ij}) = \chi(a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots, a_r).$$

With this observation we have a small window into a thorny problem: *Given an oriented matroid, find the space of all matrices with this oriented matroid.* This is essentially the question of determining the *realization space* of an oriented matroid, the subject of Chapter 7.

1.4.4 Aside: Topology of the Grassmannian

We defined the Grassmannian $G(r, \mathbb{R}^n)$ as a set to be the quotient of $\operatorname{Mat}(r,n)$ by the left action of GL_r. $\operatorname{Mat}(r,n)$ has a topology as a subspace of $\mathbb{R}^{r \times n}$, and so $G(r, \mathbb{R}^n)$ has a topology as a quotient of $\operatorname{Mat}(r,n)$. The Plücker embedding $P\colon G(r, \mathbb{R}^n) \to \mathbb{P}(\bigwedge^r \mathbb{R}^n)$ is a homeomorphism, so this topology on the Grassmannian coincides with the subspace topology in $\mathbb{P}(\bigwedge^r \mathbb{R}^n)$. It also coincides with the intuitive topology on the Grassmannian: If $V \in G(r, \mathbb{R}^n)$ then the set of r-dimensional subspaces of $\mathbb{R}^n$ that we get by wiggling V around just a bit is an open neighborhood of V.

The topology of the real Grassmannian is important for many reasons, for example in the theory of characteristic classes, and it will play a prominent role in later parts of this book. So we'll explore this topology briefly here. See, for instance, Milnor and Stasheff (1974) for more.

We'll show that $G(r, \mathbb{R}^n)$ is a manifold of dimension $r(n-r)$. First consider

$$\begin{aligned} U_{1,\ldots,r} &= \{\operatorname{row}(I|A) : (I|A) \in \operatorname{Mat}(r,n)\} \\ &= \{\operatorname{row}(M) : M \in \mathbb{R}^{r \times n} \text{ and } M_{1,\ldots,r} \in GL_r\}. \end{aligned}$$

This is an open subset of $G(r, \mathbb{R}^n)$, and the correspondence

$$\begin{aligned} U_{1,\ldots,r} &\leftrightarrow \mathbb{R}^{r \times (n-r)} \\ \operatorname{row}(I|A) &\leftrightarrow A \end{aligned}$$

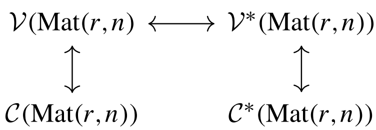
is a homeomorphism. This homeomorphism is a coordinate chart for all $V \in G(r, \mathbb{R}^n)$ such that the $(1, \ldots, r)$-coordinate of $P(V)$ is nonzero.

Generalizing this, for each $\{i_1, \ldots, i_r\} \in [n]$ we define $U_{i_1,\ldots,i_r} = \{\operatorname{row}(M) : M_{i_1,\ldots,i_r} \in GL_r\}$, and we see that $U_{i_1,\ldots,i_r}$ is homeomorphic to $\mathbb{R}^{r \times (n-r)}$. The collection of all such $U_{i_1,\ldots,i_r}$ is an open cover of the Grassmannian.

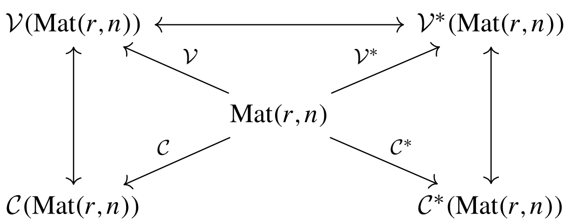
1.5 Cryptomorphisms for Realizable Oriented Matroids

The previous sections showed that the various sets $\mathcal{V}(M)$, $\mathcal{C}(M)$, $\mathcal{V}^*(M)$, and $\mathcal{C}^*(M)$ are geometrically interesting things to look at. As we are about to see, these sets encode exactly the same information about M, and χ encodes slightly more. Specifically,

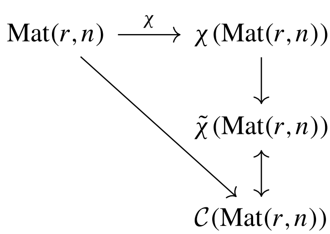
1. There are bijections



   commuting with the maps $\mathcal{V}$, $\mathcal{C}$, $\mathcal{V}^*$, and $\mathcal{C}^*$. That is, the diagram



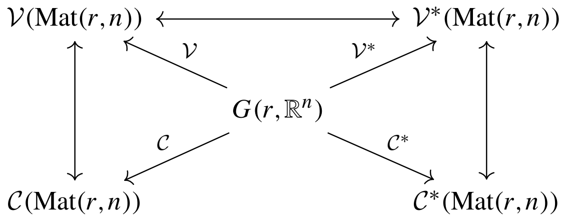
   commutes.
2. For every $M \in \mathrm{Mat}(r,n)$, let $\tilde{\chi}(M) = \{\chi(M), -\chi(M)\}$. Then there is a bijection $\tilde{\chi}(\mathrm{Mat}(r,n)) \leftrightarrow \mathcal{C}(\mathrm{Mat}(r,n))$ so that



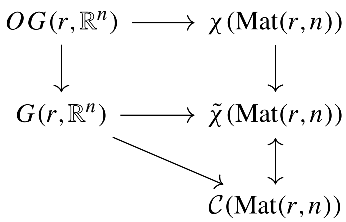
   commutes.

The following sections will establish these two points. The phrase we'll use for these results is to say the models $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{C}$, $\mathcal{C}^*$, and $\tilde{\chi}$ are **cryptomorphic** – they encode the same data in different ways.

To phrase this in terms of the Grassmannian: Recall $G(r,\mathbb{R}^n)$ is essentially the quotient of $\mathrm{Mat}(r,n)$ by the left action of GL_r, and that $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{C}$, and $\mathcal{C}^*$ are all invariant under the action of GL_r on $\mathrm{Mat}(r,n)$. Additionally, define the **oriented Grassmannian** $OG(r,\mathbb{R}^n)$ to be the quotient of $\mathrm{Mat}(r,n)$ by the left action of the group GL_r^+ of matrices with positive determinant. An element of $OG(r, \mathbb{R}^n)$ can be thought of as an r-dimensional subspace of $\mathbb{R}^n$ equipped with a distinguished orientation. Thus the previous diagrams induce diagrams



and



where the top two vertical maps $OG(r, \mathbb{R}^n) \to G(r, \mathbb{R}^n)$ and $\tilde{\chi}(\mathrm{Mat}(r,n)) \to \chi(\mathrm{Mat}(r,n))$ are double covers.

All of these results will be subsumed by results of Chapter 2, which will establish cryptomorphisms for general oriented matroids. We're doing the more limited results here to geometrically motivate our Chapter 2 results.

1.5.1 The Bijections $\mathcal{V}(\mathrm{Mat}(r,n)) \leftrightarrow \mathcal{C}(\mathrm{Mat}(r,n))$ and $\mathcal{V}^*(\mathrm{Mat}(r,n)) \leftrightarrow \mathcal{C}^*(\mathrm{Mat}(r,n))$

We'll get both of these bijections by a single argument. The forward maps $\mathcal{V}(\mathrm{Mat}(r,n)) \to \mathcal{C}(\mathrm{Mat}(r,n))$ and $\mathcal{V}^*(\mathrm{Mat}(r,n)) \to \mathcal{C}^*(\mathrm{Mat}(r,n))$ are obvious: Just take each set to its set of minimal nonzero elements. It remains to be seen how we can reconstruct elements of $\mathcal{V}(M)$ resp. $\mathcal{V}^*(M)$ from the minimal nonzero elements of these sets.

Definition 1.27 For two sign vectors $X, Y \in \{0, +, -\}^n$, define their *composition* $X \circ Y \in \{0, +, -\}^n$ by

$$X \circ Y(e) = \begin{cases} X(e) & \text{if } X(e) \neq 0, \\ Y(e) & \text{otherwise.} \end{cases}$$

We take the zero vector to be the composition of the empty sequence of elements of $\mathcal{C}$.

Problem 1.28 In Figure 1.6, in how many ways can the covector $(+, +, +)$ be expressed as a composition of two covectors?

The following proposition tells us how to reconstruct $\mathcal{V}$ from $\mathcal{C}$.

Proposition 1.29 *Let W be a subspace of $\mathbb{R}^n$. Let $\mathcal{V} = \{\mathrm{sign}(\mathbf{x}) : \mathbf{x} \in W^\perp\}$, and let $\mathcal{C}$ be the set of minimal nonzero elements of $\mathcal{V}$. Then $\mathcal{V}$ is the set of all compositions of elements of $\mathcal{C}$.*

In fact, every nonzero element X of $\mathcal{V}$ is the composition of signed circuits that are less than or equal to X.

Lemma 1.30 *For every subspace V of $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in V$, there exists $\mathbf{z} \in V$ such that* $\mathrm{sign}(\mathbf{x}) \circ \mathrm{sign}(\mathbf{y}) = \mathrm{sign}(\mathbf{z})$.

Proof: Let $\mathbf{z} = \mathbf{x} + \epsilon \mathbf{y}$, where ϵ is a sufficiently small positive real number. □

Definition 1.31 $X, Y \in \{0, +, -\}^E$ are **conformal** if their separation set $S(X, Y)$ is empty.

A set of sign vectors is **conformal** if each pair of elements is conformal.

In particular, if $Y \leq X$ and $Y' \leq X$ then Y and Y' are conformal. A set $\{Y_1, \ldots, Y_k\} \subseteq \{0, +, -\}^E$ is conformal if and only if, for each e, $\max(Y_i(e) : i \in [k])$ exists. In this case $Y_1 \circ \cdots \circ Y_k(e) = \max(Y_i(e) : i \in [k])$.

Proof of Proposition 1.29: Lemma 1.30 shows that every composition of elements of $\mathcal{C}$ is in $\mathcal{V}$. Conversely, let $X \neq \mathbf{0}$ be an element of $\mathcal{V}$. Let $Y_1, \ldots, Y_k$ be the elements of $\mathcal{C}$ that are less than or equal to X, and let $Y = Y_1 \circ \cdots \circ Y_k$. Notice that Y does not depend on the order of Y_i and that $Y \leq X$. We'll induct on the height of X in the poset $\mathcal{V}$ to see that $Y = X$. The minimal case is when $X \in \mathcal{C}$, so that $X = Y_1 = Y$.

Above this, assume by way of contradiction that $\mathrm{supp}(X) \backslash \mathrm{supp}(Y) \neq \emptyset$. Let $\mathbf{x}, \mathbf{y} \in W^\perp$ with $X = \mathrm{sign}(\mathbf{x})$ and $Y = \mathrm{sign}(\mathbf{y})$. Consider the ray $\{\mathbf{x} - \lambda \mathbf{y} : \lambda \geq 0\}$ in $W^\perp$. All points on this ray have the same eth coordinate for each $e \in \mathrm{supp}(X) \backslash \mathrm{supp}(Y)$. If $f \in \mathrm{supp}(Y)$ then $\mathrm{sign}(x_f - \lambda y_f) = \mathrm{sign}(x_f)$ for small values of λ, and $\mathrm{sign}(x_f - \lambda y_f) = -\mathrm{sign}(x_f)$ for large values of λ. Thus the ray leaves the orthant $\{\hat{\mathbf{x}} : \mathrm{sign}(\hat{\mathbf{x}}) = X\}$ containing $\mathbf{x}$ at a point $\mathbf{z}$ with $z_e = x_e$ for each $e \in \mathrm{supp}(X) \backslash \mathrm{supp}(Y)$, and $z_f = 0$ for some $f \in \mathrm{supp}(Y)$. Let $Z = \mathrm{sign}(\mathbf{z})$. Then $\mathbf{0} \neq Z < X$, so by our induction hypothesis Z is a composition of circuits that are less than or equal to Z, and hence are less than or equal to X. But $Z \not\leq Y$, a contradiction. □

For another perspective on (co)vectors as compositions of signed (co)circuits, let $\mathcal{V}^*$ and $\mathcal{C}^*$ be the covector and signed cocircuit sets of a signed hyperplane arrangement. Consider a picture of the arrangement in an affine space $\mathbb{A}$. Then $X \in \mathcal{V}^*$ indexes a convex set c_X in the partition of $\mathbb{A}$ given by the arrangement, and a signed cocircuit $Y \leq X$ indexes a point p_Y in the closure of c_X. For covectors Y and Y' and points $p_Y \in c_Y$ and $p_{Y'} \in c_{Y'}$, the set $c_{Y \circ Y'}$ is the part of our partition containing a point obtained by starting at p_Y and moving a tiny step toward $p_{Y'}$. Here "tiny," means the step moves us off of every hyperplane of the arrangement that contains p_Y but not $p_{Y'}$, but is not so big that we cross any hyperplane of the arrangement. Proposition 1.29 says that if we start at some vertex of the closure of c_X and successively take tiny steps toward each remaining vertex, we will end in c_X.

1.5.2 $\mathcal{V}(\mathrm{Mat}(r,n)) \leftrightarrow \mathcal{V}^*(\mathrm{Mat}(r,n))$: Duality

Recall (Section 1.4) that for any linear subspace W of $\mathbb{R}^n$ we have $\mathcal{V}^*(W) = \mathcal{V}(W^\perp)$. Here we introduce a notion of orthogonality for sign vectors under which $\mathcal{V}(W)$ and $\mathcal{V}^*(W)$ are "orthogonal complements" to each other.

Algebra and Dot Product for Signs

Define an operation $\cdot$ on $\{0, +, -\}$ by

$$+ \cdot + = - \cdot - = +$$

$$+ \cdot - = - \cdot + = -$$

$$i \cdot 0 = 0 \cdot i = 0$$

for each i. (Often we'll suppress the "$\cdot$.") Thus for $s_1, s_2, s_3 \in \{0, +, -\}$ we have $s_1 \cdot s_2 = s_3$ if and only if there are elements $r_1, r_2, r_3 \in \mathbb{R}$ such that $\mathrm{sign}(r_i) = s_i$ for each i and $r_1 r_2 = r_3$.

A **hyperoperation** on a set S is a function $S \times S \to \mathcal{P}(S) - \{\emptyset\}$. We define a hyperoperation on $\{0, +, -\}$, called the **hypersum** and denoted $\boxplus$, by

$$+ \boxplus + = \{+\}$$

$$- \boxplus - = \{-\}$$

$$x \boxplus 0 = 0 \boxplus x = \{x\} \text{ for all } x$$

$$+ \boxplus - = - \boxplus + = \{0, +, -\}.$$

Thus for $s_1, s_2, s_3 \in \{0, +, -\}$ we have $s_3 \in s_1 \boxplus s_2$ if and only if there are elements $r_1, r_2, r_3 \in \mathbb{R}$ such that $\mathrm{sign}(r_i) = s_i$ for each i and $r_1 + r_2 = r_3$. When a set has a single element we will frequently omit the braces, for example, by denoting $x \boxplus 0$ by x. For every $S, T \subseteq \{0, +, -\}$ we define $S \boxplus T$ to be $\bigcup_{t\in T}^{s\in S} s \boxplus t$, and for $S \subseteq \{0, +, -\}$ and $x \in \{0, +, -\}$ we let $S \boxplus x = x \boxplus S := S \boxplus \{x\}$. With these definitions, $\boxplus$ is commutative and associative: In fact,

$$\boxplus_{s\in S} s = \begin{cases} \max(S) & \text{if } \{+,-\} \not\subseteq S \\ \{0, +, -\} & \text{otherwise.} \end{cases}$$

Also, $a \cdot (b \boxplus c) = (a \cdot b) \boxplus (a \cdot c)$.

Define the **inner product** of $X, Y \in \{0, +, -\}^m$ to be

$$X \cdot Y = \boxplus_{i=1}^{m} X(i) \cdot Y(i).$$

Define two sign vectors X and Y in $\{0, +, -\}^n$ to be **orthogonal**, written $X \perp Y$, if $0 \in X \cdot Y$. Thus $X \perp Y$ if $\{X(i) \cdot Y(i) : i \in [n]\}$ either is $\{0\}$ or contains both $+$ and $-$. For any set $\mathcal{F}$ of sign vectors, let $\mathcal{F}^{\perp}$ denote $\{Y : Y \perp X \ \forall X \in \mathcal{F}\}$.

Duality

If $\mathbf{x} \perp \mathbf{y}$ are elements of $\mathbb{R}^n$, then $\sum_i x_i y_i = 0$, and so either the terms in this sum are all zero or some terms are positive and some negative. Thus, $\text{sign}(\mathbf{x}) \perp \text{sign}(\mathbf{y})$. Of course, from the hypothesis $\text{sign}(\mathbf{x}) \perp \text{sign}(\mathbf{y})$ we can't conclude $\mathbf{x} \perp \mathbf{y}$. But kind of amazingly, orthogonality of sign vectors corresponds to orthogonality of vector spaces exactly as it should:

Proposition 1.32 *If W is a subspace of $\mathbb{R}^n$, then* $\{\text{sign}(\mathbf{y}) : \mathbf{y} \in W^{\perp}\}^{\perp} = \{\text{sign}(\mathbf{x}) : \mathbf{x} \in W\}$.

In other words, $\mathcal{V}(W)^{\perp} = \mathcal{V}(W^{\perp}) = \mathcal{V}^*(W)$.

We'll prove this in a moment.

This result generalizes to arbitrary oriented matroids (Section 2.3), giving a natural notion of "orthogonal pairs of oriented matroids." (The actual term used is *dual pairs of oriented matroids.*)

Proposition 1.32 follows from a result, important in linear programming, that does not initially look inspiring. We first state it in one form it is commonly seen:

Farkas Lemma 1[3] Let $A \in \mathbb{R}^{r\times n}$ and $\mathbf{b} \in \mathbb{R}^r$. Then exactly one of the following holds:

[3] Hungarian tutorial: The *a*'s in "Farkas" are pronounced like the English long *o*, and the *s* is pronounced like the English *sh*. Farkas's name will be coming up a lot.

1. There exists $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
2. There exists $\mathbf{z} \in \mathbb{R}^r$ such that $\mathbf{z}A \geq \mathbf{0}$ and $\mathbf{z}\mathbf{b} < 0$.

Geometrically, this is easy to believe: The first alternative says that $\mathbf{b}$ is in the closed cone, $\mathrm{cone}(A)$, consisting of nonnegative linear combinations of the columns of A, while the second alternative says that there is a hyperplane $\mathbf{z}^\perp$ separating $\mathrm{cone}(A)$ and $\mathbf{b}$.

Part of the proof is clear as well: Both alternatives can't be true simultaneously, because if $\mathbf{b} \in \mathrm{cone}(A)$ and $\mathbf{z} \cdot \mathbf{a} \geq 0$ for every column $\mathbf{a}$ of A then $\mathbf{z} \cdot \mathbf{b} \geq 0$. The remainder of the proof – showing that at least one of the two alternatives hold – is substantially trickier. See, for instance, Ziegler (1995) for a proof.

To interpret the Farkas Lemma in terms of subspaces, let W be the nullspace of the matrix $(A| - \mathbf{b})$. Then the two alternatives of the Farkas Lemma can be stated in terms of W: Either

1. there exists $\mathbf{x} \in W$ such that $\mathbf{x} \geq \mathbf{0}$ and $x_{n+1} > 0$, or
2. there exists $\mathbf{y} \in W^\perp$ such that $\mathbf{y} \geq \mathbf{0}$ and $y_{n+1} > 0$.

(Here $\mathbf{y} = \mathbf{z}(A| - \mathbf{b}) \in \mathrm{row}(A| - \mathbf{b})$.) So, the Farkas Lemma says that for any subspace W of $\mathbb{R}^{n+1}$, exactly one of W, $W^\perp$ intersects the positive closed orthant $\{\mathbf{z} : \forall i \ \mathrm{sign}(z_i) \in \{0, +\}\}$ in a point whose last coordinate is positive.

Let's adapt Farkas to consider more general orthants. Notice that the form of the second alternative changes when we consider nonmaximal orthants.

Farkas Lemma 2 For every linear subspace W of $\mathbb{R}^{n+1}$ and every $Z = A^+B^-C^0 \in \{0, +, -\}^{n+1}$ with $Z(n+1) \neq 0$, exactly one of the following holds:

1. There exists $\mathbf{x} \in W$ such that $\mathrm{sign}(\mathbf{x}) \leq Z$ and $\mathrm{sign}(x_{n+1}) = Z(n+1)$.
2. There exists $\mathbf{y} \in W^\perp$ such that $\mathrm{sign}(y_i) \leq Z(i)$ for every $i \in A \cup B$ and $\mathrm{sign}(y_{n+1}) = Z(n+1)$.

Proof: Both alternatives can't hold simultaneously because if $\mathrm{sign}(\mathbf{x}) \leq Z$ and $\mathrm{sign}(y_i) \leq Z(i)$ for every $i \in A \cup B$ and $\mathrm{sign}(x_{n+1}) = \mathrm{sign}(y_{n+1}) \neq 0$ then $x_i y_i \geq 0$ for all i and $x_{n+1} y_{n+1} > 0$, so $\mathbf{x} \cdot \mathbf{y} > 0$. Thus if $\mathbf{x} \in W$ then $\mathbf{y} \notin W^\perp$.

Now, say $M = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$ is a matrix such that $W = \mathrm{null}(M)$. Let D denote the diagonal matrix with

$$D_{ii} = \begin{cases} 1 & \text{if } Z(i) = +, \\ 0 & \text{if } Z(i) = 0, \\ -1 & \text{if } Z(i) = -. \end{cases}$$

and let $W' = \text{null}(MD)$. Applying our previous version of Farkas to W', we have that exactly one of the following is true:

1. There exists $\mathbf{x}' \in W'$ such that $\mathbf{x}' \geq \mathbf{0}$ and $x'_{n+1} > 0$.
2. There exists $\mathbf{y}' \in (W')^\perp$ such that $\mathbf{y}' \geq \mathbf{0}$ and $y'_{n+1} > 0$.

If Alternative (1) holds then $\mathbf{x} := D\mathbf{x}'$ satisfies Alternative (1) in the statement of the proposition.

If Alternative (2) holds then $\mathbf{y}' = \mathbf{z}MD$ for some $\mathbf{z} \in \mathbb{R}^r$. Now consider $\mathbf{y} := \mathbf{z}M \in W^\perp$. For each i we have $y_i = \mathbf{z} \cdot \mathbf{v}_i$ and $y'_i = D_{ii}\mathbf{z} \cdot \mathbf{v}_i$. Thus

$$y'_i = \begin{cases} y_i & \text{if } Z(i) = +, \\ -y_i & \text{if } Z(i) = -, \\ 0 & \text{if } Z(i) = 0. \end{cases}$$

Since $y'_i \geq 0$ for each i, we see $\mathbf{y}$ is an element of $W^\perp$ satisfying Alternative (2) of the proposition. □

Of course, there is nothing special about the final coordinate: We have the following mild generalization.

Farkas Lemma 3 For every linear subspace W of $\mathbb{R}^{n+1}$, every $Z = A^+B^-C^0 \in \{0, +, -\}^{n+1}$, and every $j \in [n+1]$ with $Z(j) \neq 0$, exactly one of the following holds:

1. There exists $\mathbf{x} \in W$ such that $\text{sign}(x_i) \leq Z(i)$ for every i and $\text{sign}(x_j) = Z(j)$.
2. There exists $\mathbf{y} \in W^\perp$ such that $\text{sign}(y_i) \leq Z(i)$ for every $i \in A \cup B$ and $\text{sign}(y_j) = Z(j)$.

From this version of the Farkas Lemma we can prove Proposition 1.32.

Proof of Proposition 1.32: The discussion on orthogonality earlier in this section shows that $\{\text{sign}(\mathbf{x}) : \mathbf{x} \in W\} \subseteq \{X : X \perp Y \text{ for every } Y \in \text{sign}(W^\perp)\}$. Conversely, say $X \perp Y$ for every $Y \in \text{sign}(W^\perp)$. Consider a j such that $X(j) \neq 0$. Applying our final version of Farkas to X and j, we see that Alternative $\{2\}$ can't hold, and so there exists $\mathbf{x}^{(j)} \in W$ such that $\text{sign}(\mathbf{x}^{(j)}) \leq X$ and $\text{sign}(x_j^{(j)}) = X(j)$. Choose such an $\mathbf{x}^{(j)}$ for each j, and let $\mathbf{x} = \sum_j \mathbf{x}^{(j)}$. Then this $\mathbf{x}$ is the element of W we want. □

See Exercise 1.7 for a fourth version of the Farkas Lemma that removes the asymmetry between the two alternatives.

In Section 2.3.3 we will give a combinatorial version of the Farkas Lemma for families of sign vectors, rather than subspaces. As we shall see, the families that are vector sets of oriented matroids can be characterized as those which satisfy this combinatorial Farkas Lemma and a few other conditions. This will lead to a simple proof of cryptomorphism between vector sets and covector sets for general oriented matroids.

1.5.3 $\mathcal{C}(\mathbf{Mat}(r,n)) \leftrightarrow \tilde{\chi}(\mathbf{Mat}(r,n))$

This discussion will introduce some ideas that are useful in a lot of contexts:

- the Pivoting Property,
- the Dual Pivoting Property, and
- the Basis Exchange Principle.

The Dual Pivoting Property, which will be introduced in Problem 1.36, actually leads to a cryptomorphism $\mathcal{C}^*(\mathrm{Mat}(r,n)) \leftrightarrow \tilde{\chi}(\mathrm{Mat}(r,n))$ by a shorter argument than the one we'll give. We'll do the longer argument to highlight some geometric ideas.

Definition 1.33 An arrangement $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ is **linearly independent** if $\mathbf{v}_i \neq \mathbf{v}_j$ whenever $i \neq j$ and $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is linearly independent. An arrangement is **linearly dependent** if it is not linearly independent.

Let $M = (\mathbf{v}_1,\ldots,\mathbf{v}_n) \in \mathrm{Mat}(r,n)$. Our first step in showing that $\mathcal{C}(M)$ determines $\pm\chi(M)$ and vice versa is to show that $\{\mathrm{supp}(X) : X \in \mathcal{C}(M)\}$ determines $\mathrm{supp}(\chi(M))$ and vice versa.

Since $\mathcal{C}(M)$ is the set of elements of $\mathcal{V}(M)$ of minimal support, we see that a set S is $\mathrm{supp}(X)$ for some $X \in \mathcal{C}(M)$ if and only if $(\mathbf{v}_s : s \in S)$ is a minimal dependent subarrangement of M. But the minimal dependent subarrangements of M determine the maximal independent subarrangements of M and vice versa, and $(\mathbf{v}_i : i \in S)$ is a maximal independent subarrangement of M if and only if $|S| = r$ and $\chi(M)(s_1,\ldots,s_r) \neq 0$ for all orderings $(s_1,\ldots,s_r)$ of S.

Our next step begins by noting that to determine a pair X, $-X$ of sign vectors it's enough to determine $\mathrm{supp}(X)$ and the values $X(e)X(f)$ for each $e, f \in \mathrm{supp}(X)$. We will show that $C(M)$ determines the product

$$\chi(M)(e,x_2,\ldots,x_r)\chi(M)(f,x_2,\ldots,x_r)$$

for each $e, f, x_2,\ldots,x_r$, and vice versa.

If $X \in \mathcal{C}(M)$ and $\mathrm{supp}(X) = \{e\}$ then $\mathbf{v}_e = \mathbf{0}$, $\{X, -X\} = \{e^+, e^-\}$, and $\chi(M)(e,x_2,\ldots,x_r) = 0$ for all $x_2,\ldots,x_r$.

Now consider $X \in \mathcal{C}(M)$ with larger support, say $\mathrm{supp}(X) = \{e, f, i_2,\ldots,i_k\}$. Since $\{\mathbf{v}_e, \mathbf{v}_{i_2},\ldots,\mathbf{v}_{i_k}\}$ is linearly independent, it is contained in a basis $\{\mathbf{v}_e, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$. Also, since $\{\mathbf{v}_e, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}\}$ and $\{\mathbf{v}_f, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_k}\}$ span the same subspace of $\mathbb{R}^n$, $\{\mathbf{v}_f, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ is also a basis.

Proposition 1.34 *Let $\{\mathbf{v}_e, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ and $\{\mathbf{v}_f, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ be distinct independent sets of columns of M. Let $X \in \mathcal{C}(M)$ such that the support of X is contained in $\{e, f, i_2, \ldots, i_r\}$. Then*

$$\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r) = -X(e) \cdot X(f).$$

This relationship between chirotopes and signed circuits will reappear for general oriented matroids in Theorem 2.54 as the *Pivoting Property*.

Proof: This just expresses a simple geometric idea: $\{\mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ spans a hyperplane H not containing $\mathbf{v}_e$ or $\mathbf{v}_f$, and the independence of the two sets tells us that neither $\mathbf{v}_e$ nor $\mathbf{v}_f$ lie on H. From our discussion of the chirotope in Section 1.3.3 we know that $\mathbf{v}_e$ and $\mathbf{v}_f$ lie on the same side of H if and only if $\chi(M)(e, i_2, \ldots, i_r) = \chi(M)(f, i_2, \ldots, i_r)$, that is, if $\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r) = +$. Also, they lie on the same side of H if and only if there is no positive linear combination of $\mathbf{v}_e$ and $\mathbf{v}_f$ lying on H, that is, there are no scalars a_e, a_f, b_2, b_r with $a_e > 0$, $a_f > 0$, and

$$a_e \mathbf{v}_e + a_f \mathbf{v}_f = \sum_{j=2}^{r} b_j \mathbf{v}_{i_j},$$

$$a_e \mathbf{v}_e + a_f \mathbf{v}_f + \sum_{j=2}^{r} b_j \mathbf{v}_{i_j} = \mathbf{0}.$$

Thus the signed circuit X with $\mathrm{supp}(X) \subseteq \{e, f, i_2, \ldots, i_r\}$ must satisfy $X(e) = -X(f)$. To summarize, if $\mathbf{v}_e$ and $\mathbf{v}_f$ lie on the same side of H then

$$\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r) = +$$

and $X(e)X(f) = -$. Likewise if $\mathbf{v}_e$ and $\mathbf{v}_f$ lie on opposite sides of H then

$$\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r) = -$$

and $X(e)X(f) = +$. □

Thus for each $X \in \mathcal{C}(M)$ for which we know $\mathrm{supp}(X)$, the pair $\{X, -X\}$ is determined by values $\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r)$. Conversely, if we know $\mathcal{C}(M)$ and we have $(e, i_2, \ldots, i_r), (f, i_2, \ldots, i_r) \in \mathrm{supp}(\chi(M))$ with $e \neq f$, the arrangement $(\mathbf{v}_e, \mathbf{v}_f, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r})$ is dependent, and a minimal dependent subarrangement contains $\mathbf{v}_e$ and $\mathbf{v}_f$. A corresponding signed circuit X satisfies the hypothesis of Proposition 1.34, so from $X(e)X(f)$ we can find $\chi(M)(e, i_2, \ldots, i_r) \cdot \chi(M)(f, i_2, \ldots, i_r)$.

The last step of the cryptomorphism is to show that from the values of products of the form

$$\chi(M)(e, i_2, \ldots, i_r)\chi(M)(f, i_2, \ldots, i_r),$$

we can determine $\chi(M)(b_1, \ldots, b_r)\chi(M)(b'_1, \ldots, b'_r)$ for all pairs of bases $B = \{\mathbf{v}_{b_1}, \ldots, \mathbf{v}_{b_r}\}$, $B' = \{\mathbf{v}_{b'_1}, \ldots, \mathbf{v}_{b'_r}\}$.

This follows from a linear algebra observation that's also a fundamental principle in the theory of ordinary matroids:

Proposition 1.35 (Basis Exchange Principle – realizable case) *If B and B' are bases for a vector space W and $\mathbf{x} \in B \backslash B'$ then there is a $\mathbf{y} \in B'$ such that $(B \cup \{\mathbf{y}\}) \backslash \{\mathbf{x}\}$ and $(B' \cup \{\mathbf{x}\}) \backslash \{\mathbf{y}\}$ are bases for W.*

Proof: $B \backslash \{\mathbf{x}\}$ spans a hyperplane H. Let $S = B' \cup \{\mathbf{x}\}$ and $T = (B' \cap H) \cup \{\mathbf{x}\}$. S spans W, and T is an independent subset of S, so T extends to a basis $(B' \cup \{\mathbf{x}\}) \backslash \{\mathbf{y}\}$ for W. Since $\mathbf{y} \notin H$, we also have that $(B \backslash \{\mathbf{x}\}) \cup \{\mathbf{y}\}$ is a basis for W. □

Using this, we can induct on $|B \backslash B'|$. Given B and B' with $|B \backslash B'| \geq 2$, we do a basis exchange to get a basis $\{\mathbf{v}_c : c \in (B \cup \{b'\}) \backslash \{b\}\}$ with $b' \in B'$. Without loss of generality assume $b = b_1$ and $b' = b'_1$. Then by our induction hypothesis we know the values of

$$\chi(M)(b_1, \ldots, b_r)\chi(M)(b'_1, b_2, \ldots, b_r)$$

and

$$\chi(M)(b'_1, b_2, \ldots, b_r)\chi(M)(b'_1, \ldots, b'_r).$$

Since $+ \cdot + = - \cdot -$ is the multiplicative identity in $\{0, +, -\}$ and the factor $\chi(M)(b'_1, b_2, \ldots, b_r)$ is in $\{+, -\}$, the product of these two values is $\chi(M)(b_1, \ldots, b_r)\chi(M)(b'_1, \ldots, b'_r)$.

Finally, there's a similar connection between χ and $\mathcal{C}^*$, whose abstract combinatorial analog will come up in Chapter 2.

Problem 1.36 If $\{\mathbf{v}_e, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ and $\{\mathbf{v}_f, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}\}$ are bases with $e \neq f$ and Y is a signed cocircuit with support contained in the complement of $\{i_2, \ldots, i_r\}$, show that

$$\chi(M)(e, i_2, \ldots, i_r)\chi(M)(f, i_2, \ldots, i_r) = Y(e) \cdot Y(f).$$

This relationship between chirotopes and signed cocircuits will reappear for general oriented matroids in Theorem 2.54 as the *Dual Pivoting Property*.

1.5.4 Conclusions on Cryptomorphism

To summarize: The functions $\mathcal{C}$, $\mathcal{V}$, $\mathcal{C}^*$, and $\mathcal{V}^*$ all encode the same data about matrices (while χ encodes this data plus an orientation of $\mathbb{R}^r$). This data about a matrix M is called the **oriented matroid corresponding to** M. We'll also refer to oriented matroids of vector or hyperplane arrangements, or of subspaces of $\mathbb{R}^n$. $\mathcal{C}(M)$ is called the set of **signed circuits** of the oriented matroid corresponding to M. $\mathcal{C}^*(M)$ is called the set of **signed cocircuits** of the oriented matroid corresponding to M. $\mathcal{V}(M)$ is called the set of **vectors** of the oriented matroid corresponding to M, and $\mathcal{V}^*(M)$ is called the set of **covectors** of the oriented matroid corresponding to M. (Ordinary (unoriented) matroid theory has (unsigned) circuits and cocircuits. See Section 1.8 for details.) $\chi(M)$ and $-\chi(M)$ are called the two **chirotopes** of the oriented matroid corresponding to M. We will use $\mathcal{M}(M)$ to denote the oriented matroid corresponding to M. The **dual** of $\mathcal{M}(M)$, denoted $\mathcal{M}^*(M)$, is the oriented matroid with vector set $\mathcal{V}^*(M)$ and covector set $\mathcal{V}(M)$.

This is not yet the definition of oriented matroids – it's only the special case of oriented matroids arising from matrices (these are called **realizable oriented matroids**). In Chapter 2 we will define oriented matroids in general, by first defining sets of signed circuits, sets of vectors, and so on. All of these objects will have purely combinatorial definitions inspired by the realizable case. We will see that in general signed circuits, vectors, and so on are cryptomorphic.

Every property we have described in terms of one of $\mathcal{C}(M)$, $\mathcal{V}(M)$, $\mathcal{C}^*(M)$, $\mathcal{V}^*(M)$, or $\pm\chi(M)$ can be thought of as a property of $\mathcal{M}(M)$. For instance, in Section 1.3.3 we defined the rank of $\mathcal{V}(M)$; henceforward we will call this the rank of $\mathcal{M}(M)$.

For ease of linear algebra, so far we have dealt with arrangements whose objects are indexed by $[n]$, resulting in oriented matroids defined in terms of $\{0,+,-\}^n$ and $\{0,+,-\}^{\binom{n}{r}}$. Going forward, this convention is unnecessary and occasionally inconvenient, so we will typically index the elements of an arrangement by a finite set E. Thus instead of working with a vector arrangement $M \in \mathrm{Mat}(r,n)$, we will work with an arrangement $\mathcal{A} = (\mathbf{v}_e : e \in E)$. The resulting $\mathcal{C}(\mathcal{A})$, $\mathcal{V}(\mathcal{A})$, $\mathcal{C}^*(\mathcal{A})$, $\mathcal{V}^*(\mathcal{A})$ are subsets of $\{0,+,-\}^E$. We say E is the set of **elements** of $\mathcal{M}(\mathcal{A})$.

1.6 Convex Polytopes

Let's take a brief digression to see the usefulness of oriented matroids as a tool for studying convex polytopes. We'll examine the interaction between convex polytopes and oriented matroids in more depth in Chapter 8.

Recall the definition of convex polytope from Section 1.3.1. It's not hard to see that a convex polytope P is the convex hull of its vertex set. The set of faces of P is a partially ordered set, ordered by inclusion. The **combinatorial type** of a convex polytope is the isomorphism class of its face poset.

Consider a convex polytope P in affine space with vertex set $V = \{p_e : e \in E\}$. As noted in Section 1.3.3, from the oriented matroid $\mathcal{M}$ of V we can read off the faces of P. For a subset F of V, F is the set of vertices of a face of V if and only if there is a signed hyperplane $\mathcal{H}$ with $H^0 \cap V = F$ and $H^- \cap V = \emptyset$. Further, this face is the convex hull of F. Thus we have a bijection between the covectors of $\mathcal{M}$ of the form A^+ and the faces of P, sending a covector A^+ to $\text{conv}(p_e : e \in E - A)$. This suggests realizable oriented matroids as a natural tool for studying face posets of convex polytopes.

Oriented matroid duality allows us to study high-dimensional convex polytopes with relatively few vertices using low-dimensional arrangements of points. Consider a d-dimensional convex polytope P in $\mathbb{A}^d \subset \mathbb{R}^{d+1}$ with vertex set $V = \{p_e : e \in E\}$, where $|V| \leq d + 4$. The oriented matroid $\mathcal{M}$ associated to V is rank $d + 1$ with $|V|$ elements, and so $\mathcal{M}^*$ is rank $|V| - (d + 1) \leq 3$. We can realize $\mathcal{M}^*$ in $\mathbb{R}^3$, but strictly speaking we can't realize it in $\mathbb{A}^2$: Since $E^+ \in \mathcal{V}^*(\mathcal{M}) = \mathcal{V}(\mathcal{M}^*)$, the elements of a realization can't all lie in a common plane not through the origin. We can get around this by introducing the notion of a *signed affine point arrangement*: This is an arrangement $((p_e, s_e) : e \in E)$, where each p_e is a point in $\mathbb{A}$ and each s_e is either $+$ or $-$. Given a realization $(\mathbf{w}_e : e \in E)$ of $\mathcal{M}^*$ in $\mathbb{R}^3$ and an affine plane $\mathbb{A} \subset \mathbb{R}^3$ that is not parallel to any $\mathbf{w}_e$, for each $e \in E$ there is a unique multiple $\lambda_e \mathbf{w}_e$ of $\mathbf{w}_e$ in $\mathbb{A}$: We let $(p_e, s_e) = (\lambda_e \mathbf{w}_e, \text{sign}(\lambda_e))$. The resulting signed affine point arrangement encodes our realization of $\mathcal{M}^*$, up to positive scaling: We call this arrangement an **affine Gale Diagram** for P. (This is actually weaker than the usual definition, which predates oriented matroids. Normally "Gale diagram" refers to a *dual vector arrangement* to V, obtained by treating V as the columns of a matrix M and finding a matrix N such that $\text{row}(N) = \text{row}(M)^\perp$.)

Problem 1.37 Find necessary and sufficient conditions on a realizable oriented matroid $\mathcal{M}$ for $\mathcal{M}$ to be the oriented matroid of a polytope.

Note that this also gives necessary and sufficient conditions for an oriented matroid to be the oriented matroid of the Gale diagram of a polytope.

Problem 1.38 Use your solution to Problem 1.37 to count the number of combinatorial types of four-dimensional convex polytopes on seven vertices.

To give a charming application, due to Perles, we consider the following question (cf. Ziegler 1995).

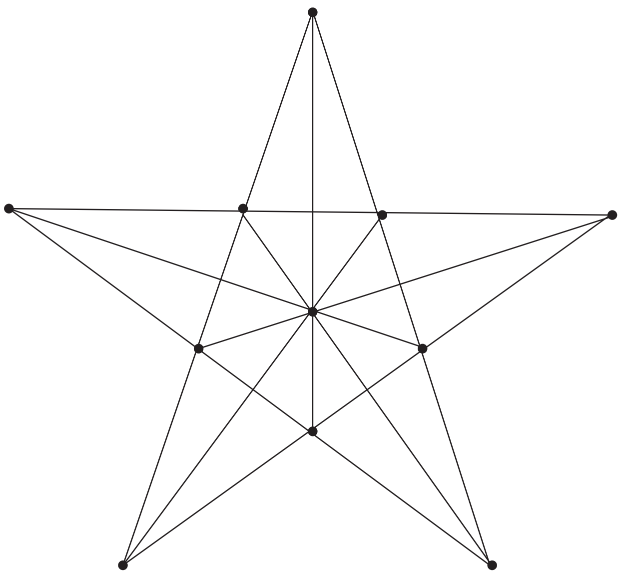


Figure 1.7 The "Betsy Ross" arrangement.

> Given a convex polytope P in $\mathbb{R}^n$, is there some convex polytope P' in $\mathbb{Q}^n$ with the same face poset as P? That is, is every combinatorial type of real convex polytope **realizable over** $\mathbb{Q}$?

It's not hard to see that any combinatorial type of convex polytope of dimension less than or equal to three is realizable over $\mathbb{Q}$, as is any combinatorial type of convex polytope whose proper faces are all simplices. However, in general the answer to the question is No, and we'll give an example here.

Consider the affine point arrangement known as the *Betsy Ross arrangement*, shown in Figure 1.7. Make a signed affine point arrangement S with two elements $(p,+)$ and $(p,-)$ for each point p shown. Thus this arrangement describes 22 elements of $\mathbb{R}^3$ that come in pairs $\mathbf{v}, -\mathbf{v}$. Let $\mathcal{M}(S)$ be the oriented matroid of this arrangement.

Problem 1.39 Use the results of Problem 1.37 to verify that S is the Gale diagram of a convex polytope P in affine space. Determine the dimension of P. Show that any convex polytope in affine space of the same combinatorial type as P has oriented matroid isomorphic to $\mathcal{M}^*(S)$.

At the center of the argument are two observations.

- The face poset of P can be read off from the positive circuits of $\mathcal{M}(S)$.
- Any set T of vectors in $\mathbb{R}^3$ such that $\mathcal{M}(T)$ has the same positive circuits as $\mathcal{M}(S)$ (up to relabeling the elements of the oriented matroid) consists of

pairs $\{\mathbf{x}_e, -\mathbf{x}_e\}$, where, up to scaling, $T' := \{\mathbf{x}_e : e \in E\}$ is an affine point arrangement, and the set of positive circuits of $\mathcal{M}(T)$ determines the set of all circuits of $\mathcal{M}(T')$.

Note the many dependent triples among elements of S. Whenever a set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^3$ is dependent, the determinant $|\mathbf{x}, \mathbf{y}, \mathbf{z}| = 0$, and this equality is a polynomial equation in the coordinates for $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. In the case of Betsy Ross, from the system of equations we can deduce that the only fields over which this collection of dependencies is realizable are fields containing $\sqrt{5}$. In particular, no vector arrangement in $\mathbb{Q}^3$ has oriented matroid $\mathcal{M}(S)$ – that is, $\mathcal{M}(S)$ is not realizable over $\mathbb{Q}$.

Problem 1.40 Show that an oriented matroid is realizable over $\mathbb{Q}$ if and only if its dual is as well.

Thus, P is not realizable over $\mathbb{Q}$!

Remark 1.41 The operation of replacing each element of our oriented matroid with two antiparallel copies is closely related to the *Lawrence construction*, which we will discuss more in Section 8.4. (The Lawrence construction performs this operation on $\mathcal{M}^*$ rather than on $\mathcal{M}$.)

Remark 1.42 For examples of nonrational combinatorial types of convex polytopes of dimension 4, see Richter-Gebert (1996a) and Dobbins (2011).

For much more on Gale diagrams, see chapter 6 of Ziegler (1995).

1.7 Deletion and Contraction

This section will discuss two operations on oriented matroids that are the basis for many inductive arguments.

From here on it will be convenient to index the columns of a matrix by an arbitrary finite set E, not necessarily $[n]$.

Given a matrix M with columns indexed by E and given $e \in E$, let $M \backslash e$ denote the matrix obtained from M by deleting the column indexed by e. For $X \in \{0, +, -\}^E$, let $X \backslash e \in \{0, +, -\}^{E \backslash e}$ be the restriction of X. Then from the definition of $\mathcal{V}^*$ it's clear that

$$\mathcal{V}^*(M \backslash e) = \{X \backslash e : X \in V^*(M)\}.$$

It would be easy to jump to the conclusion that

$$\mathcal{C}^*(M \backslash e) = \{X \backslash e : X \in C^*(M)\},$$

but this isn't quite true, for two reasons:

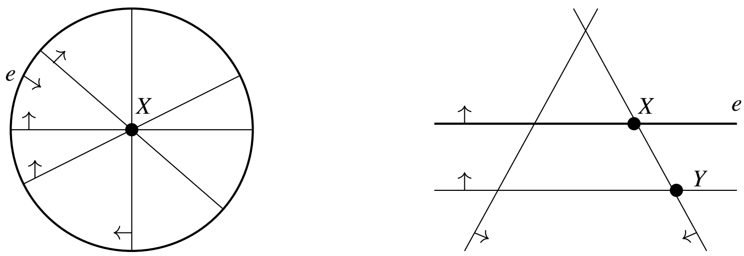


Figure 1.8 Deletions of circuits are not necessarily circuits of deletion.

1. If $X = \{e\}^+ \in \mathcal{V}^*(M)$ (i.e., row(M) has an element $\mathbf{x}$ whose only nonzero entry is x_e), then $X \backslash e = \mathbf{0} \notin \mathcal{C}^*$.
2. If there are $X, Y \in \mathcal{C}^*(M)$ so that $Y(e) \neq 0$, $X(e) = 0$, $Y(f) \leq X(f)$ for all $f \neq e$, and $Y(g) < X(g)$ for some g, then $\mathbf{0} \neq Y \backslash e < X \backslash e$, and thus $X \backslash e \notin \mathcal{C}^*$.

Illustrations of both issues are given in Figure 1.8. The figure on the left shows a representation of rank 3 oriented matroid by equators in S^2, and the figure on the right shows a representation of a rank 3 oriented matroid by hyperplanes in $\mathbb{A}^2$.

The correct description of the signed cocircuit set of the deletion is

$$\mathcal{C}^*(M \backslash e) = \min\{X \backslash e : X \in \mathcal{C}^*(M), X \backslash e \neq \mathbf{0}\}.$$

Problem 1.43 Describe $\mathcal{V}(M \backslash e)$ and $\mathcal{C}(M \backslash e)$ in terms of $\mathcal{V}(M)$ and $\mathcal{C}(M)$.

$\mathcal{M}(M \backslash e)$ is called the **deletion** of e from $\mathcal{M}(M)$, denoted $\mathcal{M}(M) \backslash e$. In Chapter 2 we'll define deletion for arbitrary oriented matroids in the way suggested by realizable oriented matroids.

In terms of vector and signed hyperplane arrangements, deletion of e corresponds to removing the vector resp. signed hyperplane corresponding to e from the arrangement.

The second operation to look at is *contraction*. The **contraction** of e from $\mathcal{M}(M)$, written $\mathcal{M}(M)/e$, is defined as $((\mathcal{M}^*(M)) \backslash e)^*$. That is, $\mathcal{M}(M)/e$ is obtained from $\mathcal{M}(M)$ by deleting e in the dual. This operation has a direct interpretation in M as well, given in the following exercise.

Problem 1.44 (i) Let $\{\mathcal{H}_f : f \in E\}$ be an arrangement of signed hyperplanes with oriented matroid $\mathcal{M}$ and let $e \in E$. Show that the arrangement $\{(H_f^0 \cap H_e^0, H_f^+ \cap H_e^0, H_f^- \cap H_e^0) : f \in E \backslash e\}$ has oriented matroid $\mathcal{M}/e$.

(ii) Let $\{\mathbf{v}_f : f \in E\}$ be an arrangement of vectors in $\mathbb{R}^n$ with oriented matroid $\mathcal{M}$, and let $e \in E$. Let π_e be the orthogonal projection map from $\mathbb{R}^n$ to $(e^\perp)^0$. Show that the arrangement $\{\pi_e(\mathbf{v}_f) : f \in E \backslash e\}$ has oriented matroid $\mathcal{M}/e$.

This is an important idea to get used to: For oriented matroid purposes,

- deleting an element H_e in a signed hyperplane arrangement is equivalent to restricting to the subspace H_e in the dual, and
- deleting an element e in a vector arrangement is equivalent to projecting to the subspace $(e^\perp)^0$ in the dual.

1.8 A Few Words about Unoriented Matroids

For the reader who has never encountered (unoriented) matroids, here is a very, very brief introduction. Indeed, we won't get as far as a definition. For any real understanding, see, for instance, Oxley (1992). This section exists purely to satisfy a reader's mild curiosity – we will not use it elsewhere.

We have already seen that oriented matroids are modeling vector or hyperplane arrangements over *ordered* fields – fields with a natural partition into positive elements, negative elements, and 0. If we forget the data in an oriented matroid that arises from this order – by replacing each "+" and "−" by the word "nonzero" – we get a *matroid*. For instance:

- Instead of recording the signed circuit set of a finite arrangement $\mathcal{A}$ of vectors in $\mathbb{R}^r$, we look only at the supports of these signed circuits. This set of supports is called the set of *(unsigned) circuits* of the matroid of $\mathcal{A}$.
- Instead of recording the entire chirotope of $\mathcal{A}$, we record only which sets of elements are bases for $\mathbb{R}^r$.

This is a very oriented-matroid-chauvinist way to present the situation. In actuality, matroid theory (developed in Whitney 1935) much predates oriented theory (formalized independently in Folkman and Lawrence 1978 and Bland and Las Vergnas 1978, Bland and Las Vergnas 1979[4]), and was conceived of as a combinatorial abstraction of linear dependence over arbitrary fields. A matroid may be realizable as a vector arrangement over, for instance, a finite field without being realizable over $\mathbb{R}$.

Every oriented matroid has an underlying matroid, as described above, but not every matroid arises in this way – that is, not every matroid is *orientable*.

[4] See section 3.9 in Björner et al. (1999) and section 5.12 in Bachem and Kern (1992) for a more complete sketch of the origins of oriented matroid theory.

Examples can be found via vector arrangements over finite fields. See Ziegler (1991) for more discussion.

Exercises

1.1 Use the results of Section 1.2.1 to write (in a systematic way) a list of 2×4 rank 2 matrices giving each possible $\mathcal{C}(M)$ for this rank and size subject to the condition that the first two columns of the matrix are independent. Verify that these are also giving each possible $\chi(M)$, up to global change of sign.

1.2 (i) Let $\mathcal{C} = \{\{1,3\}^+\{2,4\}^-, \{1,3\}^-\{2,4\}^+, \{1,3\}^+\{5\}^-, \{1,3\}^-\{5\}^+, \{1,2,4\}^+\{5\}^-, \{1,2,4\}^-\{5\}^+, \{2,4\}^+\{3,5\}^-, \{2,4\}^-\{3,5\}^+\}$. Find a vector arrangement whose circuit set is $\mathcal{C}$. Then find another vector arrangement whose cocircuit set is $\mathcal{C}$.

(ii) Do the same for $\mathcal{C} = \{\{1,2,3\}^+\{4,5\}^-, \{1,2,3\}^-\{4,5\}^+\}$.

1.3 Let $p_1, \ldots, p_n$ be distinct points on an affine line, and let $\mathcal{M}$ be the rank 2 oriented matroid determined by these points. Prove that $\mathcal{M}$ depends only on the order of p_i along the line, and that $\mathcal{M}$ determines this order (up to global reversal).

1.4 Consider the vertex set of a regular pentagon in the plane. (There is nothing special about pentagons: We just want a concrete example.) By choosing a coordinate system for the plane, we can view these vertices as five vectors in $\mathbb{R}^2$ and get a rank 2 oriented matroid $\mathcal{M}^2$. On the other hand, by viewing our plane as an affine subspace of $\mathbb{R}^3$ we can view these vertices as five vectors in $\mathbb{R}^3$ and get a rank 3 oriented matroid $\mathcal{M}^3$. What can you say about the relationship between the oriented matroids $\mathcal{M}^2$ and $\mathcal{M}^3$ arising this way? For instance, what is the relationship between $\mathcal{V}(\mathcal{M}^2)$ and $\mathcal{V}(\mathcal{M}^3)$? What is the relationship between $\mathcal{V}^*(\mathcal{M}^2)$ and $\mathcal{V}^*(\mathcal{M}^3)$?

1.5 This is a generalization of Exercise 1.4. Let $M = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathrm{Mat}(r,n)$, and let $A \in \mathrm{Mat}(k,r)$ for some $k \leq r \leq n$. What can you say about the relationship between the oriented matroid associated to M and the oriented matroid associated to AM?

1.6 (i) Consider a convex hexagon in the affine plane with vertices labeled $1, 2, \ldots, 6$ in cyclical order. The vertex set gives a rank 3 oriented matroid on elements $[6]$. Show that this oriented matroid $\mathcal{M}_{\mathrm{hex}}$ is the same for all convex hexagons.

($\mathcal{M}_{\text{hex}}$ illustrates an important issue: The oriented matroid of an affine set of points tells you which *pairs* of subsets have intersecting convex hulls, but not which *triples* of subsets have intersecting convex hulls. For instance, a regular convex hexagon has three diagonals that intersect at a point, while a generic hexagon does not.)

(ii) If $W \subseteq V$ are subspaces of $\mathbb{R}^n$ then by definition $\mathcal{V}^*(W) \subseteq \mathcal{V}^*(V)$. Show that the converse fails by finding a rank 2 realizable oriented matroid $\mathcal{N}$ such that $\mathcal{V}^*(\mathcal{N}) \subseteq \mathcal{V}^*(\mathcal{M}_{\text{hex}})$ and a subspace V of $\mathbb{R}^6$ such that $\mathcal{V}^*(V) = \mathcal{V}^*(\mathcal{M}_{\text{hex}})$ but there is no subspace W of V such that $\mathcal{V}^*(W) = \mathcal{V}^*(\mathcal{N})$.

1.7 Prove the following version of the Farkas Lemma from the previous versions. (Here $\dot\cup$ denotes disjoint union.)

Farkas Lemma 4 For every subspace W of $\mathbb{R}^n$, every $Z = A^+B^-(C\dot\cup D)^0 \in \{0, +, -\}^n$, and every $j \in A \cup B$, exactly one of the following holds.

1. There exists $\mathbf{x} \in W$ such that $\text{sign}(x_i) \leq Z(i)$ for each $i \in A \cup B \cup C$ and $x_j \neq 0$.
2. There exists $\mathbf{y} \in W^\perp$ such that $\text{sign}(y_i) \leq Z(i)$ for each $i \in A \cup B \cup D$ and $y_j \neq 0$.

1.8 Let $t_1 < t_2 < \cdots < t_n$ be real numbers. For each $i \in [n]$ let $\mathbf{v}_i = (1, t_i, t_i^2, \ldots, t_i^{r-1})$. The oriented matroid of $(\mathbf{v}_i : i \in [n])$ is called the **alternating oriented matroid** $\mathcal{M}_{\text{alt}}^{n,r}$. This oriented matroid is independent of the choice of t_j, as the first part of this exercise will show.

1. Show that $\mathcal{M}_{\text{alt}}^{n,r}$ has a chirotope χ such that $\chi(i_1, i_2, \ldots, i_r) = +$ whenever $i_1 < \cdots < i_r$. Once you've shown this, you know that $\mathcal{M}_{\text{alt}}^{n,r}$ is *uniform*, i.e., every r-tuple of elements is a basis, so you know the sizes of the supports of signed circuits and signed cocircuits. The following parts of the exercise fill out our knowledge.
2. Let $\{i_0, \ldots, i_r\} \subseteq [n]$ with $i_0 < \cdots < i_r$. Let X be the signed circuit of $\mathcal{M}_{\text{alt}}^{n,r}$ with support $\{i_0, \ldots, i_r\}$ and with $X(i_0) = +$. Prove that $X(i_j) = (-1)^j$ for each j.
3. Let $\{i_0, \ldots, i_{n-r}\} \subseteq [n]$ with $i_0 < \cdots < i_{n-r}$. Let Y be the signed cocircuit with support $\{i_0, \ldots, i_{n-r}\}$ and with $Y(i_0) = +$. For each $j \in \{0, \ldots, n-r\}$, let $\eta(j) = |\{k : i_0 < k < i_j \text{ and } k \notin \{i_0, \ldots, i_{n-r}\}\}|$. Prove that $Y(i_j) = (-1)^{\eta(j)}$.

Remark 1.45 Some sources use the term "alternating oriented matroid" for any oriented matroid on $[n]$ with a chirotope χ such that $\chi(i_1, \ldots i_r) \in \{0, +\}$ for all $i_1 < \cdots < i_r$. More recently, these have been called *positively oriented matroids*, or *positroids*. See Section 7.7 for more about these.

1.9 Use results from this chapter to prove *Carathéodory's Theorem*: If P is a set of points in a d-dimensional space and q is in the convex hull of P then q is in the convex hull of some subset of P of size at most $d + 1$.