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A STRUCTURE THEOREM IN FINITE TOPOLOGY

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ABSTRACT. Simple structure theorems or representation theorems make their appearance at a very elementary level in subjects such as algebra, but one infrequently encounters such theorems in elementary topology. In this note we offer one such theorem, for finite topological spaces, which involves only the most elementary concepts: discrete space, indiscrete space, product space and homeomorphism. Our theorem describes the structure of those finite spaces which are homogeneous.

In this short note we will establish a simple representation theorem for finite topological spaces which are homogeneous. We recall that a topological space X is said to be *homogeneous* if for all points x and y of X there is a homeomorphism from X onto itself which maps x to y. For any natural number k we let $D(k)$ and $I(k)$ denote the set $\{1, 2, \ldots, k\}$ with the discrete topology and the indiscrete topology respectively. As usual, if X and Y are topological spaces then $X \times Y$ is regarded as a topological space with the product topology, which has as a basis all sets of the form $G \times H$ where G is open in X and H is open in Y. If the spaces X and Y are homeomorphic we write $X \simeq Y$. The number of elements of a set S is denoted by $|S|$.

THEOREM. *Let X be a finite topological space. Then X is homogeneous if and only if there exist integers m and n such that $X \simeq D(m) \times I(n)$.*

Proof. Suppose X is a finite homogeneous space. For each x in X we let $M(x) = \bigcap\{G : G$ is open in X and $x \in G\}$. Since X is finite, $M(x)$ is an intersection of finitely many open sets, and is therefore open. Note that if G is any open set in X then, for all x in X, $x \in G$ implies that $M(x) \subseteq G$. We let $n(x) = |M(x)|$. Next we observe that, since X is homogeneous, $n(x) = n(y)$ for all x and y in X: in fact if f is a homeomorphism from X onto itself, then $f(M(x)) = M(f(x))$, as follows from the definition of $M(x)$. Since X is homogeneous this shows that for all x and y there is a bijection from $M(x)$ to $M(y)$ and hence these sets have the same number of elements. We let n denote the common value of $n(x)$ for x in X. We now show that for each x in X, $M(x)$ contains no nonempty proper open subset: for, suppose H is an open subset of

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X with $\phi \subset H \subset M(x)$. Let y be any element of H. Then $M(y) \subseteq H$ by definition of $M(y)$. Therefore $M(y)$ is a proper subset of $M(x)$. But this is impossible since $M(x)$ and $M(y)$ have the same number of elements. This establishes our assertion: $M(x)$ is a minimal nonempty open set.

Next we note that the family of sets $P = \{M(x) : x \in X\}$ is a partition of X: for, if $M(x) \cap M(y) \neq \phi$ then by the minimality of $M(x)$, we have $M(x) \cap M(y) = M(x)$, and hence $M(x) \subseteq M(y)$. Similarly, $M(y) \subseteq M(x)$, and so $M(x) = M(y)$. Therefore any two distinct members of P are disjoint. Since $x \in M(x)$, every element of X belongs to some member of P. Therefore P is a partition of X. Let $m = |P|$. We now show that $X \simeq D(m) \times I(n)$.

Write the family P as $P = \{M_1, M_2, \ldots, M_m\}$. For each $i = 1, 2, \ldots, m$ the set M_i contains n elements. Denote the elements of M_i by $M_i = \{x_{ij} : j = 1, 2, \ldots, n\}$. Thus $X = \{x_{ij} : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$. It is easy to see that the mapping from $D(m) \times I(n)$ to X which sends (i, j) to x_{ij} is a homeomorphism: if G is an open set in X then $x \in G$ implies $M(x) \subseteq G$ and so $G = \bigcup \{M(x) : x \in G\}$. Therefore G is a union of members of P. The inverse image of M_i under the mapping is $\{i\} \times I(n)$, which is open in $D(m) \times I(n)$. Thus the inverse image of every member of P is open and hence so is the inverse image of G. This shows the mapping to be continuous. To show that its inverse is continuous, note that the sets of the form $\{i\} \times I(n)$ are a basis for $D(m) \times I(n)$; the image of $\{i\} \times I(n)$ is M_i, which is open in X, and therefore the image of every open set is open. It follows that $X \simeq D(m) \times I(n)$.

To prove the converse, observe the following. Discrete spaces and indiscrete spaces are homogeneous. (This follows from the fact that any bijection from a discrete space to itself, or from an indiscrete space to itself, is a homeomorphism.) It is easy to show that the product of two homogeneous spaces is homogeneous, and that any space which is homeomorphic to a homogeneous space is itself homogeneous. The converse then follows directly.

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