A CLASS OF SELF-DUAL MAPS

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1. Introduction. A *dissection* of a rectangle R into a finite number n of non-overlapping squares is called a *squaring* of R of *order n*. The n squares are called the *elements* of the dissection. If there is more than one element and the elements are all unequal the squaring is called *perfect* and R is a *perfect rectangle*. (We use R to denote both a rectangle and a particular squaring of it). If a squared (perfect) rectangle is a square we call it a *squared* (*perfect*) *square*.

In the course of an investigation of squared rectangles it was found that the theory reduced to that of certain "flows of electricity" in networks (linear graphs) on the sphere. An account of this work has been given elsewhere ([1]). The connection between squared rectangles and electrical networks is discussed later on in the present paper.

We have observed that the methods for the construction of a perfect square briefly described in [1] depend on the properties of networks of a particular kind. The characteristic property of a network of this kind is that the map on the sphere which it defines is combinatorially equivalent to its dual map. For this reason we have made an investigation into the properties of such "self-dual" maps. We give our results below.

Apart from the connection of self-dual maps with squared squares, some of them give rise to a particularly interesting class of perfect rectangles. These rectangles are discussed at the end of sec. **5**.

A detailed discussion of the problem of constructing a perfect square is given in a companion paper by one of us.

Before going on to the study of self-dual maps we collect some results on electrical networks in general which will be useful later.

Let N be a connected network whose vertices are $P_1, P_2, \ldots, P_n$ $(n \geqslant 2)$. The 1-cells are called *wires*; there may be more than one wire joining two vertices, and there may be wires whose two ends coincide. With each wire is associated a non-zero real number, its *conductance*. In [1] all conductances are positive. In the present paper also we are only interested in positive conductances; but negative conductances are employed in the companion paper by Tutte. We define a matrix $\{c_{rs}\}$ as follows.

If $r \neq s$,

$$-c_{rs} = \begin{cases} \text{sum of conductances of all wires joining } P_r \text{ to } P_s. \\ 0 \text{ if there are no such wires.} \end{cases}$$

$$c_{rr} = \text{sum of conductances of all wires joining } P_r \text{ to } \textit{other} \text{ vertices.}$$

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Thus

(A) $$c_{rs} = c_{sr}, \qquad \sum_r c_{rs} = 0.$$

From (A) we can readily show that all the first cofactors of $\{c_{rs}\}$ are equal. We call their common value the *complexity* of the network, and denote it by C. It is known that $C > 0$ when all the conductances are positive. (There is a proof of this result in [1].)

The second cofactor obtained by taking the cofactor of the element c_{su} in the cofactor of c_{rt} in $\{c_{rs}\}$ is denoted by $(rs\,.\,tu)$. (If $n = 2$, $(12.12) = 1 = -(21.12)$.) We put $(rr.tu) = 0 = (rs.tt)$. We call the $(rs.tu)$'s the *transpedances* of N.

We also write the transpedance $(rs.tu)$ as $(P_rP_s.P_tP_u)$.

Consider a flow of current from P_x to P_y (the *poles*). The currents in the wires then satisfy (except at the poles) Kirchhoff's Laws, which we state as follows.

(i) The total current flowing into P_i is zero.

(ii) The algebraic sum of the EMF's round any circuit is zero.

The EMF in a wire in the direction of the current may be defined as the current in the wire divided by the conductance of the wire. The EMF in the opposite direction is the negative of this. If (ii) is satisfied for all circuits we may associate a *potential* v_r with each node P_r so that the EMF in a wire with ends P_i and P_j in the direction P_i to P_j is $v_i - v_j$.

It is known that these conditions determine the flow uniquely when the total current I (flowing in at P_x and out at P_y) is given and the conductances are all positive ([2], 324-331).

Then the fall in potential from P_r to P_s is given by

(B) $$\frac{(xy.rs)I}{C}.$$

It is convenient to take $I = C$, thus fixing the values of the currents and potential differences of the network. The flow with $I = C$ is called the *full flow*; we speak of "full currents", etc.

From the definition of a transpedance it follows that

(C) $$(rs.tu) = (tu.rs) = -(sr.tu).$$

Using (B) we may restate Kirchhoff's Laws for the full flow as

(D) $$\sum_x c_{tx}\,(rs.tx) = C\,(\delta_{ts} - \delta_{tr}),$$

(E) $$(rs.tu) + (rs.uv) = (rs.tv).$$

The function δ_{rs} is equal to 1 if $r = s$, and to 0 otherwise.

Another general property of transpedances is the following:

(F) $$C \textit{ divides } \quad (rs.rs)(tu.tu) - (rs.tu)^2$$

(for integral conductances).

To prove this we use Jacobi's Theorem on determinants ([3], p. 98). This states that if Δ is a determinant, A_{ij} the cofactor of the element a_{ij} of Δ,

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and $\Delta_{pq,rs}$ the determinant obtained from Δ by striking out the pth and qth rows and the rth and sth columns, then

$$\Delta\,\Delta_{pq,\,rs} = A_{pr}A_{qs} - A_{ps}A_{qr}.$$

If we apply this result to the determinant X which is the minor of the element c_{ij} in the matrix $\{c_{rs}\}$, and if we assume that all the conductances are integers we find that

(G) $\qquad C$ *divides* $(ip.jr)(iq.js) - (ip.js)(iq.jr)$.

This proof assumes that $p \neq q$, $r \neq s$ and that X has at least three rows. But (G) is trivially true when one of these conditions is not satisfied. It is also trivially true when $i = p$ or q, or $j = r$ or s. It is thus a general property of transpedances. If we replace p by p' in (G) and then subtract (G) from the resulting formula we obtain the result

(H) $\qquad C$ *divides* $(pp'.jr)(iq.js) - (pp'.js)(iq.jr)$

by (E). Next we replace q by q' in (H) and then subtract (H) from the resulting formula. After four operations of this kind we have the result

(I) $\qquad C$ *divides* $(pp'.rr')(qq'.ss') - (pp'.ss')(qq'.rr')$,

where P_p, $P_{p'}$, etc. are any vertices of N. (F) is a special case of (I).

2. Self-dual maps. We define a *map* as a dissection of the surface of a sphere into a finite number of simple polygons $P_1, P_2, \ldots, P_n$, called *faces*. The boundary of each face is a simple closed curve, subdivided by a finite number $\geqslant 2$ of points called *vertices* into simple arcs called *edges*. It is supposed that

(i) No two faces have any interior point in common.
(ii) Each edge is common to just two faces.
(iii) Each vertex is a vertex of every face in whose boundary it lies.
(iv) The union of the faces, edges, and vertices is the whole sphere.

We shall speak of vertices, edges, and faces collectively as *cells*.

A cell will be said to be *incident* with any cell which is contained in its boundary or in whose boundary it is contained. (The boundary of an edge is its pair of end-points.)

The vertices and edges of a map constitute a network or linear graph which we call the 1-*section* of the map.

Two maps M_1 and M_2 are *combinatorially equivalent* if there is a 1-1 correspondence f between the set of cells X of M_1 and the set of cells $f(X)$ of M_2 such that $f(X)$ is a vertex, edge, or face according as X is a vertex, edge or face, and such that $f(X)$ is incident with $f(Y)$ if and only if X is incident with Y. We call the correspondence f a *combinatorial equivalence*.

We shall be interested in a *regular subdivision* $Z(M)$ of the map M. We define this as follows. In each edge W_i of M we select just one interior point of this edge, and denote it by w_i. In each face P_j we select just one interior point which we denote by P^*_j. We subdivide P_j into triangles by joining P^*_j to each vertex and each point w_i in the boundary of P_j. The resulting map is $Z(M)$. Its vertices are the vertices of M together with all the points w_j and P^*_j, its edges are the simple arcs into which the points w_j divide the edges of M and the joins made from the points P^*_j, and its faces are the triangles into which the faces P_j are subdivided. Since its faces are triangles it is called a *triangulation* or *simplicial dissection* of the sphere.

As consequences of the method of construction we see that

(i) Each face of $Z(M)$ is incident with one vertex of M, one of the points w_j and one of the points P^*_j;

(ii) Each vertex w_j of $Z(M)$ is incident with just four edges.

Let us enumerate the vertices of M as $V_1, V_2, \ldots, V_n$. Consider the union of all the faces of $Z(M)$, with their boundaries, which are incident with V_j. The boundary of this set is a simple closed curve. We denote the set by V^*_j. As a consequence of this result we can define a new map M^* as follows:

(i) The vertices of M^* are the points P^*_j.

(ii) For each point w_j we denote by W^*_j the union of the two edges of $Z(M)$ joining w_j to points P^*_h. The edges of M^* are the arcs W^*_j.

(iii) The faces of M^* are the sets V^*_j.

It is easily verified that M^* satisfies the definition of a map. It is a *dual map* of M.

We observe that the combinatorial structure of M^* (given by the incidence relations) is fixed by that of M, that $Z(M)$ is a regular subdivision of M^*, and that M is a dual map of M^*.

A map M is *self-dual* if there is a combinatorial equivalence f transforming M into M^*.

An edge W_j of a self-dual map M is *self-dual* under the combinatorial equivalence f transforming M into M^* if $f(W_j) = W^*_j$.

For any combinatorial equivalence f transforming a map M_1 into a map M_2, let Y be a point set which is the union of a set of cells $X_1, X_2, \ldots, X_k$ of M_1. Then we define $f(Y)$ as the union of the set of cells $f(X_1), f(X_2), \ldots, f(X_k)$ of M_2.

THEOREM 1. *Let f be a combinatorial equivalence transforming a map M into its dual M^*. Then there is a combinatorial equivalence f_z transforming $Z(M)$ into itself such that if X is any cell of M, $f_z(X) = f(X)$.*

For vertices of $Z(M)$ we define f_z as follows:

(i) If V_i is a vertex of M, $f_z(V_i) = f(V_i)$.

(ii) If W_j is an edge of M, and $f(W_j) = W^*_k$, then $f_z(w_j) = w_k$.

(iii) If P_k is any face of M, and if V_q is the vertex of M which is contained in $f(P_k)$, then $f_z(P^*_k) = V_q$.

Consider a face F of $Z(M)$ with incident vertices V_i, w_j, P^*_k. $f(P_k)$ is a face of M^* whose boundary contains $f(W_j)$ and its end-point $f(V_i)$. Hence just one of the triangles into which $f(P_k)$ is subdivided has the vertices $f_z(V_i)$, $f_z(w_j)$ and $f_z(P^*_k)$. So just one of the faces of $Z(M)$ has these three vertices.

We take this face as $f_z(F)$. If G is any edge of F, with end-points A and B, we define $f_z(G)$ as the edge of $f_z(F)$ with end-points $f_z(A)$ and $f_z(B)$. This defines $f_z(G)$ uniquely, for it is clear that not more than one edge can join two given vertices of $Z(M)$. It follows that f_z is a 1-1 correspondence, and that it preserves incidence relations.

THEOREM II. *Let f_z be defined as above. Then there is a positive integer n, and a homeomorphism H of the sphere onto itself such that $H^n = I$ (the identical mapping) and such that $H(X) = f_z(X)$ where X is any cell of $Z(M)$.*

We begin by making a further subdivision of $Z(M)$. For each edge E_r having one end a vertex of M and the other a vertex of M^* we select just one interior point e_r. We then subdivide each face of $Z(M)$ into two triangles by making a join from its vertex w_i to the opposite point e_r.

Each face of the new map Z' is a triangle incident with one vertex either of M or M^*, one vertex w_i and one vertex e_r. (We take the vertices of Z' to be the vertices of $Z(M)$ together with the points e_r.)

Clearly there is a combinatorial equivalence f' transforming Z' into itself such that if X is any cell of $Z(M)$, $f'(X) = f_z(X)$. (We take $f'(e_r)$ to be e_s, where $f_z(E_r) = E_s$.)

The correspondence f' has the property that if any iteration $(f')^m$ of f' transforms the cell X of Z' into itself, then $(f')^m$ transforms every cell of Z' in the boundary of X into itself. For no iteration of f' can map a vertex V_i or P^*_j onto a vertex w_i, or either onto an e_r.

Since f' is a 1-1 correspondence, if X is any cell of Z' some iteration $(f')^m$ of f' will map X into itself. (The number of cells of Z' is finite.) The least positive integer m for which this is so will be denoted by $\pi(X)$.

In constructing the homeomorphism H we use the topological theorem that any homeomorphism of the boundary $F(X)$ of an n-simplex X into the boundary $F(Y)$ of an n-simplex Y can be extended as a homeomorphism of X onto Y. We begin by defining H for the vertices v of Z' by $H(v) = f'(v)$. If X is any edge of Z' this gives us a homeomorphism H of $F(X)$ onto $F(f'(X))$ which we can extend as a homeomorphism of X onto $f'(X)$. Actually if $\pi(X) = 1$ we define the extension of H to X as the identity mapping. This is possible since H does not interchange the points of $F(X)$. If $\pi(X) > 1$ we define the extension of H to $X, f'(X), \ldots, (f')^{\pi(X)-2}(X)$ as we please, and then define it for $(f')^{\pi(X)-1}(X)$ by postulating that $H^{\pi(X)}$ reduces to the identity mapping in X. The extension of H to the faces of Z' is analogous. From this construction it follows that H is a homeomorphism of the sphere onto itself which satisfies $H^n = I$, where n is the L.C.M. of the numbers $\pi(X)$. Moreover, if X is any cell of $Z(M)$, $H(X) = f'(X) = f_z(X)$.

Now it is known that any homeomorphism of the sphere onto itself is topologically equivalent to a rotation, a reflection, or a rotation followed by a reflection ([4]). Any self-dual map M is topologically equivalent therefore to one which is transformed into its dual by one of these three operations.

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3. Dual flows. Consider an edge W_j of any map M, not necessarily self-dual, with ends V_r, V_s and incident faces P_u, P_v. We *orient* W_j by specifying one end, V_r say, as its *positive* end and the other as its *negative* end.

Since each face P_q has a boundary which is a simple closed curve we may *orient* P_q by specifying a particular sense of description of this curve, (clockwise or anti-clockwise). For this it is enough to give the cyclic order of the vertices of $Z(M)$ in the curve.

We say that W_j is *positively* or *negatively* incident with an incident face P_q according as its positive end immediately precedes or immediately succeeds w_j in the chosen cyclic order of vertices of $Z(M)$ in the boundary of P_q.

From now on the symbols W_j, P_q will denote edges or faces taken with some fixed orientation. For the same edges or faces taken with opposite orientation we use the symbols $-W_j$ and $-P_q$.

We shall in fact take for the orientation of each face P_q or V^*_i that cyclic order of vertices which agrees with some fixed positive sense of rotation about simple polygons on the sphere. We can think of this sense as the clockwise direction as seen from the centre of the sphere. It is evident that W_j is positively incident with one of its incident faces of M and negatively incident with the other.

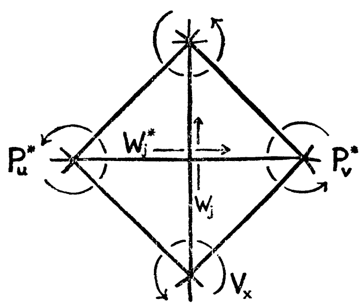

FIGURE 1

Suppose the edge W_j considered in the first paragraph of this section is positively incident with P_u and negatively incident with P_v. Then we define the orientation of W^*_j *dual* to that of W_j by taking P^*_u as its positive end and P^*_v as its negative end. Fig. 1 shows the state of affairs in the region defined by the four faces of $Z(M)$ which meet at w_j. The orientations of W_j and W^*_j are indicated by arrows directed from positive to negative ends. The curved arrows show the positive sense of rotation. From this figure we observe that

(i) If W_j is positively (negatively) oriented with respect to P_x, then P^*_x is the positive (negative) end of W^*_j;

(ii) If V_x is the positive (negative) end of W_j, then W^*_j is negatively (positively) oriented with respect to V^*_x.

We shall now consider the 1-sections of M and M^* as electrical networks in which every "wire" (edge) has conductance 1. Consider any distribution (flow) F of currents in M. We denote the current in W_j from positive to negative end by I_j, so the current in the opposite direction is $-I_j$. If P_k is any face of M we denote by $E(P_k)$ the sum of the currents in the edges of P_k, an edge W_j contributing I_j or $-I_j$ to this sum according as it is positively or negatively oriented with respect to P_k.

We define the *dual flow* F^* in M^* by taking the current in W^*_j from positive to negative end to be I_j. Then from (i) and (ii) we find

(iii) $E(P_k)$ = (algebraic sum of the current *from* P^*_k in the incident edges of M^*) and

(iv) $E(V^*_i) = -$(algebraic sum of the currents *from* V_i in the incident edges of M.)

Now the full flow in the 1-section of M whose positive and negative poles are the positive and negative ends respectively of W_j will be called the full flow in M with *polar edge* W_j. The current I_k in this flow is denoted by the "transpedance" $(W_j.W_k)$. Then using (C) we have

$$(W_j.W_k) = (W_k.W_j) = -(W_j.(-W_k)). \tag{1}$$

THEOREM III. *For any edges W_j and W_k of a map M,*

$$\begin{aligned} (W^*_j.W^*_k) &= -\frac{C^*}{C}(W_j.W_k) \quad \text{if } j \neq k, \\ (W^*_j.W^*_j) &= C^* - \frac{C^*}{C}(W_j.W_j), \end{aligned} \tag{2}$$

where C is the complexity[1] of M and C^* is the complexity of M^*.

Consider the full flow in M with polar edge W_j. The total current flowing from the positive end of W_j is C, by (D). Let F be the flow obtained from this by replacing each I_k for which $k \neq j$ by $\frac{C^*}{C} I_k$, and replacing I_j by $-\frac{C^*}{C}(C - I_j)$.

Then F satisfies Kirchhoff's Laws at each vertex and in the boundary of each face, save only for the two faces incident with W_j. Consequently F^* satisfies the Laws in the boundary of each face and at each vertex not incident with W^*_j, by (iii) and (iv). Now if P_u is the face of M with which W_j is positively incident we evidently have

$$E(P_u) = (-(W_j.W_j) + (-(C - (W_j.W_j))))\frac{C^*}{C} = -C^*$$

for the flow F. It readily follows, using (iii) and (iv), that F^* is the full flow in M^* with polar edge $-W^*_j$. The Theorem follows.

[1]More precisely we should say the complexity of the 1-section of M. It is clear that this 1-section is connected. Hence we can suppose $C > 0$. (See sec. **1**).

Corollary. *If M and M^* are combinatorially equivalent we may replace the above results by*

$$(3)\qquad \begin{cases} (W^*_j.W_k) = -(W_j.W_k) & \text{if } j \neq k, \\ (W^*_j.W^*_j) = C - (W_j.W_j). \end{cases}$$

For then $C = C^*$.

As a matter of fact, C^* is equal to C for all maps M, so that equations (3) are of general application. There is a proof in [1] that $C^* = C$ for all maps. In this paper we shall be concerned mainly with the case in which M and M^* are combinatorially equivalent, and we shall not need a proof of the general theorem $C^* = C$.

4. Reflexes. Let W_j be any edge of a map M, and V_x the positive end of W_j. Then (by (i) and (ii) of sec. **3**) we find first that W^*_j is negatively oriented with respect to V^*_x and thence that $((V_x)^*)^* = V_x$ is the negative end of $(W^*_j)^*$. Now $(W^*_j)^*$ is an oriented edge of M which contains w_j, and so we have

$$(W^*_j)^* = -W_j.$$

We return to the case of a self-dual map M transformed into its dual map by an operation ϕ which is either a rotation, a reflection, or a rotation followed by a reflection.

Now it is easily verified that if we define a positive sense of rotation on a sphere, then the effect of a rotation is to map any positively oriented simple polygon onto another positively oriented simple polygon, but the effect of a reflection is to map positively oriented simple polygons onto negatively oriented ones. As a consequence of this and the definition of duality, we have

(i) If ϕ is a reflection, or a rotation followed by a reflection, then for each edge W_j of M

$$\phi(W^*_j) = -(\phi(W_j))^*,$$

and

(ii) If ϕ is a rotation, then for each edge W_j of M

$$\phi(W^*_j) = (\phi(W_j))^*.$$

Now if W_j is any edge of M, $\phi^{-1}(W^*_j)$ is also an edge of M. We denote it by $\tilde{W}_j$. We say that M is a *reflex* with respect to ϕ if it has more than two edges and satisfies

$$\tilde{\tilde{W}}_j = W_j.$$

Now in order that M shall be a reflex with respect to ϕ it is evidently necessary that $\phi^2(w_j)$ shall be equal to w_j for each w_j. Whether we are dealing with case (i) or with case (ii), ϕ^2 must be a rotation (it must map positively oriented simple polygons onto positively oriented ones). As M has at least three points w_j, ϕ^2 is a rotation which leaves three distinct points w_j invariant. Hence $\phi^2 = I$, the identity mapping.

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We define ϵ to have the value -1 in case (i) and $+1$ in case (ii). Then since $\phi^2 = I$ we have in either case

$$\begin{aligned}\tilde{\tilde{W}}_j = \phi^2(\tilde{\tilde{W}}) &= \phi^2(\phi^{-1}((\phi^{-1}(W^*_j))^*) = \phi((\phi^{-1}(W^*_j))^*)\\ &= \epsilon\,.\,(\phi(\phi^{-1}(W^*_j))^* \quad = \epsilon\,.\,(W^*_j)^*\\ &= -\,\epsilon\,.\,W_j.\end{aligned}$$

Thus M is a reflex with respect to ϕ in case (i) but not in case (ii).

We deduce that there are essentially only two different kinds of reflexes, those in which ϕ is a reflection in a plane through the centre of the sphere, which we call *planar* reflexes, and those in which ϕ is a reflection followed by a rotation (through an angle not 2π) which we call *central* reflexes. (It is easily verified from the relation $\phi^2 = I$ that for a central reflex ϕ is a rotation through an angle π about an axis through the centre of the sphere followed by a reflection in the plane through the centre perpendicular to that axis. ϕ is thus a "reflection in the centre," transforming each point of the sphere into its diametrically opposite point.)

THEOREM IV. *Let W_j be any edge of a reflex. Then if W_j is not self-dual,* $(W_j.\tilde{W}_j) = 0$.

For then

$$\begin{aligned}(W_j.\tilde{W}_j) &= -(W^*_j.\tilde{W}^*_j), && \text{Theorem III, corollary;}\\ &= -(\phi W^*_j.\phi\tilde{W}^*_j), && \text{symmetry of } Z(M);\\ &= -(\tilde{W}_j.\tilde{\tilde{W}}_j) = -(\tilde{W}_j.W_j), && \text{definition of a reflex;}\\ &= -\,(W_j.\tilde{W}_j) && \text{by (1).}\end{aligned}$$

Consider any reflex M. Let m be the number of its vertices. Then m is also the number of its faces by the symmetry of $Z(M)$. By the Euler polyhedron formula it follows that the number of its edges is $2m - 2$. Let the vertices, edges and faces be enumerated as $V_1, V_2, \ldots, V_m$, as $W_1, W_2, \ldots, W_{2m-2}$, and as $P_1, P_2, \ldots, P_m$ respectively.

The structure of the 1-section of M can be represented by its incidence matrix $H_1 = \{\eta_{ij}^1\}$. Here η_{ij}^1 is $+1$, -1 or 0 according as V_i is the positive end of, the negative end of, or not incident with, W_j.

Let K be the matrix $\{c_{hk}\}$ defined in sec. **1**, for M.

Then

(4) $$K = H_1H'_1$$

where H'_1 denotes the transpose of H_1. For the (h,h)th element of $H_1H'_1$ is the sum of the squares of the elements of the hth row of H_1, that is the number of edges incident with V_h. And the (h,k)th element $(h \neq k)$ is evidently $-J$ where J is the number of edges which join V_h and V_k.

The incidence matrix $H_2 = \{\eta_{jk}^2\}$ is defined as follows: η_{jk}^2 is $+1$, -1 or 0 according as W_j is positively incident, negatively incident, or not incident with P_k. By elementary combinatorial topology we have ([5], p. 68)

(5) $$H_1H_2 = 0.$$

Now we divide the edges of M into four disjoint classes: S_1, S_2, S_3, S_4. S_1 is the class of all self-dual edges[2] W_j which satisfy $\tilde{W}_j = W_j$, S_2 is the class of all self-dual edges which satisfy $\tilde{W}_j = -W_j$, and finally the non-self-dual edges are partitioned among S_3 and S_4 in such a way that $\tilde{W}_j$ belongs to S_4 when W_j belongs to S_3. Let p denote the number of members of S_1, q the number of members of S_2, r the number of members of S_3 and therefore also of S_4. With a suitable ordering of the edges of M we can partition the matrix H_1 as follows

(6) $$H_1 = \{L_1|L_2|L_3|L_4\}.$$

Here L_i is the submatrix of H_1 defined by the columns corresponding to members of S_i. We imply by (6) that the edges of M are so ordered that the columns of L_1 come first in H_1, then those of L_2, and so on.

We now obtain a similar expression for H_2. We retain the same order for the edges, and we take the ith row of H_2 to correspond to the face ϕV^*_i. The edge $\tilde{W}_j$ is positively incident, negatively incident, or not incident with ϕV^*_i according as W^*_j is negatively incident, positively incident, or not incident with V^*_i (since ϕ reverses the orientation of a face), that is according as V_i is the positive end, the negative end, or not an end of W_j (by sec. **3**, prop. (ii)). Hence

(7) $$H_2 = \{L_1| -L_2|L_4|L_3\}'.$$

By (4), (5), (6) and (7) we have

$$K = L_1L'_1 + L_2L'_2 + L_3L'_3 + L_4L'_4,$$
$$0 = L_1L'_1 - L_2L'_2 + L_3L'_4 + L_4L'_3,$$

whence

(8) $$K = 2L_1L'_1 + [L_3 + L_4][L_3 + L_4]'$$

(9) $$= 2L_2L'_2 + [L_3 - L_4][L_3 - L_4]'.$$

We can write (8) and (9) in the forms

(10) $$K = (\sqrt{2}\, L_1|[L_3 + L_4])(\sqrt{2}\, L_1|[L_3 + L_4])',$$

(11) $$K = (\sqrt{2}\, L_2|[L_3 - L_4])(\sqrt{2}\, L_2|[L_3 - L_4])'.$$

Each of these expresses K as a product of a matrix with its transpose.

Now the rank of K cannot exceed that of $(\sqrt{2}\, L_1|[L_3 + L_4])$, or that of $(\sqrt{2}\, L_2|[L_3 - L_4])$. But the first of these matrices has $p + r$ and the second $q + r$ columns. Further the rank of K is $m - 1$, since $C > 0$ and $|K| = 0$. (By the definition of K its columns sum to zero.) As the sum $(p + r) + (q + r)$ is equal to the number of edges of M, which is $2m - 2$, it follows that

(12) $$p = q.$$

The number of edges $\tilde{W}_j$ of M such that $W_j = \pm W_j$ is thus even. We denote it by $2n$.

We see also that the matrices L_1 and L_2 have each m rows and n columns, while L_3 and L_4 have each m rows and $(m - n - 1)$ columns. Therefore

[2]Here by "self-dual" we mean "self-dual with respect to the operation ϕ".

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$U = (\sqrt{2}L_1 | [L_3 + L_4])$ and $V = (\sqrt{2}L_2 | [L_3 - L_4])$ have each m rows and $(m - 1)$ columns. Since the sum of the elements in any column of H_1 is zero, the same is true of U and V. These matrices accordingly have the property that all their minor determinants obtained by striking out one row are equal apart from sign; let us say that the minor determinants are equal to $\pm u$ for U and $\pm v$ for V. Since $K = UU' \doteq VV'$ we have, taking any first co-factor in K (which will by definition be the complexity of the 1-section of M),

$$C = u^2 = v^2.$$

But from its definition $u = (\sqrt{2})^n X$, where X is some integer. Hence we have

THEOREM V. *The complexity of M is $2^n X^2$, where X is an integer.*

Thus the complexity C of a reflex M is either of the form Y^2 or else of the form $2Y^2$, where Y is an integer.

THEOREM VI. *If C is of the form Y^2 the transpedances of the 1-section of M all divide by Y; if C is of the form $2Y^2$, where Y is even, they all divide by $2Y$.*

Let Z denote Y in the first case, and $2Y$ in the second case.

First, if $\tilde{W}_j = \pm W_j$, we have $(\tilde{W}_j.\tilde{W}_j) = (W^*_j.W^*_j) = (W_j.W_j)$. Hence by (3), $C = 2(W_j.W_j)$ and so Z divides $(W_j.W_j)$.

For any other edge W_j we have by (F),

$$C \text{ divides } ((W_j.W_j)(\tilde{W}_j.\tilde{W}_j) - (W_j.\tilde{W}_j)^2).$$

But $(W_j.\tilde{W}_j) = 0$, by Theorem IV, and $(\tilde{W}_j.\tilde{W}_j) = (W^*_j.W^*_j) = C - (W_j.W_j)$, by (3). Hence C divides $(W_j.W_j)^2$ and therefore Z divides $(W_j.W_j)$.

Next, if W_j and W_k are distinct edges of M then by (F)

$$C \text{ divides } ((W_j.W_j)(W_k.W_k) - (W_j.W_k)^2).$$

But by our previous result Z divides $(W_j.W_j)$ and $(W_k.W_k)$. Hence C divides $(W_j.W_j)(W_k.W_k)$ and therefore C divides $(W_j.W_k)^2$. Consequently Z divides $(W_j.W_k)$.

This proves the theorem for transpedances $(ab.cd)$ in which V_a and V_b are joined by an edge and V_c and V_d are joined by an edge. We can complete the proof by showing that each transpedance is a sum of transpedances of this form. This readily follows from (C) and (D).

5. Squared rectangles. Let M be any map. We orient the edges and faces of M as in sec. **3**. Let W_j be any edge of M. Let the positive and negative ends of W_j in M be V_p and V_q respectively. Let the faces of M incident with W_j be P_r and P_s. We may suppose that W_j is positively incident with P_r and negatively incident with P_s.

It is clear that the 1-sections of M and M^* are connected. Hence the complexities of these maps are positive. We shall denote the complexities of these maps by C and C^* respectively.

Let F be the full flow in M with polar edge W_j and let F_1 be the full flow in M^* with polar edge W^*_j.

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We may suppose that V_q has zero potential in F and that P^*_s has zero potential in F_1. Then the potential of V_p in F is $(W_j.W_j)$ and the potential of P^*_r in F_1 is $(W^*_j.W^*_j)$.

A vertex V_1 of M is said to be *active* in F if there is a non-zero current (in F) in some edge incident with V_i in M. Since $C > 0$ it follows from (D) that V_p and V_q are active in F. If V_i is not a pole of F and is active in F it is evident from Kirchhoff's Laws that V_i is incident with an edge in which a positive current flows to V_i and an edge in which a positive current flows from V_i. Consequently V_i is then joined by edges of M to one vertex of M of higher potential and one vertex of M of lower potential than V_i.

It follows that, in the flow F, the active vertices of highest and lowest potential are the poles. Since $C > 0$ it follows from (D) that V_p is incident with an edge in which a positive current flows from V_p in F. The other end of this edge is either V_q or an active vertex of lower potential than V_p. From these observations we may deduce the physically obvious result that

(13) $$(W_j.W_j) \geqslant v \geqslant 0,$$

where v is the potential of any active vertex of M in F.

Similarly we have

(14) $$(W^*_j.W^*_j) \geqslant w \geqslant 0,$$

where w is the potential of any active vertex of M^* in F_1.

Let ξ be any real number. We say that an edge W_k of M *comprises* ξ if ξ lies between the potentials in F of the ends of W_k. If W_k comprises ξ the ends of W_k are active in F. So by (13) we have

(15) $$(W_j.W_j) > \xi > 0.$$

Similarly W^*_k *comprises* the real number η if η lies between the potentials in F_1 of the ends of W^*_k, and if W^*_k comprises η we have

(16) $$(W^*_j.W^*_j) > \eta > 0.$$

Suppose that W_k is not W_j and that the current of F in W_k is non-zero. Then the set of all points

$$\left(\frac{C}{C^*}\eta, \xi\right)$$

in the (x, y) plane such that W_k comprises ξ and W^*_k comprises η is the interior of a square E_k of side $(W_j.W_k)$, by (2). By (2), (15) and (16), E_k is contained in the rectangle

$$(W_j.W_j) \geqslant y \geqslant 0, \qquad C - (W_j.W_j) \geqslant x \geqslant 0.$$

We call this rectangle R.

Let ξ be any real number satisfying (15) and not equal to the potential in F of any vertex M. Let S be the set of all vertices of M whose potential in F exceeds ξ, and let T be the set of all other vertices of M. Let X be the set of all edges of M which have one end in S and the other in T. Thus X is the set of all edges of M which comprise ξ. X is non-null, for $W_j \in X$, by (15).

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Let P_t be any face of M. Each vertex incident with P_t is either in S or in T. From this it follows that the number ν_t of members of X incident with P_t is even. Also in consecutive members of X in the boundary of P_t the positive currents flow in opposite directions in this boundary. The current in a member of X is non-zero by the definition of X.

Let X^* be the set of edges of M^* dual to the members of X. We say that a vertex P^*_t of M^* is *ξ-active* in M^* if it is incident with a member of X^*. By the preceding paragraph it follows that the number of edges of X^* incident with a ξ-active vertex of M^* is even. Since $W_j \in X$, P^*_r and P^*_s are ξ-active in M^*. If P^*_t is any other ξ-active vertex of M^* it follows from the preceding paragraph, and from equations (2), that in the flow F_1 the positive current in half the members of X^* incident with P^*_t flows to P^*_t, and the positive current in the other half flows from P^*_t. So then P^*_t is joined by edges of M^* to one ξ-active vertex of higher potential in F_1 and to one ξ-active vertex of lower potential in F_1.

We can construct a simple arc L in the 1-section of M^*, with ends P^*_r and P^*_s having the following properties:

(i) Each edge of L is in X^*, and L does not contain W^*_j;

(ii) The potentials in F_1 of the vertices of M^* in L, taken in order from P^*_r to P^*_s in L, form a strictly decreasing sequence.

To construct L we first observe that P^*_r, being incident with an even number of members of X^*, is incident with one edge K_1 of X^* other than W^*_j. Let U_1 be the other end of K_1. By the definition of X the current of F in K^*_1 is non-zero, and therefore the current of F_1 in K_1 is non-zero. So by (14) the potential in F_1 of U_1 is less than that of P^*_r. If U_1 is not P^*_s it is joined by a member of X^*, K_2 say, to a ξ-active vertex U_2 of M^* of lower potential in F_1. Similarly if U_2 is not P^*_s it is joined by a member K_3 of X^* to a ξ-active vertex of M^* of lower potential in F_1, and so on. The sequence $K_1, K_2 \ldots$ must terminate since the number of edges of M^* is finite. Clearly the union of the edges $K_1, K_2 \ldots$ is a simple arc L in the 1-section of M^*, with ends P^*_r and P^*_s having properties (i) and (ii).

Let L' be the simple closed curve in the 1-section of M^* obtained by adjoining W^*_j to L. Then L' is a union of members of X^*.

Any vertex of M must be contained in one of the two residual domains in the sphere of the simple closed curve L'. The two ends of a member J of X lie in different faces of M^* incident with J^*. Hence if J^* is contained in L' they lie in different residual domains of L'. In particular V_p and V_q lie in different residual domains of L'. We denote the residual domains containing V_p and V_q by D_+ and D_- respectively.

The potential in F of any active vertex of M which is in D_+ must exceed ξ. For let V_k be a vertex of M which is in D_+, is active in F, and has the lowest possible potential in F consistent with these conditions. Since V_k is in D_+ it is not V_q. Hence it is joined by an edge H of M to an active vertex of lower potential. This vertex must be in D_-. Hence H intersects L'. Thus H must be a member of X. Consequently H comprises ξ and therefore the potential in F of V_k exceeds ξ.

A similar argument shows that the potential in F of any active vertex of M which is in D_- must be less than ξ.

We conclude that if J is any member of X its two ends must be in different residual domains of L'. Hence J intersects L' and therefore J^* is one of the edges of M^* in L'. So L' is the union of all the members of X^*.

Now let η be any real number satisfying (16), and not equal to the potential in F_1 of any vertex of M^*. By properties (i) and (ii) of the arc L it follows that there is just one edge W_k of N other than W_j such that W_k comprises ξ and W^*_k comprises η.

From this result it is easily seen that no two of the squares E_k have any interior point in common, and that each point of the rectangle R belongs to at least one of the squares E_k. We recall that E_k is defined only when W_k is not W_j and the current I_k of F is non-zero.

Thus the squares E_k define a squaring of the rectangle R.

We say that the 1-section of M is a *c-net* of the resulting squared rectangle. The network obtained from this 1-section by suppressing the edge W_j is a *p-net* of the squared rectangle. The highest common factor of the lengths of the sides of the squares E_k, i.e., the highest common factor of the transpedances $(W_j.W_k)$ taken for the given value of j and all values of k, is called the *reduction* of the squared rectangle.

Segments parallel to the x axis will be called *horizontal*. Segments parallel to the y axis will be called *vertical*. The lengths of the vertical and horizontal sides of the squared rectangle R are $(W_j.W_j)$ and $C - (W_j.W_j)$ respectively. The numbers obtained by dividing these by the reduction of the squared rectangle are called the *reduced* horizontal and vertical sides respectively. The numbers $(W_j.W_j)$ and $C - (W_j.W_j)$ are called the *full* vertical and horizontal sides respectively.

A point in R which is common to four of the elements E_k is called a *cross* of the squared rectangle.

As a consequence of Theorem VI (leaving aside the case $C = 2Y^2$, Y odd) we see that the reduction of any squared rectangle having a reflex as c-net is a multiple of the reduced horizontal side. This property also holds for the reduction of a squared square ([1]). It seems plausible that if one made a list of a few hundreds of such rectangles one would discover some perfect squares among them. At least the possibility of deriving a perfect square from a given reflex cannot be excluded, as it can for most networks, by the reduction theorems of [1]. We have not made such a long list; we merely draw the attention of more industrious squarers of rectangles to the possibility.

We have evaluated a few squared rectangles of fairly small order having 1-sections of reflexes as c-nets. The perfect ones in our list all correspond to central reflexes. They are given below in the notation of C. J. Bouwkamp ([6], pp. 1179-1180).

In Bouwkamp's notation the top left-hand corner of each component square of a squared rectangle is taken as its "representative point." The lengths of the sides of those squares for which the representative points lie in the same horizontal segment (connected component of the union of horizontal sides of the elements of the squared rectangle) are bracketed together in the order of the representative points from left to right. The brackets read in order from top to bottom of the rectangle. When several brackets correspond to collinear horizontal segments they are written in the order of these segments from left to right.

We remark that each one of the rectangles listed below has a cross. It can be shown that this is a consequence of Theorem IV.

Rectangle (1). Order XXII. Full horizontal side $(271)^2$. Reduction 271. Reduced sides 271, 257.

(91, 80, 100), (11, 49, 20), (67, 35), (29, 30, 61), (32, 3),
(52, 28, 1), (31), (24, 4), (99), (96), (76).

Rectangle (2). Order XXIV. Full horizontal side $(480)^2$. Reduction 480. Reduced sides 480, 456. Side-ratio 20:19.

(158, 160, 162), (118, 40), (38, 91, 31), (29, 133), (60,) (78),
(25, 66, 34, 26), (180, 41), (8, 18), (32, 10), (161), (139).

Rectangle (3). Order XXIV. Full horizontal side $(494)^2$. Reduction 494. Reduced sides 494, 418. Side-ratio 13:11.

(183, 149, 162), (34, 102, 13), (59, 116), (113, 104), (30, 29),
(1, 28,) (36, 66, 31), (4, 140), (35), (9, 131), (122), (101).

Rectangle (4). Order XXIV. Full horizontal side $(459)^2$. Reduction 459. Reduced sides 459, 401.

(118, 107, 123, 111), (11, 80, 16), (12, 99), (129), (64, 87),
(35, 45, 41, 23), (18, 191), (25, 10), (59), (55), (154), (114).

Rectangle (5). Order XXIV. Full horizontal side $(463)^2$. Reduction 463. Reduced sides 463, 464.

(200, 134, 129), (45, 84), (94, 40), (54, 31), (109, 63, 28),
(23, 8), (92), (35, 87, 77), (46, 52), (10, 159), (155), (149).

Rectangle (6). Order XXIV. Full horizontal side $(473)^2$. Reduction 473. Reduced sides 473, 435.

(166, 138, 169), (57, 81), (137, 29), (50, 119), (86), (62, 69),
(27, 59, 55, 7), (48, 147), (132, 5), (32), (4, 99), (95).

Rectangle (7). Order XXIV. Full horizontal side $(399)^2$. Reduction 399. Reduced sides 399, 429. Side-ratio 133:143.

(137, 120, 142), (17, 81, 22), (154), (59, 41, 64), (18, 23),
(106, 34, 13, 5), (8, 84), (21), (55), (139), (138, 16), (122).

Rectangle (8). Order XXIV. Full horizontal side $(424)^2$. Reduction 848. Reduced sides 212, 214. Side-ratio 106:107.

(79, 62, 71), (17, 36, 9), (27, 20, 33), (53, 24, 19),
(7, 13), (5, 50, 34), (29), (46), (82), (16, 18), (66), (64).

The last of these deserve special comment. It is remarkable that any perfect rectangle of the twenty-fourth order should have such small elements. Even in the thirteenth order most of the perfect rectangles have larger reduced elements than this. (A list of all the simple squared rectangles of order less than 14 is given in [6]).

6. Central and planar reflexes. For a central reflex the number n is zero, since ϕ can transform no point w_j into itself. Hence, by Theorem V, the complexity of a central reflex is of the form X^2 where X is an integer. This is exemplified by the full sides given in the above list.

For any planar reflex M, let Q be the great circle in which the plane of symmetry of $Z(M)$ cuts the sphere. By symmetry considerations Q can meet the boundary of a face F of $Z(M)$ only in the vertex w_j or in the mid-point of the opposite side ($V_i P^*_k$ say). It follows that Q cuts the 1-section of M only in points w_j. An arc in Q joining two consecutive points w_j,—let us say w_{j_1} and w_{j_2}—on Q is evidently a diagonal of a quadrilateral $w_{j_1} w_{j_2} V_1 P^*_k$ composed of two faces of $Z(M)$.

From this we deduce that if $w_1, w_2, \ldots, w_k$ are points w_j on Q, taken in their cyclic order on Q, then $W_1, W_2, \ldots, W_k$ is a cyclic sequence of self-dual edges in which each edge has one end in common with its successor, and the other with its predecessor. We say that these edges constitute the *girdle* of M. The girdle of M^* evidently consists of the edges $W^*_1, W^*_2, \ldots, W^*_k$.

From these considerations it is evident that an edge W_j of M satisfies $W_j = \pm \tilde{W}_j$ if and only if it is in the girdle of M.

THEOREM VII. *The edges W_j of the girdle satisfy alternately $\tilde{W}_j = W_j$ and $\tilde{W}_j = -W_j$.*

Consider the quadrilateral $w_{j_1} w_{j_2} V_i P^*_k$ mentioned above. If V_i is the positive (negative) end of both W_{j_1} and W_{j_2} then $W^*_{j_1}$ and $W^*_{j_2}$ are both negatively (positively) oriented with respect to V^*_i. (Sec. **3**, Prop. (ii)). Then P^*_k must be the positive end of one of them and the negative end of the other, by (5). It follows that V_i is the positive end of the edges $\tilde{W}_{j_1}$, $\tilde{W}_{j_2}$ and the negative end of the other, whence the theorem is true for W_{j_1} and W_{j_2}. The argument when V_i is the positive end of one of the edges W_{j_1} and W_{j_2} is analogous.

Thus the edges of the girdle belong alternately to S_1 and S_2. (See sec. **4**).

THEOREM VIII. *If W_j and W_k are distinct edges belonging to the same class S_1 or S_2, then $(W_j . W_k) = 0$.*

For then
$$\begin{aligned}(W_j . W_k) &= -(W^*_j . W^*_k) \text{ by (3)},\\ &= -(\phi W^*_j . \phi W^*_k) = -(\tilde{W}_j . \tilde{W}_k)\\ &= -(W_j . W_k), \text{ by the definition of } S_1 \text{ and } S_2.\end{aligned}$$

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THEOREM IX. *Let W_j be an edge of the girdle of a planar reflex M. Then the squared rectangle corresponding to the full flow in the 1-section of M with polar edge W_j is a diagonally symmetric squared square.*

We use the notation of sec. **5**.

Suppose there is an edge W_k of M, other than W_j, and real numbers ξ and η, such that W_k comprises ξ and W^*_k comprises η. Then by the symmetry of $Z(M)$, $\tilde{W}_k$ comprises η and $(\tilde{W}_k)^*$ comprises ξ. Hence the elements of the rectangle corresponding to W_k and $\tilde{W}_k$ are reflections of one another in the line $y = x$. The theorem follows.

It is this theorem that underlies the methods for the construction of perfect squares given in [1]. It is easily seen that we can construct a planar reflex by the following sequence of operations. First we draw the girdle. Then we arbitrarily fix the 1-section of M in the "northern hemisphere," arranging that it shall fit the given girdle and not meet the "equator" Q. Then we subdivide this so as to form the part of $Z(M)$ in the northern hemisphere. Finally we complete $Z(M)$ by reflecting in the equator.

Fig. 2 shows a planar reflex as seen from above the "north pole". The full lines represent the part of M in the northern hemisphere. The broken lines represent equally well the part of M in the southern hemisphere or the part of M^* in the northern hemisphere.

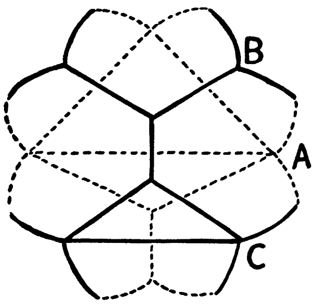

FIGURE 2

The device adopted in [1] amounts to taking a "rotor" for the part of the 1-section of M, apart from the edges of the girdle, in the northern hemisphere and then in the resulting planar reflex replacing this rotor, in one hemisphere only, by its mirror image. It is found that this destroys the symmetry of the squares of Theorem IX without affecting their squareness. For details the reader is referred to [1] and to the companion paper which follows immediately.

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