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MODULES OVER INFINITE NILPOTENT GROUPS

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To Laci Kovács on his 65th birthday

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Abstract

The paper discusses modules over free nilpotent groups and demonstrates that faithful modules are more restricted than might appear at first glance. Some discussion is also made of applying the techniques more generally.

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The aim of this note is to discuss the nature of modules over finitely generated nilpotent groups. More particularly, we shall state some results for finitely generated faithful modules over free nilpotent groups and then briefly conjecture how these ideas might apply more generally. The techniques involve the application of results of Brookes [3] and of Brookes and the author [4, 5, 6].

In contrast to the situation for modules over free abelian groups, it is not clear how to write down a wide range of faithful modules over an infinite nilpotent group. In order to avoid the complicated detail that comes with greater generality, we will restrict our consideration to faithful modules over free nilpotent groups of finite rank. Let H denote a free nilpotent group of finite rank with centre $\zeta(H)$ and let k be a field. How can we construct a faithful module for kH?

We clearly have free modules as well as, possibly, other torsion-free modules. We also have some straightforward quotients of these. If P is a prime ideal of $k\zeta(H)$ satisfying $(P + 1) \cap \zeta(H) = 1$ then, by a theorem of Zalesskii (see [9, Corollary 11.4.6]), $kH/P.kH$ is a faithful module for H. Alternatively, we can regard such modules as being induced from modules over the centre of H (where it

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is relatively easy to construct faithful modules). The latter also suggests the more general possibility of inducing modules from subgroups of infinite index.

Another, less obvious, way to construct a faithful module is to start with a normal subgroup H_1 of H so that H/H_1 is infinite cyclic with $H = \langle H_1, h_0\rangle$. Let M be a finitely generated torsion-free module for kH_1. Such M, if chosen correctly, can have a k-automorphism ρ so that, for $m \in M$ and $\alpha \in kH_1$, we have $(m\alpha)\rho = (m\rho)h_0^{-1}\alpha h_0$. Then we can define a kH action on M via $m.(h_1h_0^l) = (m.h_1)\rho^l$ for all $m \in M$, $h_1 \in H_1$ and integers l. Careful choice of the module M and automorphism ρ will ensure that the resulting kH-module is faithful.

Our claim is that all faithful modules for free nilpotent groups can, in some sense, be built by combining the methods we have described. We need a definition to make this precise.

DEFINITION. We shall say that the module M is *reduced* if there is a prime ideal P of $k\zeta(H)$ so that M is annihilated by P and is torsion-free as $k\zeta(H)/P$-module.

It is easily verified that every finitely generated kH-module has a finite series in which each section is reduced.

Our statement when the free nilpotent group H is of class 2 is simpler as well as easier to prove.

THEOREM 1. *Let H be a free nilpotent group of class 2 and let M be a cyclic reduced kH-module which is faithful for H. Let P denote the annihilator of M in $k\zeta(H)$. Then either*:

(1) *$M \cong kH/P.kH$; or*

(2) *there is a normal subgroup H_1 of H so that H/H_1 is infinite cyclic and M is torsion-free as kH_1/PkH_1-module; further M has a finitely generated kH_1-submodule N so that M/N is $k\zeta(H)/P$-torsion.*

When H is not of class 2, we must be more careful with the statement of the result, although the general tenor remains:

THEOREM 2. *Let H be a free nilpotent group of finite rank with derived group H' and let M be a finitely generated reduced kH-module which is faithful for H. Let P denote the annihilator of M in $k\zeta(H)$. Then either*

(1) *M has a non-zero submodule of the form $L \otimes_{kH'} kH$ for some kH'-submodule L of M; or*

(2) *there is a normal subgroup H_1 of H so that H/H_1 is infinite cyclic and M is torsion-free as kH_1/PkH_1-module; further M has a finitely generated kH_1-submodule N so that M/N is kH'/PkH'-torsion.*

We shall prove Theorem 1 in Section 1, Theorem 2 in Section 2 and discuss generalisations to non-free groups in the final Section 3.

1. Free nilpotent groups of class 2

We shall prove Theorem 1 by using available results ([4, 5]) on crossed products of division rings D with abelian groups A. Such a crossed product is an A-graded D-algebra in which each graded component has D-dimension 1. We shall identify D with the homogeneous component corresponding to the identity of A. For each $a \in A$, we can choose a unit $\bar{a}$ in the homogeneous component corresponding to a; and we can choose $\bar{1} = 1$.

A DA-module M is said to be of *dimension* l if l is greatest such that M embeds a copy of DB for some subgroup B of A having rank l. We refer to [5] for some basic properties of this dimension (including a proof that it co-incides with Gelfand–Kirillov dimension).

The main result (Theorem 4.4) of [5] is a description of a geometrical invariant $\Delta(M) \subseteq \operatorname{Hom}(A, \mathbb{R})$ associated with a finitely generated DA-module M. The main facts we require here are that, if m denotes the dimension of M, then $\Delta(M)$ lies in a finite union of polyhedral cones of dimension no more than m and that there is a canonical subset $\Delta^*(M)$ which is a non-empty finite union of rational polyhedral cones of dimension exactly m so that the difference between $\Delta(M)$ and $\Delta^*(M)$ can be covered by a finite union of cones of dimension less than m. A convex polyhedron of dimension m spans a subspace of $\operatorname{Hom}(A, \mathbb{R})$ having dimension m and this subspace can be identified as

$$B^\circ = \{\chi \in \operatorname{Hom}(A, \mathbb{R}) : \chi(b) = 0 \text{ for all } b \in B\}$$

for some isolated subgroup B of A. We call an isolated subgroup B which arises in this way from a convex polyhedral constituent of $\Delta^*(M)$, a *carrier space* subgroup, or *CS*-subgroup, corresponding to M.

A major tool in this section will be Brookes [3, Theorem 3], recast in a more technical statement suited to our purposes.

THEOREM 3 (Brookes). *Let N be a finitely generated DA-module and let B be a CS-subgroup corresponding to N. Then DB has a non-zero module of finite D-dimension. If also D is central in DB, then DB_1 is commutative for some subgroup B_1 of finite index in B.*

The proof is essentially already present in the original proof by Brookes. We need observe only two extra facts. The centrality of D, assumed in the statement of [3,

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Theorem 3], is used only in the last paragraph of the proof, to show that the last sentence of our statement follows from the second sentence. Also, if B is a CS-subgroup then B° is spanned by one of the polyhedra of $\Delta^*(N)$. Such a polyhedron contains internal points χ such that the rank of $A/\ker\chi$ is the dimension of the polyhedron and so $\ker\chi = B^\circ$. Thus Brookes' proof applies to CS-subgroups.

We now consider some modules over crossed products which arise from the study of modules over free nilpotent groups.

PROPOSITION 1. *Let D be a division ring and A a free abelian group of finite rank n. Let DA be a crossed product of D with A and let N be a finitely generated DA-module. Suppose that every non-zero submodule of N has dimension at least $n-1$. Then one of the following holds*:

(1) *N has a non-zero submodule which is torsion-free as DA-module*; *if N is cyclic then N is torsion-free*;

(2) *for some $C \leq A$ with A/C infinite cyclic, N is finitely generated and torsion-free as DC-module.*

PROOF. If N has dimension n, then N embeds a copy of DA and so has a torsion-free submodule. If also N is cyclic, then N must actually be a copy of DA.

If N has dimension $n-1$, then $\Delta(N)$ is of dimension $n-1$ and there is a line in $\mathrm{Hom}(A, \mathbb{R})$, having equation with rational coefficients, which meets $\Delta(N)$ only in the origin. This line can then be written as C° for some subgroup C of rank $n-1$ in A. By [5, Proposition 3.8], N is finitely generated as DC-module; when we regard N as C-module, we shall denote it by N_C. But then, if π_C denotes the natural restriction map $\mathrm{Hom}(A, \mathbb{R}) \to \mathrm{Hom}(C, \mathbb{R})$ we have, by [5, Proposition 3.7], $\pi_C(\Delta(N_A)) = \Delta(N_C)$ and so $\Delta(N_C)$ has dimension $n-1$. By [5, Theorem 4.4], N_C is not a torsion DC-module.

We claim that N_C is torsion-free. If not, then it contains a non-zero torsion submodule which itself contains a critical non-zero torsion submodule L, by [5, Proposition 2.5]. As L is a torsion DC-module and C has rank $n-1$, L must have dimension at most $n-2$. But then, by [5, Lemma 2.4], $L \cdot DA$ either has dimension $n-2$ or is of the form $L \otimes_{DC} DA$. The former case is disallowed by the hypotheses of the theorem. The latter case would imply that N_C contained an infinite direct sum of DC-modules and this would imply that N_C was not Noetherian. But we have shown that N_C is finitely generated and so Noetherian. This contradiction completes the proof that N is finitely generated and torsion-free as DC-module. □

PROOF OF THEOREM 1. We begin by translating the problem to one involving crossed products. Let K denote the field of fractions of $k\zeta(H)/P$ and let $\widehat{M}$ denote

$$\widehat{M} = M \otimes_{kH/PkH} K.$$

Observe that $\widehat{M}$ is a cyclic module for $(kH/PkH) \otimes_{k\zeta(H)/P} K$ and the latter has a natural structure as a crossed product of K with $A = H/\zeta(H)$; we shall write this ring as KA. It is easily seen that $\widehat{M} = MK$ and, because M is torsion-free over $k\zeta(H)/P$, the map $m \mapsto m \otimes 1$ is an embedding of M into $\widehat{M}$.

We can now apply Theorem 3 to the module $\widehat{M}$; observe that K is central in KA. It tells us that, if B is a CS-subgroup of A then KB_1 is commutative for some subgroup B_1 of finite index in B. Thus, if H_1 is the subgroup of H so that $H_1/\zeta(H) = B_1$, then H_1 is commutative. Because H is free, the centre of H equals the derived group and any abelian subgroup of H lies in an abelian normal subgroup which is a cyclic extension of the centre. Thus any subgroups B_1 of A with KB_1 commutative must be of rank at most 1. In particular, any CS-subgroup of $\widehat{M}$ must have rank at most 1 and so the corresponding convex polyhedron must have dimension at least $n - 1$. Hence $\widehat{M}$ has dimension at least $n - 1$.

Observe that every submodule of $\widehat{M}$ is of the form $\widehat{N} = N \otimes_{kH/PkH} K$ for some submodule N of M. If N is non-zero then N is also reduced with annihilator P. Thus N is faithful over $\zeta(H)$ and so also over H. We can repeat the previous arguments to show that, if $\widehat{N}$ is non-zero, then it has dimension at least $n - 1$.

By Proposition 1, $\widehat{M}$ is either KA or, for some B with A/B cyclic, is finitely generated and torsion-free as KB-module. In the former case, $\widehat{M}$ is certainly kH/PkH-torsion-free and hence so also is M. But M is cyclic and so $M \cong kH/PkH$.

In the second case let H_1 be the subgroup of H corresponding to B. We are assuming that $\widehat{M}$ is KB-torsion-free and we know that $KB = (kH_1/PkH_1) \otimes_{k\zeta(H)/P} K$. Thus $\widehat{M}$ will also be kH_1/PkH_1-torsion-free and hence M will also be kH_1/PkH_1-torsion-free.

We know that $\widehat{M}$ is finitely generated as KB-module, and since $\widehat{M} = M \otimes K$, there is no loss in assuming that these generators lie in M. Thus M will contain a finitely generated kH_1-submodule N with $N \otimes_{kH} KA = \widehat{M} = M \otimes_{kH} KA$. Thus M/N is $(k\zeta(H)/P)$-torsion. □

2. Other free nilpotent groups

We begin with a technical result on commutation in free nilpotent groups.

LEMMA 2. *Let H be a free nilpotent group and let K be a subgroup of H which does not lie in the derived group H' of H. Suppose that L is a subgroup of H which satisfies $[K, L] \le L'$. Then either L lies in the centre of H or L does not lie in H'.*

PROOF. Let F be a free group of the same rank as H and let $\pi : F \to H$ be the natural projection. Suppose that $L \subseteq \gamma_t(H)$ but $L \not\subseteq \gamma_{t+1}(H)$. Choose $U \in F$ so that $\pi(U) \in K \setminus H'$ and V so that $\pi(V) \in L \setminus \gamma_{t+1}(H)$. Because $L \subseteq \gamma_t(H)$ we will have that $L' \subseteq \gamma_{2t}(H)$ and so $\pi([U, V]) \in \gamma_{2t}(H)$. Thus $[U, V] \in \gamma_{\min(2t, c+1)}(F)$ where c is the class of H.

We now use [8, Corollary 5.12 (iii)] to show that either $t = 1$ or $[U, V] \in \gamma_{t+1}(F) \setminus \gamma_{t+2}(F)$. Suppose that $t \neq 1$. Then $\min(2t, c+1) \leq t+1$ and so $c \leq t$.

Thus $t = 1$ or $t = c$. That is, either L lies in $\gamma_c(H) = \zeta(H)$ or L does not lie in H'. □

The next result aims to use the results of Brookes and Brown described in [1] and Brookes and Groves in [6]. We recall that every module M has an *injective hull* $E(M)$ which has the property that M is an essential submodule of $E(M)$. That is, every non-zero submodule of $E(M)$ has non-zero intersection with M. Suppose that M is a kH-module (here H could be any group). Let ρ be an automorphism of H. Then we can define a new kH-module $M\rho$ with elements $m\rho$ in one-to-one correspondence with those of M and, for $m \in M$ and $h \in H$, $m\rho.h = m\rho(h)$. We say that ρ *stabilises* M if $E(M)$ and $E(M\rho) = E(M)\rho$ are kH-isomorphic. A further discussion of the necessary ideas can be found in [1].

PROPOSITION 3. *Suppose that H is a free nilpotent group and that M is a faithful reduced kH-module. Denote by P the annihilator of M in $k\zeta(H)$. If M has no non-zero submodule which is induced from the derived group H' then M is torsion-free as $kH'/P.kH'$-module.*

PROOF. Suppose that the result is false. Then we can suppose that M has a non-zero $kH'/P.kH'$-torsion submodule M_1 which is finitely generated as kH-module. Let $E(M_1)$ denote the injective hull of M_1 and let W be an indecomposable injective kH'-submodule of $E(M_1)$ which is minimal in the sense of [1]. We aim to show that $E(M_1)$ has a non-zero kH-submodule N which is induced from H'. Because a module is essential in its injective hull, it will follow that $N \cap M_1$ is non-zero. But, it then follows from (for example) the Lemma of [1] that $N \cap M_1$ contains a non-zero submodule induced from H'. This contradiction will then complete the proof.

It follows from [1, Proposition 1] that, for some isolated subgroup L of H' and some impervious indecomposable injective kL-submodule V, the module W is the injective hull of $V \otimes_{kL} kH'$. From the discussion preceding Theorem 2 of [1], we can assume that V is the injective hull of a finitely generated impervious kL-module V_1. We can now apply [1, Theorem 2] to deduce that

$$V_1.kH \cong V_1.kS \otimes_{kS} kH$$

where S is the stabiliser in $N_H(L)$ of V_1 and so of V. Observe that if $S \leq H'$ then $V_1.kH$ is also induced from H' and the proof is complete.

We can now apply [6, Theorem 3.1] to V_1 to show that the stabiliser in Aut L of V_1 acts finitely on L/L'. Thus S also acts finitely on L/L' and so some subgroup S_1 of finite index in S acts trivially on L/L'. That is, $[S_1, L] \leq L'$.

Now we can apply Lemma 2 to show that either $S_1 \leq H'$ or $L \leq \zeta(H) = \gamma_c(H)$. As H' is isolated and S_1 has finite index in S, the former case implies that $S \leq H'$. The latter case implies that W is induced over $\zeta(H)$ and so is torsion-free over kH'/PkH'. In either case, the contradiction completes the proof. □

We continue with a technical result which will enable us to replace the arguments used by Brookes in the proof of the third sentence of Theorem 3.

PROPOSITION 4. *Let DA be a crossed product with D a division ring and A a free abelian group of finite rank. Suppose that there is a homomorphism $v : D^* \to \mathbb{R}$ from the multiplicative group of D to the additive group of the real numbers.*

Suppose also that there are $x, y \in A$ so that, if we set $z = [\bar{x}, \bar{y}] \in D$ then $v(z) \neq 0$ and, for each $d \in D$ and $w \in \{\bar{x}, \bar{y}\}$, $v(d^w) = v(d)$.

Then DA has no non-zero module of finite D-dimension.

PROOF. Suppose, on the contrary, that DA satisfies the assumptions but has a non-zero module V of finite D-dimension. Let $v_1, \dots, v_m$ be a D-basis for V. Then, for $\alpha \in DA$, we can write

$$v_i\alpha = \sum_j v_j d_{ji}(\alpha) \quad \text{with } d_{ji} \in D$$

and so we obtain a function (not necessarily a homomorphism) $\sigma : DA \to GL_m(D)$ which sends α to the matrix with i, j entry equal to $d_{j,i}$. It is easily verified that

(1) if $\beta \in DA$ is invertible then $\sigma(\alpha\beta) = \sigma(\beta)\sigma(\alpha)^\beta$;
(2) if $d \in D$ then $\sigma(d)$ is the scalar matrix with entry d,

where $\sigma(\alpha)^\beta$ denotes the matrix obtained from $\sigma(\alpha)$ by replacing each entry d by $\beta^{-1}d\beta$.

We recall the Dieudonne determinant on $GL_m(D)$. This is a multiplicative function to the abelianisation D_{ab} of the multiplicative group of D, together with 0 (see [7, Chapter IV] for more detail). Let $\tau(\alpha)$ be the function $DA \to D_{ab}$ which sends α to the determinant of $\sigma(\alpha)$. Then we have

(1) if β is invertible then $\tau(\alpha\beta) = \tau(\alpha)^\beta\tau(\beta)$;
(2) if $d \in D$ then $\tau(d) = d^m$,

where $\tau(\alpha)^\beta$ denotes $(\beta D')^{-1}\tau(\alpha)(\beta D')$.

Observe also that the function v is zero on the kernel D' of the map from the multiplicative group of D to its abelianisation D_{ab}.

Set $\tau(\bar{x}) = \lambda$ and $\tau(\bar{y}) = \mu$, where $\lambda, \mu \in D_{ab}$. Then

$$\tau(\bar{y}\bar{x}) = \tau(\bar{y})^{\bar{x}}\tau(\bar{x}) = \mu^{\bar{x}}\lambda$$

and

$$\tau(\bar{y}\bar{x}) = \tau(\bar{x}\bar{y}z) = \tau(\bar{x}\bar{y})^{z}\tau(z) = \tau(\bar{x}\bar{y})\tau(z) = \tau(\bar{x})^{\bar{y}}\tau(\bar{y})\tau(z) = \lambda^{\bar{y}}\mu\hat{z}^{m}$$

where $\hat{z}$ is the image of z in D_{ab}. Thus

$$\mu^{\bar{x}}\lambda = \lambda^{\bar{y}}\mu\hat{z}^{m}.$$

We now return to D rather than its abelianisation. Choose elements $\lambda_1, \mu_1 \in D$ so that $\lambda_1 D' = \lambda$ and $\mu_1 D' = \mu$. Then, recalling that D' is a characteristic subgroup of the multiplicative group of D, the equation above can be replaced by

$$\mu_1^{\bar{x}}\lambda_1 d = \lambda_1^{\bar{y}}\mu_1 z^{m}$$

for some element d of D'. We now apply v to this equation. We obtain

$$v(\mu_1) + v(\lambda_1) + v(d) = v(\lambda_1) + v(\mu_1) + v(z^{m}).$$

That is, $v(d) = mv(z)$. But, as observed above, $v(d)$ is zero whereas we have assumed that $v(z)$ is non-zero. This contradiction completes the argument. □

PROPOSITION 5. *Let H be a free nilpotent group of finite rank and let M be a finitely generated, reduced, faithful kH-module. Let P denote the annihilator in $k\zeta(H)$ of M and suppose that M is torsion-free as kH'/PkH'-module. Then either*

(1) *M has a non-zero submodule which is $kH/P.kH$-torsion-free; or*

(2) *there is a normal subgroup H_1 of H so that H/H_1 is infinite cyclic and M is torsion-free as kH_1/PkH_1-module and M has a submodule N so that N is finitely generated as kH_1-module and M/N is kH'/PkH'-torsion.*

PROOF. Let T denote the set of non-zero elements of $kH'/P.kH'$. Then T is a right denominator set in $kH/P.kH$ and so we can form the ring of quotients $(kH/PkH)T^{-1}$. We can regard this as a crossed product DA of the division ring $D = (kH'/PkH')T^{-1}$ by the free abelian group $A = H/H'$. Set $\widehat{M} = M \otimes_{kH} DA$ and observe that, as M is torsion-free for $kH'/P.kH'$, M embeds into $\widehat{M}$. We refer to the proof of [4, Theorem 3.2] for a more detailed discussion of this.

We can now apply Theorem 3 to show that, if B is a CS-subgroup of A then DB has a non-zero module of finite D-dimension. We aim to show that if B has rank greater than 1 then the combination of this with Proposition 4 leads to a contradiction. Let B be a CS-subgroup of A and suppose that B has rank at least 2. Choose elements x, y in H which form part of a free generating set of H and so that xH' and yH' lie in B (observe that B is necessarily isolated). Let $\bar{x}$ be the element of DB satisfying $\bar{x} = x + PkH'$ and define $\bar{y}$ similarly. Then $z = [\bar{x}, \bar{y}] = [x, y] + PkH'$ is an element of D. Observe that $[x, y]$ is necessarily part of a basis for $H'/\gamma_3(H)$ and so we can find an isolated normal subgroup H_1 of H with $\gamma_3(H) \leq H_1 \leq H'$ and so that $H' = \langle H_1, [x, y]\rangle$.

Let T_1 be the set of non-zero elements of $kH_1/P.kH_1$. By methods similar to those of [9, Lemma 13.3.5 (ii)], it follows that T_1 is an Ore set in kH/PkH. Then $D_1 = (kH_1/P.kH_1)T_1^{-1}$ is a division ring and $(kH'/P.kH')T_1^{-1}$ is a crossed product of D_1 with an infinite cyclic group generated by the image of z. We write this as $(kH'/P.kH')T_1^{-1} = D_1\langle z\rangle$ which is a skew Laurent polynomial ring. Thus we can regard an element of $(kH'/P.kH')T_1^{-1}$ as a Laurent polynomial in z with coefficients from D_1 and so we can refer to the degree of such an element. This degree function on $(kH_1/P.kH_1)T_1^{-1}$ extends to a degree function on the division ring of quotients D. We omit the essentially routine verification of these facts and of the fact that this yields a multiplicative function $v : D \setminus \{0\} \to \mathbb{Z}$ with $v(z) = 1$. Observe that $[x, y]^w[x, y]^{-1} \in \gamma_3(H) \leq H_1$ for $w \in \{x, y\}$ and so conjugation by w does not change the degree of an element of $D_1\langle z\rangle$. Thus conjugation by w will not change the v-value of any element of D. Hence we can apply Proposition 4 to show that DB has no non-zero module of finite D-dimension. But then, from Theorem 3, it follows that B cannot be a CS-subgroup.

Thus the rank of a CS-subgroup is at most 1 and so the dimension of $\widehat{M}$ is at least $n - 1$ where n is the rank of A. Observe that the properties assumed for M pass to non-zero submodules and that any submodule of $\widehat{M}$ is of the form $N \otimes_{kH} DA$ for some submodule N of M. Thus we can deduce that all non-zero submodules of $\widehat{M}$ have dimension at least $n - 1$. We can now apply Proposition 1 to show that either $\widehat{M}$ has a non-zero module $\widehat{N}$ which is torsion-free as DA-module or, for some C with A/C infinite cyclic, $\widehat{M}$ is finitely generated and torsion-free as DC-module.

In the first case, $\widehat{N}$ will be of the form $N \otimes_{kH} DA$ and N will embed into $\widehat{N}$ (as N is $kH'/P.kH'$-torsion-free). Thus N is $kH/P.kH$-torsion-free. In the second case M is torsion-free as $kH_1/P.kH_1$-module where H_1 is the normal subgroup of H corresponding to C. The existence of the finitely generated kH_1 submodule N follows in a similar way to the proof used in Theorem 1. □

PROOF OF THEOREM 2. We suppose that conclusion (1) of the theorem fails. Thus, by Proposition 3, M is torsion-free as $kH'/P.kH'$-module. Thus we can apply Proposition 5 to show that either conclusion (2) holds or else M contains a non-zero submodule which is torsion-free over $kH/P.kH$. This submodule, in turn, contains a copy of $kH/P.kH$ which as module is induced over $\zeta(H)$ and so over H'. This contradicts our assumption that (1) fails and completes the proof. □

3. A conjecture

The results presented here have been stated for free nilpotent groups because such groups have very heavy restrictions on their structure which make it simpler to use the available results. We believe, however, that similar, albeit more complicated, results hold for all nilpotent groups. In this section we shall make one conjecture that expresses a little of what we believe may be true. We shall restrict ourselves to class 2 since even here the conjecture seems very difficult to prove.

We fix some notation; as before k is a field. Let H denote a finitely generated torsion-free nilpotent group of class 2 and let $\zeta(H)$ be the centre of H. Let M denote a finitely generated reduced kH-module which is faithful for H and let P denote the annihilator of M in $\zeta(H)$. Let K denote the field of fractions of $k\zeta(H)/P$ and set $K[H] = kH \otimes_{k\zeta(H)/P} K$. Then M is torsion-free over $k\zeta(H)/P$ and so embeds into $\widehat{M} = M \otimes_{k\zeta(H)/P} K[H]$. By the *dimension* of M, we shall mean the Gelfand–Kirillov dimension (or that described in the second paragraph of section 1) of $\widehat{M}$ considered as $K[H]$-module. Observe that [4, Theorem 3.1] shows that if H is non-abelian then M has dimension at least 1.

CONJECTURE. *With notation as above, suppose that M is impervious. Suppose that $H/\zeta(H)$ has rank n and that M has dimension m. Then H has a subgroup of finite index which is a central product of $n - m$ factors, each of which contains $\zeta(H)$ as a subgroup of infinite index.*

Observe that each central factor in this product must have quotient by the centre of torsion-free rank at least one if it is not virtually central and torsion-free rank at least 2 if it is not virtually abelian.

Observe two extreme cases of this. If H has a reduced faithful impervious module of dimension $n - 2$ or less, then we conjecture that H is a central product, non-trivial in the above sense. Also, [3, Corollary 5] implies that any finitely generated reduced, impervious and faithful module for H must have dimension at least $n/2$ and the arguments of Brookes in [2] show that, if some extension of M by H is finitely presented, then we must have $m = n/2$. The conjecture claims in this case that H is virtually a central product of 2-generator groups. This has been conjectured by Ure in [10] for modules arising from finitely presented groups and has been proved in some special cases.

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