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PROJECTIONS ON TREE-LIKE BANACH SPACES

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1. In this paper, we investigate the ranges of projections on certain Banach spaces of functions defined on a diadic tree. The notion of a "tree-like" Banach space is due to James [**4**], who used it to construct the separable space JT which has nonseparable dual and yet does not contain l_1. This idea has proved useful. In [**3**], Hagler constructed a hereditarily c_0 tree space, HT, and Schechtman [**6**] constructed, for each $1 \leqq p \leqq \infty$, a reflexive Banach space, ST_p, with a 1-unconditional basis which does not contain l_p, yet is uniformly isomorphic to $\left(\sum_{i=1}^{n} \oplus ST_p\right)_{l_p^n}$ for each n.

In [**1**] we showed that if U is a bounded linear operator on JT, then there exists a subspace $W \subset JT$, isomorphic to JT such that either U or $(I - U)$ acts as an isomorphism on W and UW or $(I - U)W$ is complemented in JT. In this paper, we establish this result for the Hagler and Schechtman tree spaces.

By arguments of Casazza and Lin [**2**], this implies that if X is either the Hagler or one of the Schechtman tree spaces, $X = Z \oplus W$, and either Z or W is isomorphic to its square, then either Z or W is itself isomorphic to X. Although in both this paper and in [**1**] and [**2**], great use is made of the symmetry properties of the unit vector basis, the arguments of [**1**] are not sufficient for analyzing the Hagler or Schechtman tree spaces. The new idea which is used is that of a banded subtree (see Definition 1), and in the case of these spaces, we show that the unit vector basis is equivalent to any subsequence of it which is supported on a banded subtree. Roughly speaking, bandedness means that for each n, when levels in the original tree are considered, the n-th subtree level is completed before the $(n + 1)$-st subtree level is begun.

In Section 2, we present the terminology and elementary lemmas concerning trees, as well as the definitions of the tree-like spaces of Hagler and Schechtman. We analyze the spaces in Sections 3 and 4, respectively.

Our notation is standard in Banach space theory, as may be found in [**5**]. If A is a subset of a Banach space, we denote the closed linear span of A by $[A]$. The greatest integer function is also denoted by $[\cdot]$. Standard results concerning perturbations of Schauder bases are used in several places.

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2. The standard tree is

$$\mathscr{T} = \{ (n, i):0 \leqq n < \infty, 0 \leqq i < 2^n\}.$$

The points (n, i) are called *nodes*, and we say (n, i) is on the *n-th level of* $\mathscr{T}$. We denote the level of a node t by lev t. We say that $(n+1, 2i)$ and $(n+1, 2i+1)$ are the *successors* of (n, i). A *segment* is a finite set $S = \{t_1, t_2, \dots, t_k\}$ of nodes such that for each j, t_{j+1} is a successor of t_j. If $\mathrm{lev}(t_1) = m$ and $\mathrm{lev}(t_k) = n$, we say the segment $\{t_1, \dots, t_k\}$ is an $m - n$ *segment*. A family of segments $\{S_1, \dots, S_r\}$ is *admissible* if the segments are mutually disjoint and there exist integers m and n such that each S_i is an $m - n$ segment. $\mathscr{T}$ is partially ordered by the relation $<$ defined by $t_1 < t_2$ if and only if $t_1 \neq t_2$ and there is a segment with first element t_1 and last element t_2. If $t_2 \geqq t_1$, we say t_2 is a *follower* of t_1. A sequence of nodes $\{t_i\}$ is *strongly incomparable* provided $i \neq j$ implies t_i and t_j are not comparable and no more than two of the t_i are contained in the segments of any admissible family. An *n-branch* is a totally ordered set $\{ (m, l_m) \}_{m=n}^{\infty}$, and a *branch* is a set which is an n-branch for some n.

A *tree* is a partially ordered set $\mathscr{S}$ which is order isomorphic to $\mathscr{T}$. If $\mathscr{S}_1$ and $\mathscr{S}_2$ are trees with $\mathscr{S}_1 \subset \mathscr{S}_2$, we say that $\mathscr{S}_1$ is a *subtree* of $\mathscr{S}_2$. If $\mathscr{S}$ is a tree and $\psi:\mathscr{S} \to \mathscr{T}$ is an order isomorphism, we may use ψ to carry the above terminology from $\mathscr{T}$ to $\mathscr{S}$. In particular, for $s \in \mathscr{S}$, we define

$$\mathrm{lev}_{\mathscr{S}}(s) = \mathrm{lev}(\psi(s)).$$

If $\mathscr{S} \subset \mathscr{T}$ is a subtree of $\mathscr{T}$ and S is a segment of $\mathscr{T}$, we say S is *compatible with* $\mathscr{S}$ if there exist $s_1, s_2 \in \mathscr{S}$ such that $s_1 \leqq t \leqq s_2$ for all $t \in S$.

For ease of referral, we isolate the next notions in

Definition 1. Let $\{m_i\}$, $\{n_i\}$ be sequences of natural numbers such that $m_i \leqq n_i < m_{i+1}$ for all i. We say the subtree $\mathscr{S} \subset \mathscr{T}$ is *banded by* $\{m_i\}$, $\{n_i\}$ (or *banded*) if

1. $\mathrm{lev}_{\mathscr{S}}(t) = i$ implies $m_i \leqq \mathrm{lev}(t) \leqq n_i$,

2. $\mathrm{lev}_{\mathscr{S}}(t) = i$ implies there is a unique $m_i - n_i$ segment S_t of $\mathscr{T}$ which contains t and is compatible with $\mathscr{S}$, and

3. $\mathrm{lev}_{\mathscr{S}}(t) = i$ implies there exist precisely two $n_i - m_{i+1}$ segments S_j, which are compatible with $\mathscr{S}$ and such that $s \in S_j$ implies $t \leqq s$.

We shall omit the proofs of the following propositions. Proposition 4 is a strengthened version of Proposition 5 of **[1]**.

PROPOSITION 2. *If $\mathscr{S}$ is a tree and A is a subset of $\mathscr{S}$, then there exists a subtree $\mathscr{S}_1 \subset \mathscr{S}$ such that either $\mathscr{S}_1 \subset A$ or $\mathscr{S}_1 \subset \tilde{A}$.*

PROPOSITION 3. *Each subtree of $\mathscr{T}$ contains a banded subtree.*

PROPOSITION 4. *Let f be bounded real valued function on a tree. Then for any $\epsilon > 0$, there exists a subtree $\mathscr{S}$ such that*

a. *for any branch B of $\mathscr{S}$*

$$\lim_{\substack{t\to\infty \\ t\in B}} f(t) = L_B \text{ exists, and}$$

b. *if, for each* $t \in \mathcal{S}$, B_t *is a branch of* $\mathcal{S}$ *containing* t, *then*

$$\sum_{t\in\mathcal{S}} |F(t) - L_{B_t}| < \epsilon.$$

Let L denote the space of finitely nonzero functions on $\mathcal{T}$. The unit vectors are

$$x_t(s) = \begin{cases} 1 & s = t \\ 0 & s \neq t, \end{cases}$$

and we denote the sequence of biorthogonal functionals by $\{x_t^*\}$. We shall use the projections and functionals on L, or any completion of L, defined by the following formulas. In these, N is a natural number, S is either a segment or a branch, and t is a node.

$$\langle S^*, x\rangle = \sum_{t\in S} \langle x_t^*, x\rangle,$$

$$P_S x = \sum_{t\in S} \langle x_t^*, x\rangle x_t,$$

$$P_t x = \sum_{s\geqq t} \langle x_t^*, x\rangle x_t,$$

$$P_N = \sum_{\substack{t \\ \operatorname{lev}(t)\leqq N}} \langle x_t^*, x\rangle x_t, \quad \text{and}$$

$$Q_N = \sum_{\operatorname{lev}(t)\geqq N} P_t = I - P_{N-1}.$$

The Hagler tree space, HT, is the completion of L with respect to the norm

$$\|x\| = \sup \sum_{i=1}^{r} |\langle S_i^*, x\rangle|,$$

where the supremum is taken over all r and all admissible families $\{S_1, \ldots, S_r\}$. The unit vectors, in the order $x_{0,0}, x_{1,0}, x_{1,1}, x_{2,0}, \ldots$, are a Schauder basis for HT. We shall discuss this space in Section 3.

The spaces ST_p were constructed by Schechtman after an analysis of several tree spaces. For $\lambda > 1$, define a sequence of norms on L by

$$\|x\|_0 = \|x\|_{l_1},$$

$$\|x\|_m = \inf\left\{\|x_0\|_{m-1} + \lambda \sum_{k=1}^{K} \max_{0 \leqq i < 2^k} \|P_{k,i}x_k\|_{m-1}\right\}$$

where the infinum is taken over all K and all sequences $x_0, \ldots, x_K$ in L such that

$$\sum_{k=0}^{K} x_k = x \quad \text{and} \quad Q_k x_k = x_k \quad \text{for } k = 0, \ldots, K.$$

Let

$$\|x\| = \lim_{m\to\infty} \|x\|_m,$$

and denote by Y_m and Y the completions of L with respect to the norms $\| \ \|_m$ and $\| \ \|$, respectively. The norms dual to these are

$$|x|_0 = \|x\|_{c_0}$$

$$|x|_m = \max\left\{|x|_{m-1}, \lambda^{-1} \max_{1 \leqq k < \infty} \sum_{i=0}^{2^k-1} |P_{k,i}x|_{m-1}\right\},$$

and

$$|x| = \lim_{m\to\infty} |x|_m.$$

We shall denote by Z_m and Z the completions of L with respect to these norms.

The space ST_∞ is then the completion of L with respect to

$$\|\Sigma\, a_{n,i}x_{n,i}\| = \|\Sigma |a_{n,i}|^2 x_{n,i}\|_Y^{1/2}.$$

To define ST_p for $1 \leqq p < \infty$, let $\{x_t\}$ be the unit vector basis in ST_∞, and let $\{x_t^*\}$ be the biorthogonal sequence in ST_∞^*. Take $ST_1 = ST_\infty^*$, and for $1 < p < \infty$, let ST_p be the completion of L under the norm

$$\|\Sigma\, a_{n,i}x_{n,i}\| = \|\Sigma |a_{n,i}|^p x_{n,i}^*\|_{ST_1}^{1/p}.$$

3. In this section, we prove

THEOREM 5. *Let $U:HT \rightarrow HT$ be a bounded linear operator. Then there exists a subspace $X \subset HT$ such that X is isomorphic to HT, $U|X$ (or $(I - U)|X$) is an isomorphism, and UX (or $(I - U)X$) is complemented in HT.*

We prepare for the proof of this theorem with several propositions.

PROPOSITION 6. *Let $\mathcal{S}$ be a banded subtree of $\mathcal{T}$, and let*

$$X = [\{x_s : s \in \mathcal{S}\}].$$

Then X is isomorphic to HT and complemented in HT.

Proof. Let $\mathcal{S}$ be banded by $\{m_i\}$ and $\{n_i\}$, let $\phi:\mathcal{S} \to \mathcal{T}$ be an order isomorphism, and for each $t = (i, j) \in \mathcal{T}$, let S_t be the unique $m_i - n_i$ segment of $\mathcal{T}$ containing $\phi^{-1}(t)$, and compatible with $\mathcal{S}$.

If $\{a_t\}$ is a finite set of scalars, and $x = \sum a_t x_t$, let $\{S_1, \ldots, S_r\}$ be an admissible family such that

$$\|x\| = \sum_{i=1}^{r} |\langle S_i^*, x\rangle|.$$

Since $\{S_1, \ldots, S_r\}$ is admissible, there exist p, q such that each S_i is a $p - q$ segment. If S_i' is the unique $m_p - n_q$ segment of $\mathcal{T}$ which contains all of the $\phi^{-1}(t)$ for $t \in S_i$ and is compatible with $\mathcal{S}$, then $\{S_i'\}_{i=1}^r$ is an admissible family, and

$$\sum_{i=1}^{r} |\langle S_i'^*, \sum a_t x_{\phi^{-1}(t)}\rangle| = \sum_{i=1}^{r} |\langle S_i^*, x\rangle| = \|x\|.$$

Hence

$$\|\sum a_t x_t\| \leqq \|\sum a_t x_{\phi^{-1}(t)}\|.$$

For the reverse inequality, let $S_1, \ldots, S_r$ be $p - q$ segments with

$$\|\sum a_t x_{\phi^{-1}(t)}\| = \sum_{i=1}^{r} |\langle S_i^*, \sum a_t x_{\phi^{-1}(t)}\rangle|.$$

Since $\mathcal{S}$ is banded, we may assume there exist i and j such that $m_i \leqq p \leqq n_i$ and $m_j \leqq q \leqq n_j$, and with

$$y = \sum a_t x_{\phi^{-1}(t)},$$

we have

$$\sum_{i=1}^{r} |\langle S_i^*, y\rangle| = \sum_{i=1}^{r} |\langle S_i^*, P_{n_i} y + (P_{m_j} - P_{n_i})y + (I - P_{m_j})y\rangle| \leqq 3\, \|\sum a_t x_t\|.$$

It follows that the basic sequence $\{x_s\}_{s \in \mathcal{S}}$ is equivalent to $\{x_t\}$, and hence, that X is isomorphic to ST.

For each $t = \phi^{-1}(n, i) \in \mathcal{S}$, let S_t be the unique $(n_{i-1} + 1) - n_i$ segment containing t and compatible with $\mathcal{S}$. Define

$$Px = \sum_{t \in \mathcal{S}} \langle S_t^*, x \rangle \, x_t.$$

It is apparent that P is a projection onto X and that $\|P\| \leqq 2$.

PROPOSITION 7. *Let $U:HT \to HT$ be a bounded linear operator, $\epsilon > 0$, N an integer, $\mathcal{S} \subset \mathcal{T}$ a subtree and $t_0 \in \mathcal{S}$. Then there exists $t_1 \in \mathcal{S}$, $t_1 > t_0$, such that*

$$\|P_N U x_{t_1}\| < \epsilon.$$

Proof. If no such t_1 exists, then for any follower $t \in \mathcal{S}$ of t_0, there exists t', $\text{lev}(t') \leqq N$ with

(4) $\quad |\langle x_{t'}^*, P_N U_{x_t} \rangle| \geqq \epsilon/K,$

where $K = 2^{N+1} - 1$. Thus, for any L and any collection $\{t_l\}_{l=1}^L$ of followers in $\mathcal{S}$ of t_0, $[L/K]$ of the t_l satisfy (4) for the same node t'. Hence there is a choice of signs $\{\theta_i = \pm 1\}$ such that

$$(5) \quad \left\| \sum_{l=1}^{L} P_N U(\theta_l x_{t_l}) \right\| \geqq \langle x_{t'}^*, \sum_{l=1}^{L} U(\theta_l x_{t_l}) \rangle \geqq \frac{\epsilon}{K} \left[\frac{L}{K} \right].$$

If, however, the $\{t_l\}$ are chosen to be strongly noncomparable, we have

$$\left\| \sum_{l=1}^{L} P_N U(\theta_l x_{t_l}) \right\| \leqq \|U\| \, \|\Sigma \, \theta_l x_{t_l}\| \leqq 2 \, \|U\|.$$

Since L is arbitrary, (5) is contradicted.

PROPOSITION 8. *Let $U:HT \to HT$ be a bounded linear operator, $\epsilon > 0$, N an integer, $\mathcal{S}$ a subtree of $\mathcal{T}$, and $t_0, \ldots, t_k$ mutually noncomparable nodes of $\mathcal{S}$. Then there exists $t > t_0$, $t \in \mathcal{S}$, $M \in \mathbf{N}$, $N_1 \geqq N$, and $N_1 - (M + 1)$ segments S_i, $i = 1, \ldots, k$, of $\mathcal{T}$ having the properties*:

a. $\|P_N U x_t\| < \epsilon$,

b. $\|(I - P_M) U x_t\| < \epsilon$,

c. *For each i, there exists $t_i' \in \mathcal{S}$ such that $t_i \leqq s < t_i'$ for all $s \in S_i$,*

d. *For each i, $|\langle S^*, Ux_t \rangle| < \epsilon$ for each segment $S \supset S_i$.*

Proof. Let K satisfy

$$2^{-K}\|U\| < \epsilon/3,$$

and let

$$N_1 \geqq \max(N, \text{lev}(t_i))$$

be such that for each $i = 1, \ldots, k$ there are 2^K branches of $\mathcal{S}$ which contains t_i and pass through distinct nodes in the N_1-th level of $\mathcal{T}$. Then there exists $t > t_0$ such that $t \in \mathcal{S}$ and

$$\|P_{N_1}Ux_t\| < \epsilon/3.$$

Hence a. is satisfied. To satisfy b., choose $M > N_1$ such that

$$\|(I - P_M)Ux_t\| < \epsilon/3.$$

Now for $i = 1, \ldots, K$, let $S_i^1, \ldots, S_i^{2^K}$ be disjoint $N_1 - (M + 1)$ segments satisfying c. For fixed i, if no S_i^j satisfies

$$|\langle S_i^{j*}, Ux_t\rangle| < \epsilon/3,$$

it follows that

$$\frac{\epsilon}{3}2^K \leqq \sum_{j=1}^{2^K} |\langle S_i^{j*}, Ux_t\rangle|$$

$$\leqq \|Ux_t\| \leqq \|U\| < \frac{\epsilon}{3}2^K,$$

a contradiction. Hence for each i, there exists $S_i = S_i^j$ such that

$$|\langle S_i^*, Ux_t\rangle| < \epsilon/3.$$

Now, if $S \supset S_i$,

$$|\langle S^*, Ux_t\rangle| \leqq |\langle S^*, P_{N_1-1}Ux_t\rangle| + |\langle S_i^*, Ux_t\rangle| + |\langle S^*, (I - P_{M+1})Ux_t\rangle| < \epsilon.$$

We are now ready for the

Proof of Theorem 5. Let $0 < \gamma < 1/2$. Using standard perturbation arguments, Propositions 2, 3, 4, 7, 8, and the arguments of [**1**], we may assume the existence of a subtree $\mathscr{S} = \{t(n, i)\} \subset \mathscr{T}$ banded by sequences $\{m_i\}$ and $\{n_i\}$ such that for each $t \in \mathscr{S}$ and each $n_i - m_j$ segment S of $\mathscr{T}$ which is compatible with $\mathscr{S}$, we have

$$\langle S^*, Vx_t\rangle = \begin{cases} \gamma_t & t \in S \\ 0 & t \notin S, \end{cases}$$

where $\gamma \leqq \gamma_t \leqq \|V\|$, where V is either U or $(I - U)$. We shall assume that $V = U$, and show that $U(HT)$ contains a complemented isomorph of HT. Furthermore, we may assume that along each branch B of $\mathscr{S}$,

$$\lim_{\substack{t\to\infty \\ t\in B}} \gamma_t = \gamma_B \text{ exists}$$

and that if $\gamma_t' = \gamma_{B_t}$ for some branch containing t, then

$$\sum_{t\in\mathscr{S}} |\gamma_t - \gamma_t'| < \frac{\gamma}{6}.$$

Let $X = [\{x_t\}_{t \in \mathscr{S}}]$. By Proposition 6, X is isomorphic to HT, and we shall now show that $\{Ux_t\}_{t \in \mathscr{S}}$ is a basic sequence equivalent to $\{x_t\}_{t \in \mathscr{S}}$. It will follow that $U|X$ is an isomorphism.

Since U is bounded, if $\{a_{n,i}\}$ is a finite set of scalars,

$$\|\sum a_{n,i}\, Ux_{t(n,i)}\| \leqq \|U\|\, \|\sum a_{n,i}x_{t(n,i)}\|.$$

For the reverse inequality, let

$$x = \sum a_{n,i}x_{t(n,i)},$$

and notice that there exist disjoint $m_p - n_q$ segments $S_1, \ldots, S_k$ of $\mathscr{T}$ and branches $B_j \supset S_j$ such that

$$\|x\| \leqq 3 \sum_{j=1}^{k} |\langle S_j^*, x\rangle|$$

$$\leqq \frac{3}{\gamma} \sum_{j=1}^{k} \gamma_{B_j} \left| \sum_{t(n,i) \in S_j} a_{n,i} \right|$$

$$= \frac{3}{\gamma} \langle f, x\rangle$$

where $f \in HT^*$ is defined by

$$f = \sum_{j=1}^{k} \gamma_{B_j} \operatorname{sgn}\langle S_j^*, x\rangle S_j^*.$$

Let $\epsilon_j = \operatorname{sgn}\langle S_j^*, x\rangle$, and let

$$\tilde{\gamma}_s = \begin{cases} \gamma_t & t \in \mathscr{S} \cap B_j \\ \gamma_{B_j} & t \in B_j \setminus \mathscr{S}, \end{cases}$$

and define $g \in HT^*$ by

$$g = \sum_{j=1}^{k} \epsilon_j \sum_{s \in S_j} \tilde{\gamma}_s x_s^*.$$

Then

$$\|g - f\| \leqq \sum_{j=1}^{k} \sum_{t \in S_j \cap \mathscr{S}} |\gamma_t - \gamma_{B_j}| < \frac{\gamma}{6},$$

so for any $y \in HT$,

$$\langle f, y\rangle \leqq \langle g, y\rangle + \frac{\gamma}{6} \|y\|.$$

In particular,

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$$\|x\| \leqq \frac{3}{\gamma}\langle f, x\rangle \leqq \frac{3}{\gamma}\left[\langle g, x\rangle + \frac{\gamma}{6}\|x\|\right]$$

$$\leqq \frac{3}{\gamma}\sum_{j=1}^{k} \epsilon_j \sum_{t(n,i)\in S_j} \gamma_{n,i}a_{n,i} + \frac{1}{2}\|x\|,$$

so

$$\|x\| \leqq \frac{6}{\gamma}\sum_{j=1}^{k} \epsilon_j \sum_{t(n,\,i)\in S_j} \gamma_{n,i}a_{n,i}$$

$$\leqq \frac{6}{\gamma}\sum_{j=1}^{k} |\langle S_j^*, Ux\rangle| \leqq \frac{6}{\gamma}\|Ux\|.$$

Thus, $U|X$ is an isomorphism, and to see that UX is complemented, observe first that the preceding argument may be used to show that the multiplier operator M on X defined by $Mx_t = \gamma_t x_t$ is bounded and invertible. Denoting by P the projection onto X constructed in the proof of Proposition 6, we see that UX is complemented by $Q = (U|X)\, M^{-1}P$.

4. This section is devoted to proving

THEOREM 9. *If X is one of the Schechtman tree spaces Y, Z or ST_p, $1 \leqq p \leqq \infty$, and U is a bounded linear operator on X, then there is a subspace $W \subset X$ such that $U|W$ (or $(I - U)|W$) is an isomorphism and UW (or $(I - U)W$) is complemented in X.*

In [**6**], Schechtman proved that $\{x_{n,i}\}$ is a 1-unconditional basis for Y_m and for Y, and that c_0 does not embed in Y. From this we easily obtain

PROPOSITION 10. 1. *$\{x_{n,i}\}$ is a boundedly complete basis for Y.*
2. *$Z^* = Y$ and $\{x_{n,i}\}$ is a shrinking basis for Z.*
3. *$\{x_{n,i}\}$ is a 1-unconditional basis for Z_m and for Z.*
4. *$\{x_{n,i}\}$ converges weakly to zero in Z.*

PROPOSITION 11. *Let $\mathcal{S} = \{t(n, i)\}$ be a banded subtree of $\mathcal{T}$. Then $[\{x_t\}_{t\in S}]$ in Z is isometric to Z and $[\{x_t\}_{t\in S}]$ in Y is isometric to Y.*

Proof. We first consider the unit vectors in Z and show that for any finite scalar sequence $\{a_{n,i}\}$,

$$|\sum a_{n,i}x_{n,i}| = |\sum a_{n,i}x_{t(n,i)}|.$$

The proof is by induction and passage to the limit. Since $|\cdot|_0 = \|\cdot\|_{c_0}$, we have that

$$|\sum a_{n,i}x_{n,i}|_0 = |\sum a_{n,i}x_{t(n,i)}|_0$$

for any banded subtree $\mathscr{S} = \{t(n, i)\}$ and any sequence of scalars $\{a_{n,i}\}$. Assume that for any banded subtree $\mathscr{S} = \{t(n, i)\}$,

$$|\sum a_{n,i}x_{n,i}|_{m-1} = |\sum a_{n,i}x_{t(n,i)}|_{m-1}$$

for all scalar sequences $\{a_{n,i}\}$. Now let $\mathscr{S}$ be banded by $\{m_i\}$, $\{n_i\}$, and let

$$x = \sum a_{n,i}x_{t(n,i)}.$$

We have

$$|x|_m \geqq \max\left\{|x|_{m-1}, \lambda^{-1} \max_{m_k} \sum_{i=0}^{2^{m_k}-1} |P_{m_k,i}x|_{m-1}\right\}$$

$$= \max\left\{|\sum a_{n,i}x_{n,i}|_{m-1}, \lambda^{-1} \max_k \sum_{i=0}^{2^k-1} |P_{k,i}(\sum a_{n,i}x_{n,i})\,|_{m-1}\right\}$$

$$= |\sum a_{n,i}x_{n,i}|_{m'}$$

by the induction hypothesis. For the other inequality, we consider two cases:

(1) $\quad |x|_m = |\sum a_{n,i}x_{t(n,i)}|_{m-1}$ and

(2) $\quad |x|_m = \lambda^{-1} \max\limits_{1\leqq k<\infty} \sum\limits_{i=0}^{2^k-1} |P_{k,i}x|_{m-1}.$

In the first case, the induction hypothesis implies that

$$|x|_m = |x|_{m-1} = |\sum a_{n,i}x_{n,i}|_{m-1} \leqq |\sum a_{n,i}x_{n,i}|_m.$$

In the second case, there exists K such that

$$|x|_m = \lambda^{-1} \sum_{i=0}^{2^K-1} |P_{K,i}x|_{m-1},$$

and let j be the largest integer such that $m_j \leqq K$. If $m_j \leqq K < n_j$, then there exists l such that

$$P_{K,i}x = P_{K,i}P_{m_j,l}x,$$

and by the 1-unconditionality in $|\cdot|_{m-1}$,

$$|P_{K,i}x|_{m-1} \leqq |P_{m_j,l}x|_{m-1}.$$

Hence

$$|x|_m = \lambda^{-1} \sum_{i=0}^{2^K-1} |P_{K,i}x|_{m-1} \leqq \lambda^{-1} \sum_l |P_{m_j,l}x|_{m-1}$$

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$$= \lambda^{-1} \sum_{l} |P_{j,l} \sum a_{n,i}x_{n,i}|_{m-1} \leqq |\sum a_{n,i}x_{n,i}|_m.$$

On the other hand, if $n_j \leqq K < m_{j+1}$, then for each i, either there exist l_1 and l_2 such that

$$P_{K,i}x = P_{m_{j+1},l_1}x + P_{m_{j+1},l_2}x$$

or there exists l such that

$$P_{K,l}x = P_{m_{j+1},l}x.$$

In either case, using the triangle inequality, we have

$$\begin{aligned} |x|_m &= \lambda^{-1} \sum_{i=0}^{2^K-1} |P_{K,i}x|_{m-1} \\ &\leqq \lambda^{-1} \sum_{l} |P_{m_{j+1},l}x|_{m-1} \\ &= \lambda^{-1} \sum_{l} |P_{j+1,l}(\sum a_{n,i}x_{n,i})| \\ &\leqq |\sum a_{n,i}x_{n,i}|_m. \end{aligned}$$

The equivalence of $\{x_t\}_{t\in\mathcal{T}}$ and $\{x_t\}_{t\in\mathcal{S}}$ in the space Y follows from the equivalence in Z and the fact that $Z^* = Y$.

Proof of Theorem 9. As in the proof of Theorem 5, the argument may be carried out for one of U or $(I - U)$. We shall call that operator U, and show that UX contains a complemented isomorph of X.

If U is a bounded operator on Z, $\{Ux_{n,i}\}$ converges weakly to zero since $\{x_{n,i}\}$ converges weakly to zero, and we may assume there exists a banded subtree $\mathcal{S} = \{t(n, i)\}$ such that $t \in \mathcal{S}$ implies

$$|\langle x_t^*, Ux_t\rangle| \geqq 1/2,$$

and that the Ux_t are disjointly supported. With $W = [\{x_t\}_{t\in\mathcal{S}}]$, W is isometric to Z, and the unconditionality of $\{x_{n,i}\}$ implies that $U|W$ is an isomorphism. Again by the unconditionality, the operator M defined by

$$Mx_t = \begin{cases} \langle x_t^*, Ux_t\rangle^{-1}x_t & t \in \mathcal{S} \\ 0 & t \notin \mathcal{S} \end{cases}$$

is bounded, and UW is complemented by the projection UM.

In the case of the space Y, the unit vectors do not tend weakly to zero, and if U is a bounded linear operator on Y, in order to obtain a sequence $\{f_{n,i}\}$ for which $\{Uf_{n,i}\}$ is disjointly supported, we use differences of unit vectors. To this end, select a subtree $\mathcal{S} \subset \mathcal{T}$ such that $t \in \mathcal{S}$ implies

$$\langle x_t^*, Ux_t\rangle \geqq 1/2,$$

and inductively choose sequences $\{m_i\}$, $\{n_j\}$ and nodes $t^1(n, i)$, $t^2(n, i)$ of $\mathcal{S}$ such that

a. $t^1(n, i) < t^2(n, i)$

b. $t^2(n, i) < t^1(n + 1, 2i)$ and $t^2(n, i) < t^1(n + 1, 2i + 1)$

c. $\{t^l(n, i)\}$ is banded by $\{m_j\}$ and $\{n_j\}$, for $l = 1, 2$

d. $\langle x^*_{t^2(n,i)}, Ux_{t^1(n,i)}\rangle = 0$, and

e. with $f_{n,i} = x_{t^2(n,i)} - x_{t^1(n,i)}$, the $Uf_{n,i}$ are disjointly supported.

Now, let $W = [\{f_{n,i}\}]$. Then

$$\begin{aligned}\|\sum a_{n,i}x_{n,i}\| &= \|\sum a_{n,i}x_{t^2(n,i)}\| \\ &\leqq \|\sum a_{n,i}f_{n,i}\| \quad \text{by d,} \\ &\leqq 2\|\sum a_{n,i}x_{n,i}\|,\end{aligned}$$

so W is isomorphic to Y. Furthermore, since

$$\langle x_{t^*}, Ux_t\rangle \geqq 1/2,$$

by the unconditionality of $\{x_{n,i}\}$ and e,

$$\begin{aligned}\|\sum a_{n,i}f_{n,i}\| &\leqq 2\|\sum a_{n,i}x_{n,i}\| \\ &= 2\|\sum a_{n,i}x_{t^2(n,i)}\| \\ &\leqq 4\|\sum a_{n,i}Uf_{n,i}\| \\ &\leqq 4\|U\|\,\|\sum a_{n,i}f_{n,i}\|.\end{aligned}$$

It is easily seen that UW is complemented in Y.

As for the spaces ST_p, $1 \leqq p \leqq \infty$, it follows from Proposition 11 and the definitions of the norms that whenever $\mathcal{S}$ is a bounded subtree of $\mathcal{T}$, $\{x_t\}_{t\in\mathcal{T}}$ is isometrically equivalent to $\{x_t\}_{t\in\mathcal{S}}$. Since these spaces are reflexive, the unit vector basis is shrinking, and thus converges weakly to zero. Thus, the argument used for the space Z also proves the theorem for ST_p, $1 \leqq p \leqq \infty$.

5. A consequence of Theorems 5 and 9 is that if X is either the Hagler tree space or one of the Schechtman tree spaces, and W is complemented in X, then W contains a complemented isomorph of X. Since these spaces are isomorphic to their Cartesian squares, the arguments of [**2**] show

COROLLARY 10. *If $X = HT, Z, Y$, or ST_p, $1 \leqq p \leqq \infty$, $X = W \oplus V$, and $W \approx W \oplus W$ or $V \approx V \oplus V$, then either $W \approx X$ or $V \approx X$.*

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