RESONANT THREE-DIMENSIONAL PERIODIC SOLUTIONS ABOUT THE TRIANGULAR EQUILIBRIUM POINTS IN THE RESTRICTED PROBLEM

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ABSTRACT

In the three-dimensional restricted three-body problem, the existence of resonant periodic solutions about L_4 is shown and expansions for them are constructed for special values of the mass parameter, by means of a perturbation method. These solutions form a second family of periodic orbits bifurcating from the triangular equilibrium point. This bifurcation is the evolution, as μ varies continuously, of a regular vertical bifurcation point on the corresponding family of planar periodic solutions emanating from L_4.

1. INTRODUCTION

It is known that for values of the mass parameter less than the critical value of Routh, the general solution of the linearized equations of motion around the triangular equilibrium point L_4 has the form

$$\begin{aligned} x(t) &= A_1\cos\sigma_1 t + A_2\sin\sigma_1 t + A_3\cos\sigma_2 t + A_4\sin\sigma_2 t, \\ y(t) &= B_1\cos\sigma_1 t + B_2\sin\sigma_1 t + B_3\cos\sigma_2 t + B_4\sin\sigma_2 t\ , \qquad (1) \\ z(t) &= C_1\cos t + C_2\sin t, \end{aligned}$$

where

$$\sigma_1 = \left[\frac{1-\sqrt{\Delta}}{2}\right]^{\frac{1}{2}}, \qquad \sigma_2 = \left[\frac{1+\sqrt{\Delta}}{2}\right]^{\frac{1}{2}}, \qquad \Delta = 1 - 27\mu(1-\mu). \qquad (2)$$

Due to the difference in the values of the two frequencies, long and short period terms are recognized, corresponding to small (σ_1) and large (σ_2) values of the frequency .

By a suitable choice of the initial conditions three particular solutions of the linearized equations are obtained in three dimensions.

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These are:

1. $x(t) = 0, \quad y(t) = 0, \quad z(t) = C_1 \cos t + C_2 \sin t,$ (3)

2. $$x(t) = A_1 \cos \sigma_1 t + A_2 \sin \sigma_1 t,$$
$$y(t) = B_1 \cos \sigma_1 t + B_2 \sin \sigma_1 t, \tag{4}$$
$$z(t) = C_1 \cos t + C_2 \sin t,$$

3. $$x(t) = A_3 \cos \sigma_2 t + A_4 \sin \sigma_2 t,$$
$$y(t) = B_3 \cos \sigma_2 t + B_4 \sin \sigma_2 t, \tag{5}$$
$$z(t) = C_1 \cos t + C_2 \sin t.$$

The first solution is periodic and is continued to periodic orbits of finite size for every value of the mass parameter. A small part of this family has been given by Buck (1920). The second and the third solutions are not periodic unless the period $T_{x,y}$ of the planar motion is commensurate to the period T_z of the motion along the Oz-axis.

We suppose that

$$T_{x,y} = \frac{p}{q} T_z \tag{6}$$

where p and q are mutually prime integers, or equivalently that

$$\sigma_i(\mu) = \frac{p}{q} , \tag{7}$$

with i = 1 for the case of long period and i = 2 for the case of short period planar periodic solutions.

Relation (7) is valid for "special" values of the mass parameter μ. For these values of μ the corresponding linearized equations admit a periodic solution which, as we show in this article, is continued to a family of periodic solutions of the non-linear equations. This family which bifurcates from the triangular equilibrium point is not an isolated dynamical phenomenon occuring for these "special" values of μ but it is the "arrival" at L_4, as μ varies continuously, of a vertical bifurcation point on either the family of planar-short-period, or the planar-long-period solutions.

2. SECOND ORDER EXPANSIONS FOR THE RESONANT THREE DIMENSIONAL PERIODIC SOLUTIONS

The Equations of the three-dimensional motion of the third particle, when expanded to second order terms with respect to x,y and z, take the form

$$x'' - 2(1+\alpha)y' = (1+\alpha)^2\left[\frac{3}{4}x + \frac{3\sqrt{3}}{4}\rho y + \frac{21}{16}\rho x^2 - \frac{3\sqrt{3}}{8}xy - \frac{33}{16}y^2 + \frac{3}{4}\rho z^2\right],$$

$$y'' + 2(1+\alpha)x' = (1+\alpha)^2\left[\frac{3\sqrt{3}}{4}\rho x + \frac{9}{4}y - \frac{3\sqrt{3}}{16}x^2 - \frac{33}{8}\rho xy - \frac{9\sqrt{3}}{16}y^2 + \frac{3\sqrt{3}}{4}z^2\right], \tag{8a}$$

$$z'' = -(1+\alpha)^2\left[z - \frac{3}{2}\rho xz - \frac{3\sqrt{3}}{2}yz\right],$$

where

$$t = (1+\alpha)\tau, \qquad \alpha = \alpha_1\varepsilon + \alpha_2\varepsilon^2, \tag{8b}$$

and $\rho = 1 - 2\mu$.

The solution of Equations (8a) is expressed as

$$\begin{aligned} x(\tau) &= x_1(\tau)\varepsilon + x_2(\tau)\varepsilon^2 + \dots, \\ y(\tau) &= y_1(\tau)\varepsilon + y_2(\tau)\varepsilon^2 + \dots, \\ z(\tau) &= z_1(\tau)\varepsilon + z_2(\tau)\varepsilon^2 + \dots, \end{aligned} \tag{9}$$

where $x_i(\tau)$, $y_i(\tau)$, $z_i(\tau)$, $i = 1,2,\dots$, are functions of τ to be determined and ε is a small orbital parameter.

Expressions (9) are now substituted into Equations (8a), and the coefficients of the same powers of ε are equated.

The coefficients of the first power of ε are solutions of the "linearized" Equations:

$$x_1'' - 2y_1' = \frac{3}{4}x_1 + \frac{3\sqrt{3}}{4}\rho y_1 ,$$

$$y_1'' + 2x_1' = \frac{3\sqrt{3}}{4}\rho x_1 + \frac{9}{4}y_1 , \tag{10}$$

$$z_1'' = - z_1 .$$

We consider as a particular solution of Equations (10) the solution (4) or (5), i.e.,

$$x_1(\tau) = A_j\cos\sigma_i\tau + A_{j+1}\sin\sigma_i\tau ,$$

$$y_1(\tau) = B_j\cos\sigma_i\tau + B_{j+1}\sin\sigma_i\tau , \tag{11}$$

$$z_1(\tau) = C_1\cos\tau + C_2\sin\tau ,$$

which is assumed periodic because of condition (6) which we assume to hold. From condition (6), or (7), "special" values of the mass parameter μ are determined.

The coefficients of the second power of ε are solutions of the Equations

$$x_2'' - 2y_2' - \frac{3}{4}x_2 - \frac{3\sqrt{3}}{4}\rho y_2 = 2\alpha_1 y_1' + \frac{3}{2}\alpha_1 x_1 + \frac{3\sqrt{3}}{2}\alpha_1\rho y_1$$

$$+ \frac{21}{16}\rho x_1^2 - \frac{3\sqrt{3}}{8}x_1y_1 - \frac{33}{16}\rho y_1' + \frac{3}{4}\rho z_1^2 ,$$

$$y_2'' + 2x_2' - \frac{3\sqrt{3}}{4}\rho x_2 - \frac{9}{4}y_2 = - 2\alpha_1 x_1' + \frac{3\sqrt{3}}{2}\rho\alpha_1 x_1 + \frac{9}{2}\alpha_1 y_1 \tag{12}$$

$$- \frac{3\sqrt{3}}{16}x_1^2 - \frac{33}{8}\rho x_1y_1 - \frac{9\sqrt{3}}{16}y_1^2 + \frac{3\sqrt{3}}{4}z_1^2 ,$$

$$z_2'' + z_2 = - 2\alpha_1 z_1 + \frac{3}{2}\rho x_1 z_1 + \frac{3\sqrt{3}}{2}y_1 z_1 .$$

By substitution of expressions (11) into the second members of Equations (12) we obtain the following system of Equations

$$(D^2 - \frac{3}{4})x_2 - (2D + \frac{3\sqrt{3}}{4}\rho)y_2 = K_1\sin\sigma_i\tau + K_2\cos\sigma_i\tau + K_3\cos^2\sigma_i\tau$$

$$+ K_4\sin^2\sigma_i\tau + K_5\sin 2\sigma_i\tau + K_6\sin^2\tau \triangleq f_2(\tau) ,$$

$$(2D - \frac{3\sqrt{3}}{4}\rho)x_2 + (D^2 - \frac{9}{4})y_2 = \Lambda_1\sin\sigma_i\tau + \Lambda_2\cos\sigma_i\tau + \Lambda_3\cos^2\sigma_i\tau$$

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$$+ \Lambda_4 \sin^2 \sigma_i \tau + \Lambda_5 \sin 2\sigma_i \tau + \Lambda_6 \sin^2 \tau \triangleq g_2(\tau) , \qquad (13)$$

$$(D^2 + 1) z_2 = E_1 \sin(\sigma_i + 1)\tau - E_1 \sin(\sigma_i - 1)\tau +$$

$$+ E_2 \cos(\sigma_i + 1)\tau - E_2 \cos(\sigma_i - 1)\tau \triangleq h_2(\tau) ,$$

where we have abbreviated:

$$\begin{aligned}
K_1 &= \alpha_1(-2B_j\sigma_i + \frac{3}{2} A_{j+1} + \frac{3\sqrt{3}}{2} \rho B_{j+1}) , \\
K_2 &= \alpha_1(2B_{j+1}\sigma_i + \frac{3}{2} A_j + \frac{3\sqrt{3}}{2} \rho B_j) , \\
K_3 &= \frac{21}{16} \rho A_j^2 - \frac{3\sqrt{3}}{8} A_j B_j - \frac{33}{16} \rho B_j^2 , \\
K_4 &= \frac{21}{16} \rho A_{j+1}^2 - \frac{3\sqrt{3}}{8} A_{j+1} B_{j+1} - \frac{33}{16} \rho B_{j+1}^2 , \\
K_5 &= \frac{21}{16} \rho A_j A_{j+1} - \frac{3\sqrt{3}}{16} A_{j+1} B_j - \frac{3\sqrt{3}}{16} A_j B_{j+1} - \frac{33}{16} \rho B_j B_{j+1} , \\
K_6 &= \frac{3}{4} \rho C_1^2 , \\
\Lambda_1 &= \alpha_1(2A_j\sigma_i + \frac{3\sqrt{3}}{2} \rho A_{j+1} + \frac{9}{2} B_{j+1}) , \\
\Lambda_2 &= \alpha_1(-2A_{j+1}\sigma_i + \frac{3\sqrt{3}}{2} \rho A_j + \frac{9}{2} B_j) , \qquad (14) \\
\Lambda_3 &= -(\frac{3\sqrt{3}}{16} A_j^2 + \frac{33}{8} \rho A_j B_j + \frac{9\sqrt{3}}{16} B_j^2) , \\
\Lambda_4 &= -(\frac{3\sqrt{3}}{16} A_{j+1}^2 + \frac{33}{8} \rho A_{j+1} B_{j+1} + \frac{9\sqrt{3}}{16} B_{j+1}^2) , \\
\Lambda_5 &= -(\frac{33}{16} \rho A_j B_{j+1} + \frac{33}{16} \rho A_{j+1} B_j + \frac{3\sqrt{3}}{16} A_j A_{j+1} + \frac{9\sqrt{3}}{16} B_j B_{j+1}) , \\
\Lambda_6 &= \frac{3\sqrt{3}}{4} C_1^2 , \\
E_1 &= \frac{3}{4} C_1(\rho A_j + \sqrt{3} B_j) , \\
E_2 &= \frac{3}{4} C_1(\rho A_{j+1} + \sqrt{3} B_{j+1}) .
\end{aligned}$$

A periodic solution of Equations (14) is given by:

$$x_2(\tau) = \frac{\Gamma_1}{\Theta - 12} + \frac{\Gamma_4}{\Phi}\cos 2\sigma_i\tau + \frac{\Gamma_5}{\Phi}\sin 2\sigma_i\tau$$
$$+ \frac{\Gamma_6}{\Theta}\cos 2\tau + \frac{\Gamma_7}{\Theta}\sin 2\tau ,$$

$$y_2(\tau) = \frac{\Delta_1}{\Theta - 12} + \frac{\Delta_4}{\Phi}\cos 2\sigma_i\tau + \frac{\Delta_5}{\Phi}\sin 2\sigma_i\tau \qquad (15)$$
$$+ \frac{\Delta_6}{\Theta}\cos 2\tau + \frac{\Delta_7}{\Theta}\sin 2\tau ,$$

$$z_2(\tau) = \frac{E_1}{-\sigma_i^2 - 2\sigma_i}\sin(\sigma_i + 1)\tau - \frac{E_1}{-\sigma_i^2 + 2\sigma_i}\sin(\sigma_i - 1)\tau$$
$$+ \frac{E_2}{-\sigma_i^2 - 2\sigma_i}\cos(\sigma_i + 1)\tau - \frac{E_2}{-\sigma_i^2 + 2\sigma_i}\cos(\sigma_i - 1)\tau ,$$

where, supressing the secular terms, we have forced $\alpha_1 = \alpha_2 = 0$ (in (8b)) and, therefore, $t = \tau$. The quantities Θ and Φ are given by

$$\Theta = 12 + \frac{27}{4}\mu(1-\mu), \qquad \Phi = 16\sigma_i^4 - 4\sigma_i^2 + \frac{27}{4}\mu(1-\mu). \qquad (16)$$

We have also abbreviated:

$$\Gamma_1 = -\frac{9}{8}(K_3 + K_4 + K_6) + \frac{3\sqrt{3}}{8}\rho(\Lambda_3 + \Lambda_4 + \Lambda_6) ,$$

$$\Gamma_4 = -(2\sigma_i^2 + \frac{9}{8})(K_3 - K_4) + 4\Lambda_5\sigma_2 + \frac{3\sqrt{3}}{8}\rho(\Lambda_3 - \Lambda_4) ,$$

$$\Gamma_5 = -(4\sigma_i^2 + \frac{9}{4})K_5 - 2(\Lambda_3 - \Lambda_4)\sigma_i + \frac{3\sqrt{3}}{4}\rho\Lambda_5 ,$$

$$\Gamma_6 = \frac{25}{8}K_6 - \frac{3\sqrt{3}}{8}\rho\Lambda_6 ,$$

$$\Gamma_7 = \Lambda_6 , \qquad (17)$$

$$\Delta_1 = -\frac{3}{8}(\Lambda_3 + \Lambda_4 + \Lambda_6) + \frac{3\sqrt{3}}{8}\rho(K_3 + K_4 + K_6) ,$$

$$\Delta_4 = -(2\sigma_i^2 + \frac{3}{8})(\Lambda_3 - \Lambda_4) - 4K_5\sigma_i + \frac{3\sqrt{3}}{8}\rho(K_3 - K_4) ,$$

$$\Delta_5 = -(4\sigma_i^2 + \frac{3}{4})\Lambda_5 + 2(K_3 - K_4)\sigma_i + \frac{3\sqrt{3}}{4}\rho K_5 ,$$

$$\Delta_6 = \frac{19}{8}\Lambda_6 - \frac{3\sqrt{3}}{8}\rho K_6 ,$$

$$\Delta_7 = 2K_6 .$$

It has been verified numerically that for values of the small parameter ε in the interval $(0, 0.05]$, the periodic functions (15) represent periodic solutions of the problem to an accuracy of at least six significant figures. Furthermore, these solutions can be "corrected" and "continued" by numerical methods. In this way the existence of these resonant periodic orbits which had been questioned by the classical workers (Buck, 1920), has been demonstrated.

3. SOLUTION OF A HILL EQUATION FOR VERTICAL STABILITY ALONG THE FAMILIES OF PLANAR PERIODIC ORBITS

An important question arising here is whether the family of periodic solutions constructed in the previous paragraph exists only for the resonant value of μ.

As we shall see the answer is that it also exists for other values of μ. However, for these other values it does not bifurcate from L_4. Rather, it bifurcates from a vertical-bifurcation point on the planar family of (short - or long - period) periodic orbits. Hereafter we use the term "family of planar periodic solutions" to indicate either the short - period family or the long - period family of planar periodic solutions, the two cases been formally identical.

First we consider the family of planar periodic solutions and we derive second order expansions for them. The derivation of these expansions is similar to the above derivation of the resonant three - dimensional orbits and the resulting expressions differ from expressions (11), (15) only in the absence of the π-periodic terms.

The second order expansions for the planar orbits are:

$$x(t) = (A_j \cos\sigma_i t + A_{j+1} \sin\sigma_i t)\varepsilon + (G_1 + G_2 \cos 2\sigma_i t + G_3 \sin 2\sigma_i t)\varepsilon^2 , \quad (18a)$$

$$y(t) = (B_j \cos\sigma_i t + B_{j+1} \sin\sigma_i t)\varepsilon + (H_1 + H_2 \cos 2\sigma_i t + H_3 \sin 2\sigma_i t)\varepsilon^2 , \quad (18b)$$

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where

$$G_1 = \Gamma_1/\Theta - 12, \quad G_2 = \Gamma_4/\Phi, \quad G_3 = \Gamma_5/\Phi,$$
$$H_1 = \Delta_1/\Theta - 12, \quad H_2 = \Delta_4/\Phi, \quad H_3 = \Delta_5/\Phi, \tag{19}$$

with $i = 1$, $j = 1$ for the long period solutions and $i = 2$, $j = 3$ for the short period ones.

Along each family of planar solutions we can determine the parameter s_v which characterizes every periodic solution as vertically stable or unstable. If the periodic solution is vertically stable,

$$|s_v| < 1,$$

and if there are integers p and q such that

$$s_v = \cos 2\pi \frac{p}{q}, \tag{20}$$

then this planar periodic orbit is vertically self-resonant and a bifurcation point of a three-dimensional family.

In the present case where we know the analytical expression of the family of planar solutions we can in fact calculate s_v analytically as a function of the orbital parameter ε. Indeed, the value of s_v results from two linearly independent solutions of the Hill equation

$$\ddot{v} + Q(t)v = 0,$$

with

$$Q(t) = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}.$$

(see, e.g. Markellos, 1977). Using the second order expansions (18) for x and y we obtain for the periodic function Q the expression

$$Q(t,\varepsilon) = 1 + (Q_2 \cos\sigma_i t + Q_3 \sin\sigma_i t)\varepsilon$$
$$+ (Q_1 + Q_4 \cos 2\sigma_i t + Q_5 \cos 2\sigma_i t)\varepsilon^2 \tag{21}$$

with:

$$Q_1 = -\frac{3}{2}(1-2\mu)G_1 - \frac{3\sqrt{3}}{2}H_1 + \frac{3}{16}(A_j^2 + A_{j+1}^2) + \frac{33}{16}(B_j^2 + B_{j+1}^2)$$
$$+ \frac{15\sqrt{3}}{8}(1-2\mu)(A_j B_j + A_{j+1}B_{j+1}), \tag{22}$$

$$Q_2 = -\frac{3}{2}(1-2\mu)A_j - \frac{3\sqrt{3}}{2}B_j, \tag{23}$$

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$$Q_3 = -\frac{3}{2}(1-2\mu)A_{j+1} - \frac{3\sqrt{3}}{2}B_{j+1} , \tag{24}$$

$$Q_4 = -\frac{3}{2}(1-2\mu)G_2 - \frac{3\sqrt{3}}{2}H_2 + \frac{3}{16}(A_j^2 - A_{j+1}^2) + \frac{33}{16}(B_j^2 - B_{j+1}^2) + \frac{15\sqrt{3}}{8}(A_jB_j - A_{j+1}B_{j+1}) , \tag{25}$$

$$Q_5 = -\frac{3}{2}(1-2\mu)G_3 - \frac{3\sqrt{3}}{2}H_3 + \frac{3}{8}A_jA_{j+1} + \frac{33}{8}B_jB_{j+1} + \frac{15\sqrt{3}}{8}(A_jB_{j+1} + A_{j+1}B_j) . \tag{26}$$

We seek solutions v(t) of the Equation

$$\ddot{v} + Q(t,\varepsilon)v = 0 , \tag{27}$$

in the form

$$v(t) = v_0(t) + v_1(t)\varepsilon + v_2(t)\varepsilon^2 . \tag{28}$$

By substitution into Equation (27), neglecting terms of order higher than the second in ε and solving the resulting differential equations, we obtain as the general solution of Equation (27) the expression

$$\begin{aligned} v(t) = {} & (\mu_1\cos t + \mu_2\sin t)(1+\varepsilon+\varepsilon^2) + \Big[(w_1\cos(1+\sigma_i)t \\ & + w_2\sin(1+\sigma_i)t + w_3\cos(1-\sigma_i)t + w_4\sin(1-\sigma_i)t\Big](\varepsilon+\varepsilon^2) \\ & + \Big[w_5 t\cos t + w_6 t\sin t + w_7\cos(2\sigma_i+1)t + w_8\sin(2\sigma_i+1)t \\ & + w_9\cos(2\sigma_i-1)t + w_{10}\sin(2\sigma_i-1)t\Big]\varepsilon^2 , \end{aligned} \tag{29}$$

where we have abbreviated:

$$w_1 = -\frac{u_1}{\sigma_i(\sigma_i+2)} , \qquad w_2 = -\frac{u_2}{\sigma_i(\sigma_i+2)} ,$$

$$w_3 = -\frac{u_3}{\sigma_i(\sigma_i-2)} , \qquad w_4 = -\frac{u_4}{\sigma_i(\sigma_i-2)} ,$$

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$$w_5 = -\frac{u_5}{2}, \qquad w_6 = -\frac{u_6}{2}, \tag{30}$$

$$w_7 = -\frac{u_7}{4\sigma_i(\sigma_i+1)}, \qquad w_8 = -\frac{u_8}{4\sigma_i(\sigma_i+1)},$$

$$w_9 = -\frac{u_9}{4\sigma_i(\sigma_i-1)}, \qquad w_{10} = -\frac{u_{10}}{4\sigma_i(\sigma_i-1)},$$

and

$$u_1(\mu_1, \mu_2) = -\frac{1}{2} Q_2\mu_1 + \frac{1}{2} Q_3\mu_2,$$

$$u_2(\mu_1, \mu_2) = -\frac{1}{2} Q_3\mu_1 - \frac{1}{2} Q_2\mu_2,$$

$$u_3(\mu_1, \mu_2) = -\frac{1}{2} Q_2\mu_1 - \frac{1}{2} Q_3\mu_2,$$

$$u_4(\mu_1, \mu_2) = \frac{1}{2} Q_3\mu_1 - \frac{1}{2} Q_2\mu_2,$$

$$\begin{aligned} u_5(\mu_1, \mu_2) = {} & -Q_1\mu_1 + \frac{Q_2u_1}{2\sigma_i(\sigma_i+2)} + \frac{Q_3u_2}{2\sigma_i(\sigma_i+2)} \\ & + \frac{Q_2u_3}{2\sigma_i(\sigma_i-2)} - \frac{Q_3u_4}{2\sigma_i(\sigma_i-2)}, \end{aligned} \tag{31}$$

$$\begin{aligned} u_6(\mu_1, \mu_2) = {} & -Q_1\mu_2 - \frac{Q_3u_1}{2\sigma_i(\sigma_i+2)} + \frac{Q_2u_2}{2\sigma_i(\sigma_i+2)} \\ & + \frac{Q_3u_3}{2\sigma_i(\sigma_i-2)} + \frac{Q_2u_4}{2\sigma_i(\sigma_i-2)}, \end{aligned}$$

$$u_7(\mu_1, \mu_2) = -\frac{1}{2}\mu_1Q_4 + \frac{1}{2}\mu_2Q_5 + \frac{Q_2u_1}{2\sigma_i(\sigma_i+2)} - \frac{Q_3u_2}{2\sigma_i(\sigma_i+2)},$$

$$u_8(\mu_1, \mu_2) = -\frac{1}{2}\mu_2Q_4 - \frac{1}{2}\mu_1Q_5 + \frac{Q_3u_1}{2\sigma_i(\sigma_i+2)} + \frac{Q_2u_2}{2\sigma_i(\sigma_i+2)},$$

$$u_9(\mu_1, \mu_2) = -\frac{1}{2}\mu_1Q_4 - \frac{1}{2}\mu_2Q_5 + \frac{Q_2u_3}{2\sigma_i(\sigma_i-2)} + \frac{Q_3u_4}{2\sigma_i(\sigma_i-2)},$$

$$u_{10}(\mu_1, \mu_2) = \frac{1}{2}\mu_2Q_4 - \frac{1}{2}\mu_1Q_5 + \frac{Q_3u_3}{2\sigma_i(\sigma_i-2)} - \frac{Q_2u_4}{2\sigma_i(\sigma_i-2)}.$$

4. PARAMETER OF VERTICAL STABILITY AS FUNCTION OF ε

If $v^*(t)$ and $v^{**}(t)$ are two linearly independent solutions of Equation (27), with

$$\begin{pmatrix} v^*(0) & v^{**}(0) \\ \dot{v}^*(0) & \dot{v}^{**}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{32}$$

then

$$2s_v = v^*(t) + \dot{v}^{**}(t) , \tag{33}$$

where T is the period of the planar periodic orbit.

From Equations (29) and (33) we finally obtain, for the vertical stability parameter, the expression

$$s_v = s_0(\mu) + s_1(\mu)\varepsilon + s_2(\mu)\varepsilon^2 , \tag{34}$$

with

$$s_0(\mu) = \cos\frac{2\pi}{\sigma_i} , \tag{35}$$

$$s_1(\mu) = -\frac{Q_3}{2\sigma_i}\sin\frac{2\pi}{\sigma_i} , \tag{36}$$

$$s_2(\mu) = \frac{1}{2}\left[\Omega_1\cos\frac{2\pi}{\sigma_i} + \Omega_2\sin\frac{2\pi}{\sigma_i}\right] , \tag{37}$$

where

$$\frac{2\pi}{\sigma_i} = T ,$$

is the period of the planar periodic orbit.

The quantities Ω_1 and Ω_2 involved in Equation (37) are given by the following expressions:

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$$\Omega_1 = 4 + \frac{Q_4}{2(\sigma_i^2 - 4)} + \left[Q_1 + \frac{Q_2^2 + Q_3^2}{2(\sigma_i^2 - 4)} \right] \frac{2\pi}{\sigma_i}$$

$$+ \frac{20 Q_3^2 + 4Q_2^2 - (5Q_2^2 + 13Q_3^2)\sigma_i^2}{4\sigma_i^2(\sigma_i^2 - 4)^2} ,$$

$$\Omega_2 = \frac{-2\sigma_i^2 - \sigma_i + 5}{\sigma_i(\sigma_i^2 - 4)} Q_3 - \frac{\sigma_i}{2(\sigma_i^2 - 4)} Q_5$$

$$+ \frac{-2\sigma_i^3 + 10\sigma_i^2 - 3\sigma_i - 18}{2\sigma_i^2(\sigma_i^2 - 4)^2} Q_2 Q_3 + \frac{Q_2^2 + Q_3^2}{4(\sigma_i^2 - 4)} . \qquad (38)$$

Equating the expression for s_v to the bifurcation value (20) we obtain the relation

$$s_0(\mu) + s_1(\mu)\varepsilon + s_2(\mu)\varepsilon^2 = \cos 2\pi \frac{p}{q} , \qquad (39)$$

connecting the mass parameter μ with the orbital parameter ε for any given resonance p/q.

Thus, given a value of μ say μ* near the resonant value $\mu_{p/q}$, we can determine the value ε for which the "vertical bifurcation" occurs. In other words, we find how the resonant family has evolved in going from $\mu_{p/q}$ to μ*, i.e. from a branching at the equilibrium point to a branching at a vertical self-resonant orbit of the planar family.

We have therefore demonstrated how this "peculiar" resonant family of periodic orbits exists not only at the resonant value of μ but also at the neighboring values. It has a natural evolution as a tree-dimensional branch of the family of planar orbits. This is true for the short period family as well as for the long period family of planar periodic orbits.

Numerical results and further details of the evolution of these families of three-dimensional periodic orbits will be published elsewhere.

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