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The Uncomplemented Subspace $\mathbf{K}(X, Y)$

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Abstract. A vector measure result is used to study the complementation of the space $K(X, Y)$ of compact operators in the spaces $W(X, Y)$ of weakly compact operators, $CC(X, Y)$ of completely continuous operators, and $U(X, Y)$ of unconditionally converging operators. Results of Kalton and Emmanuele concerning the complementation of $K(X, Y)$ in $L(X, Y)$ and in $W(X, Y)$ are generalized. The containment of c_0 and ℓ_∞ in spaces of operators is also studied.

Throughout this paper X and Y denote Banach spaces. Notation is consistent with that used in Diestel [2]. Let $\mathcal{P}$ be the power class of the positive integers. Let (e_n) be the canonical base of c_0, (e_n^*) be the canonical base of ℓ_1, and (e_n^p) be the canonical base of ℓ_p, $p > 1$. The set of all bounded linear operators from X to Y will be denoted by $L(X, Y)$, and the compact, weakly compact, unconditionally converging, resp. completely continuous operators will be denoted by $K(X, Y)$, $W(X, Y)$, $U(X, Y)$, resp. $CC(X, Y)$. An operator $T\colon X \to Y$ is unconditionally converging if T maps weakly unconditionally converging series into unconditionally converging series. An operator $T\colon X \to Y$ is called completely continuous (or Dunford-Pettis) if T maps weakly Cauchy sequences to norm convergent sequences. The $w^* - w$ continuous maps from X^* to Y (resp. $w^* - w$ continuous compact) will be denoted by $L_{w^*}(X^*, Y)$ (resp. $K_{w^*}(X^*, Y)$). The bounded subset A of X is called a limited subset of X if each w^*-null sequence in X^* tends to 0 uniformly on A. If every limited subset of X is relatively compact, then we say that X has the Gelfand-Phillips property.

Numerous authors have studied the complementation of the spaces $W(X, Y)$ and $K(X, Y)$ in the space $L(X, Y)$. See Bator and Lewis [1], Kalton [12], Emmanuele [4, 5], Emmanuele and John [7], Feder [8, 9], and John [11]. Kalton [12] proved that if ℓ_1 is complemented in X and Y is infinite dimensional, then $K(X, Y)$ is not complemented in $L(X, Y)$. Emmanuele [6] showed that if ℓ_1 embeds in X and there is an operator $T\colon \ell_2 \to Y$ such that $(T(e_n^2))$ is basic and normalized, then $K(X, Y)$ is not complemented in $W(X, Y)$.

In this note we want to extend the previous results and provide sufficient conditions for $K(X, Y)$ to be uncomplemented in $W(X, Y)$, $U(X, Y)$, and $CC(X, Y)$.

Emmanuele [5] and John [11] proved that if c_0 embeds in $K(X, Y)$, then $K(K, Y)$ is not complemented in $L(X, Y)$. Emmanuele provided a useful tool for identifying copies (even complemented copies) of c_0 in spaces of operators in [5, Theorem 3]. A generalization of this theorem will be helpful in our study ([10, Theorem 20]).

We recall the following well-known isometries [14]:

(i) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$, $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \to T^*)$
(ii) $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$, and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \to T^{**})$.

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Theorem 1 ([10, Theorem 20]) *Let X and Y be Banach spaces satisfying the following assumption: there exists a Banach space G with an unconditional basis (g_n), biorthogonal coefficients (g_n^*), and two operators $R\colon G \to Y$ and $S\colon G^* \to X$ such that $(R(g_i))$ and $(S(g_i^*))$ are seminormalized sequences and either $(R(g_i))$ or $(S(g_i^*))$ is a basic sequence. Then $c_0 \hookrightarrow K_{w^*}(X^*, Y)$.*

Moreover, if $(R(g_i))$ and $(S(g_i^*))$ are basic and Y (or X) has the Gelfand–Phillips property, then $K_{w^*}(X^*, Y)$ contains a complemented copy of c_0.

The following result of Lewis and Schulle [13] plays an important role in the proof of Theorem 3 which in turn strengthens results in [6, 10, 12].

Lemma 2 ([13]) *If $\mu\colon \mathcal{P} \to X$ is bounded and finitely additive, $\mu(\{n\}) = 0$ for all n, and there are countably many functionals in X^* separating the points in $\mu(\mathcal{P})$, then there is an infinite subset M of $\mathbb{N}$ such that $\mu(B) = 0$ for all $B \subseteq M$.*

We remark that if X is separable and Y is the dual of a separable space, then there are countably many functionals separating the points of $L(X, Y)$.

Theorem 3 *Let X and Y be Banach spaces with the following properties.*

There exists a Banach space G with an unconditional basis (g_i), coefficient functionals (g_i^*), and operators $R\colon G \to Y$ and $S\colon X \to G$ such that $(R(g_i))$ is a seminormalized basic sequence in Y and $(S^*(g_i^*))$ has no norm convergent subsequence. Suppose that R (or S) is weakly compact. If (P_A) is the family of projections associated with (g_i) and $T\colon W(X, Y) \to K(X, Y)$ is an operator, then there is an $N \in \mathbb{N}$ so that

$$TRP_{\{n\}}S \neq RP_{\{n\}}S$$

for $n > N$. Thus $K(X, Y)$ is not complemented in $W(X, Y)$. Further, c_0 embeds in $K(X, Y)$ and ℓ_∞ embeds in $W(X, Y)$.

Proof Suppose (P_A) is the family of projections associated with (g_n), R and S are as in the hypothesis, R is weakly compact, and define $\mu\colon \mathcal{P} \to W(X, Y)$ by $\mu(A) = RP_AS$, $A \subseteq \mathbb{N}$. Let X_0 be a separable subspace of X such that $\|x^*\| = \|x^*|_{X_0}\|$ for all $x^* \in [S^*(g_n^*) : n \geq 1]$.

Let (y_n^*) be the sequence of biorthogonal coefficients corresponding to $(R(g_n))$ and let (f_n^*) be a sequence of Hahn-Banach extensions to Y^*. Note that $\mu(A)|_{X_0}$ is compact if and only if A is finite. Indeed, $(\mu(A)^*(f_n^*)) = (S^*(g_n^*))_{n\in A}$, which is relatively compact if and only if A is finite.

Now suppose that $T\colon W(X, Y) \to K(X, Y)$ is an operator and $B = \{n \in \mathbb{N} : T\mu(\{n\}) = \mu(\{n\})\}$ is an infinite set. Let $J\colon Y \to \ell_\infty$ be an operator that is an isometry on $[R(g_n) : n \geq 1]$. Identify $\mathcal{P}$ with $\mathcal{P}(B)$ in the obvious way, and define $\nu\colon \mathcal{P}(B) \to W(X_0, \ell_\infty)$ by

$$\nu(A) = (JT\mu(A) - J\mu(A))|_{X_0},\ A \subseteq B.$$

Apply Lemma 2 to obtain an infinite subset M of B so that $JT\mu(M) = J\mu(M)$ on X_0. Since J is an isometry on $[R(g_n) : n \geq 1]$ and $JT\mu(M)|_{X_0}$ is compact, $\mu(M)|_{X_0}$ is

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compact, a contradiction. Therefore, there does not exist a projection $P\colon W(X, Y) \to K(X, Y)$.

Since $(S^*(g_i^*))$ is w^*-null and has no norm convergent subsequence, $\|S^*(g_i^*)\| \not\to 0$, and we may assume that $(S^*(g_i^*))$ is seminormalized. Apply Theorem 1 and the preceding isometries to conclude that $c_0 \hookrightarrow K(X, Y)$. Further, note that $\mu\colon \mathcal{P} \to W(X, Y)$ is bounded and finitely aditive and $\|\mu(\{n\})\| = \|S^*(g_n^*)\|\|R(g_n)\| \not\to 0$. Apply the Diestel–Faires theorem to obtain that $\ell_\infty \hookrightarrow W(X, Y)$. ■

Remark If one assumes in the preceding theorem that $R\colon G \to Y$ (or $S\colon X \to G$) is completely continuous (resp. R (or S) is unconditionally converging) and that $T\colon CC(X, Y) \to K(X, Y)$ (resp. $T\colon U(X, Y) \to K(X, Y)$) is an operator, then the same proof shows that $K(X, Y)$ is not complemented in $CC(X, Y)$ (resp. $K(X, Y)$ is not complemented in $U(X, Y)$), c_0 embeds in $K(X, Y)$, and ℓ_∞ embeds in $CC(X, Y)$ (resp. ℓ_∞ embeds in $U(X, Y)$).

The following result contains [6, Lemma 3].

Corollary 4 *If ℓ_1 is complemented in X and Y does not have the Schur property, then $K(X, Y)$ is not complemented in $W(X, Y)$ and $\ell_\infty \hookrightarrow W(X, Y)$.*

Proof Let $G = \ell_1$ and let $P\colon X \to \ell_1$ be a projection. Since P is a projection, P^* is an isomorphism, and thus $(P^*(e_n))$ has no norm convergent subsequence. Let (y_n) be a normalized weakly null basic sequence in Y. Define $R\colon \ell_1 \to Y$ by $R(b) = \sum b_n y_n$, $b = (b_n) \in \ell_1$. Since $(R(e_n^*)) = (y_n)$ is weakly null, R is weakly compact. Apply Theorem 3. ■

Corollary 5 ([6, 10]) *If $c_0 \hookrightarrow Y$ and X^* does not have the Schur property, then $K(X, Y)$ is not complemented in $W(X, Y)$ and $\ell_\infty \hookrightarrow W(X, Y)$. Further, $K(X, Y)$ is not complemented in $U(X, Y)$ and $\ell_\infty \hookrightarrow U(X, Y)$.*

Proof Let $G = c_0$ and $R\colon c_0 \to Y$ be an embedding. Let (x_n^*) be a weakly null normalized sequence in X^* and define $S\colon X \to c_0$ by $S(x) = (x_n^*(x))$. Note that $(S^*(e_n^*)) = (x_n^*)$ has no norm convergent subsequence. Further, since $(S^*(e_n^*))$ is weakly null, S^*, thus S, is weakly compact. Since every weakly compact operator is unconditionally converging, S is unconditionally converging. Apply Theorem 3. ■

The following result contains [12, Lemma 3].

Corollary 6 *If ℓ_1 is complemented in X and Y is infinite dimensional, then $K(X, Y)$ is not complemented in $CC(X, Y)$ and $K(X, Y)$ is not complemented in $U(X, Y)$. Consequently, $K(X, Y)$ is not complemented in $L(X, Y)$. Further, $\ell_\infty \hookrightarrow CC(X, Y)$ and $\ell_\infty \hookrightarrow U(X, Y)$.*

Proof Let $P\colon X \to \ell_1$ be a projection. As in Corollary 4, $(P^*(e_n))$ has no norm convergent subsequence. Let (y_n) be a normalized basic sequence in Y. Define $R\colon \ell_1 \to Y$ by $R(b) = \sum b_n y_n$, $b = (b_n) \in \ell_1$. Note that R is completely continuous and unconditionally converging, since ℓ_1 has the Schur property. Apply Theorem 3. ■

We remark that in the previous proof both operators P and R are completely continuous. Further, $RP\colon X \to Y$ is completely continuous and non-compact, hence $K(X,Y) \neq CC(X,Y)$. Thus Corollary 6 strictly extends [12, Lemma 3].

Corollary 7 *If X is infinite dimensional, $L(X,c_0) = CC(X,c_0)$, and $c_0 \hookrightarrow Y$, then $K(X,Y)$ is not complemented in $CC(X,Y)$ and $\ell_\infty \hookrightarrow CC(X,Y)$.*

Proof Let $G = c_0$ and $R\colon c_0 \to Y$ be an embedding. Use the Josefson–Nissenzweig theorem to obtain a normalized and w^*-null sequence (x_n^*) in X^* and define $S\colon X \to c_0$ by $S(x) = (x_n^*(x))$. Note that $(S^*(e_n^*)) = (x_n^*)$ has no norm convergent subsequence. The hypothesis assures that S is completely continuous. Apply Theorem 3. ∎

We remark that in the previous argument S is completely continuous, and thus $RS\colon X \to Y$ is completely continuous. Further, RS is not compact and $K(X,Y) \neq CC(X,Y)$.

If $1 < p < \infty$, then we say that p' is conjugate to p if $\frac{1}{p} + \frac{1}{p'} = 1$, *i.e.*, $(\ell_p)^* \simeq \ell_{p'}$.

The following result extends and complements [13, Theorem 3.3].

Theorem 8 *Suppose that $1 < p < \infty$, p' is conjugate to p, and $S\colon X \to \ell_{p'}$ is a non-compact operator. Suppose $1 < p \leq q < \infty$. For $p' \leq p \leq q$ or $p \leq p' \leq q$, if $R\colon \ell_q \to Y$ is a non-compact operator, then $K(X,Y)$ is not complemented in $W(X,Y)$ and $K(X,Y)$ is not complemented in $U(X,Y)$. Further, $c_0 \hookrightarrow K(X,Y)$, $\ell_\infty \hookrightarrow W(X,Y)$, and $\ell_\infty \hookrightarrow U(X,Y)$.*

However, if $1 < p < q < p' < \infty$, then there exist Banach spaces X and Y and appropriate operators R and S such that $K(X,Y) = L(X,Y)$ and $c_0 \not\hookrightarrow K(X,Y)$.

Proof *Case 1.* Suppose $p' \leq p \leq q$. Since $S^*\colon \ell_p \to X^*$ is non-compact, we can find a $\delta > 0$ and a sequence (x_n) in the unit ball of ℓ_p such that $\|S^*(x_n) - S^*(x_m)\| > \delta$ if $n \neq m$. Since ℓ_p is reflexive ($1 < p < \infty$), without loss of generality we may assume that $(a_n) = (x_n - x_{n+1})$ is weakly null. Note that $(a_n) \not\to 0$. By the Bessaga–Pelczyinski Selection Principle, (a_n) has a subsequence (a_{n_i}) that is equivalent to a block basic sequence of (e_n^p). Note that ℓ_p is perfectly homogeneous, since $1 < p < \infty$. Thus we may assume that (a_n) is equivalent to (e_n^p).

Since $p' \leq p$, there is a natural injection $J\colon \ell_{p'} \to \ell_p$ such that $a_n = J(e_n^{p'})$ for all n. Then $(S^*(a_n)) = (S^* J(e_n^{p'}))$ is weakly null and not norm null. The Bessaga–Pelczyinski Selection Principle also applies to $(S^*(a_n))$, and without loss of generality $(S^*(a_n))$ is a seminormalized basic sequence. Note that since both $\ell_{p'}$ and ℓ_p are reflexive, J is w^*-w^* continuous, and thus an adjoint operator. Suppose that $J = T^*$ for some operator $T\colon \ell_{p'} \to \ell_p$. Hence $(S^*(a_n)) = (S^* T^*(e_n^{p'}))$ is a seminormalized basic sequence.

Similarly, since $R\colon \ell_q \to Y$ is non-compact, we can find a weakly null, seminormalized sequence (b_n) equivalent to (e_n^q) in ℓ_q such that $(R(b_n))$ is a seminormalized basic sequence. Since $p \leq q$, there is a natural injection $U\colon \ell_p \to \ell_q$ such that $b_n = U(e_n^p)$ for all n. Hence $(R(b_n)) = (RU(e_n^p))$ is basic and seminormalized. Let $G = \ell_p$. Note that RU is weakly compact and $(S^*(a_n)) = (S^* T^*(e_n^{p'}))$ has no norm convergent subsequence. Further, since $c_0 \not\hookrightarrow \ell_p$, RU is unconditionally converging. Apply Theorem 3 to $TS\colon X \to \ell_p$ and $RU\colon \ell_p \to Y$.

Case 2. Suppose $p \leq p' \leq q$. The argument is similar to that in Case 1. Apply Theorem 3 for $G = \ell_{p'}$.

Case 3. Suppose $1 < p < q < p' < \infty$. Since $q < p'$, $L(\ell_{p'}, \ell_q) = K(\ell_{p'}, \ell_q)$. Further, this space of compact operators is reflexive [12], and thus $c_0 \not\hookrightarrow K(\ell_{p'}, \ell_q)$. In this case, let $X = \ell_{p'}$, $Y = \ell_q$, and let $S\colon \ell_{p'} \to \ell_{p'}$ and $R\colon \ell_q \to \ell_q$ be the identity operators. ∎

Corollary 9 *Suppose that $2 \leq q < \infty$. If $\ell_{q'}$ is a quotient of X and there is a non-compact operator $T\colon \ell_q \to Y$, then $K(X, Y)$ is not complemented in $W(X, Y)$ and $K(X, Y)$ is not complemented in $U(X, Y)$. Further, $c_0 \hookrightarrow K(X, Y)$, $\ell_\infty \hookrightarrow W(X, Y)$, and $\ell_\infty \hookrightarrow U(X, Y)$.*

Proof If $2 \leq q < \infty$, then $1 < q' \leq q$. Let Q be a quotient map from X to $\ell_{q'}$. Then Q is non-compact. Let $p = q$. Apply Theorem 8. ∎

Corollary 10 *If $\ell_1 \hookrightarrow X$ and there is $2 \leq q < \infty$ and a non-compact operator $T\colon \ell_q \to Y$, then $K(X, Y)$ is not complemented in $W(X, Y)$ and $K(X, Y)$ is not complemented in $U(X, Y)$. Further, $c_0 \hookrightarrow K(X, Y)$, $\ell_\infty \hookrightarrow W(X, Y)$, and $\ell_\infty \hookrightarrow U(X, Y)$.*

Proof Since $\ell_1 \hookrightarrow X$, X has a quotient isomorphic to ℓ_2, by a result of [3]. Apply Theorem 8. ∎

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