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THE KERNEL OF $C(N) \longrightarrow C(N(\sqrt{-1}))$ AND THE 4-RANK OF $K_2(O)$

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ABSTRACT. An upper bound is given for the order of the kernel of the map on S-ideal class groups that is induced by $A \cdot O_N \longmapsto A \cdot O_{N(\sqrt{-1})}$. For some special types of number fields F the connection between the size of the above kernel for $N = F(\sqrt{-\sigma})$ and the units and norms in $F(\sqrt{\sigma})$ are examined. Let $K_2(O)$ denote the Milnor K-group of the ring of integers of a number field. In some cases a formula by Conner, Hurrelbrink and Kolster is extended to show how closely the 4-rank of $K_2(O_{F(\sqrt{-\sigma})})$ is related to the 4-rank of the S-ideal class group of $F(\sqrt{\sigma})$.

1. **Notation.** Let N be a number field with ring of integers O_N. Let $C(N)$ denote the *S-ideal class group* of N, where S is the set consisting of all infinite and dyadic primes of N. We examine the kernel of $C(N) \longrightarrow C(N(\sqrt{-1}))$, the map induced by $A \cdot O_N \longmapsto A \cdot O_{N(\sqrt{-1})}$. For the most part, this paper will deal only with quadratic extensions of a special type of number field. The following property is a natural generalization of properties of $\mathbb{Q}$. It is also a special case of the *regular fields* examined in [4] and [5].

(1.1) DEFINITION. A number field is said to have *property* $(*)$ if it is totally real, contains exactly one dyadic prime, has odd S-class number and contains S-units with independent signs; where S is the set of all its infinite and dyadic primes.

For a number field N, let $r_1(N)$ denote the number of its real embeddings, $r_2(N)$ the number of its pairs of complex embeddings and $g_2(N)$ the number of its dyadic primes. Let U_N denote the group of S-units of N, where S is as above. If N has $(*)$ the kernel of $U_N/(U_N)^2 \longrightarrow (\mathbb{Z}/2)^{r_1(N)}$ has order 2; *i.e.* there exists exactly one non-trivial totally positive square class of S-units. This square class, or any representative, will be denoted by τ_N. Throughout this paper F, E and L will denote very specific types of number fields, see below, while N will stand for an arbitrary number field.

(1.2) NOTATION. F is a number field with property $(*)$, D_F is the dyadic prime of F and $\sigma \in F^*/(F^*)^2$ is a non-trivial totally positive square class. $E = F(\sqrt{\sigma})$, $L = F(\sqrt{-\sigma})$, and M is their common quadratic extension. $I_*\colon C(E) \longrightarrow C(M)$ and $i_*\colon C(L) \longrightarrow C(M)$, are the maps on S-class groups induced by $A \longmapsto A \cdot O_M$. The respective norm maps are denoted by $N_{M|E}, N_{M|L}, N_{E|F}$. The cohomology group $H^0(\mathrm{Gal}(E|F), U_E)$ will be used frequently and if no confusion is possible it will be abbreviated by H^0. Its quotient with the subgroup generated by the class of τ_F will be denoted by H^0/τ.

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One of the main tools used throughout this paper will be the exact hexagon from [2], applied to the rings of S-integers of quadratic extensions of number fields. Here is an overview of this material as it pertains to this paper:

Let $M|N$ be a quadratic extension of number fields and let $C_2 = \mathrm{Gal}(M|N)$. When broken up, the exact hexagon yields the following exact sequence:

$$H^0(C_2, U_M) \xrightarrow{i_0} R^0(M|N) \xrightarrow{j_1} H^1(C_2, C(M)) \longrightarrow H^1(C_2, U_M) \xrightarrow{i_1} R^1(M|N) \longrightarrow H^0(C_2, C(M))$$

All six groups in the hexagon are elementary abelian 2-groups.

(1.3) FACTS. a) $H^0(\mathrm{Gal}(M|N), C(M)) \cong H^1(\mathrm{Gal}(M|N), C(M))$; since $C(M)$ is finite abelian.

b) The group $R^0(M|N)$ is defined as a quotient of cohomology groups. It injects into $H^0(C_2, M^*)$; the composition of this injection with i_0 commutes with the inclusion of $H^0(C_2, U_M)$ into $H^0(C_2, E^*)$.

c) In Section 6 of [2] the 2-ranks of R^0 and R^1 are computed. Let s be the number of dyadic primes of N that are inert in M, then

$$2\,\mathrm{rk}\, R^0(M|N) = \begin{cases} 0 & \text{if } M|N \text{ is unramified and } s = 0 \\ s - 1 + \#(\textit{primes of } N \textit{ that ramify in } M) & \text{otherwise} \end{cases}$$

$$2\,\mathrm{rk}\, R^1(M|N) = \begin{cases} 1 & \text{if } M|N \text{ is unramified and } s = 0 \\ \#(\textit{odd finite primes of } N \textit{ that ramify in } M) & \text{otherwise} \end{cases}$$

d) An S-version of [2] (7.1) gives an injection of $\mathrm{Ker}(C(N) \to C(M))$ into $H^1(C_2, U_M)$. Furthermore, if $M|N$ is unramified or if $M|N$ is ramified but no finite prime of F outside S is ramified in E then $H^1(C_2, U_M) \cong \mathrm{Ker}(C(N) \to C(M))$.

2. **The Kernel of C(N)$\longrightarrow$ C(N($\sqrt{-1}$))**. The kernel of $C(N) \longrightarrow C(N(\sqrt{-1}))$ is an elementary abelian 2-group, see (1.3.d). The following generalizes the bound from [6] (2.4).

(2.1) THEOREM. *Let N be a number field and $M = N(\sqrt{-1})$. Then*

$$2\,\mathrm{rk}\,\mathrm{Ker}(C(N) \to C(M)) \leq r_2(N) + g_2(M) - g_2(N).$$

PROOF. If $\sqrt{-1} \in N$ the statement is trivially true. Else: Let n be the largest integer such that N contains $\mathbb{Q}(\zeta_{2^n})^+ = \mathbb{Q}(\zeta_{2^n} + \bar{\zeta}_{2^n})$, where ζ_{2^n} denotes a primitive $(2^n)^{th}$ root of 1. The element $2 + \zeta_{2^n} + \bar{\zeta}_{2^n} \in U_N$ is the norm of $1 + \zeta_{2^n} \in U_M$. By choice of n this element is not a square in N, hence its square class is a non-trivial element in the kernel of $U_N / U_N^2 \longrightarrow H^0(C_2, U_M)$. By Dirichlet's S-unit theorem the 2-rank of U_N / U_N^2 is given by $r_1(N) + r_2(N) + g_2(N)$. This yields an upper bound on the 2-rank of $H^0(C_2, U_M)$:

$$2\,\mathrm{rk}\, H^0(C_2, U_M) \leq r_1(N) + r_2(N) + g_2(N) - 1.$$

If $M|N$ is unramified and all dyadic primes of N split in M then $2\operatorname{rk} R^0(M|N) = 0$ and $2\operatorname{rk} R^1(M|N) = 1$ and of course $r_1(N) = 0$. If $M|N$ is ramified or if there exists a dyadic prime of N that is inert in M then $2\operatorname{rk} R^0(M|N) = 2 \cdot g_2(N) - g_2(M) + r_1(N) - 1$ and $2\operatorname{rk} R^1(M|N) = 0$. In either case, taking the alternating sum of 2-ranks in the exact hexagon associated to the quadratic extension $M|N$ yields the desired bound. ∎

Now consider the maps I_* and i_* defined in (1.2). By (2.1) the kernel of I_* is either trivial or $\mathbb{Z}/2$. In fact, it can only be non-trivial if $g_2(M) - g_2(E) = 1$, which is the case exactly if $-\sigma$ is a local square at D_F. The kernel of i_* can be much bigger, as (2.3) below shows. Recall the explicit formula for the 2-rank of Ker i_* from [1] (4.1 and 4.3):

(2.2) If σ is not a local square at D_F then Ker $i_* \cong \operatorname{Ker} I_* \times$ Coker $i_0(M|E)$. Furthermore, if $-\sigma$ is not a local square either, then Coker $i_0(M|E) \cong H^0/\tau$; else Coker $i_0(M|E) \cong H/\tau$, where H is a subgroup of index 2 of $H^0 = H^0(\operatorname{Gal}(E|F), U_E)$.

(2.3) EXAMPLE. Number fields of arbitrarily large degree for which the 2-rank of the kernel of i_* achieves the upper bound from (2.1). Let F be any number field with $(*)$ and $r := r_1(F)$. Let $u_1, \ldots, u_{r+1}$ be a basis of the $\mathbb{Z}/2$ vector space U_F/U_F^2. By class field theory there exist finite non-dyadic primes P_j, $1 \leq j \leq r+1$, such that P_j is inert in $F(\sqrt{u_j})|F$ and P_j splits in $F(\sqrt{u_i})|F$ for all $i \neq j$. By the Approximation theorem there exists $\sigma \in F^*$ such that σ is totally positive, $ord_{P_j}(\sigma) \equiv 1 \bmod 2$ for all j and $ord_{D_F}(\sigma) \equiv 1 \bmod 2$. For this choice, neither σ nor $-\sigma$ is a local square at D_F. For $E = F(\sqrt{\sigma})$ the only S-units of F that are norms from E are squares, hence $U_F/U_F^2 = U_F/N_{E|F}(U_E)$, and this has a 2-rank of $r+1$. By (2.2) for $L = F(\sqrt{-\sigma})$ the 2-rank of the kernel of i_* equals the 2-rank of H^0/τ, hence $2\operatorname{rk}\operatorname{Ker} i_* = r_1(F) = r_2(L)$. This is the upper bound in this case. What does the kernel of i_* look like if σ is a local square at D_F?

(2.4) THEOREM. *Let F be a number field with property $(*)$ and $\sigma \in F^*/(F^*)^2$ a non-trivial totally positive square class such that σ is a local square at D_F. Let $E = F(\sqrt{\sigma})$, $L = F(\sqrt{-\sigma})$ and $M = E(\sqrt{-1})$. Then*

$$\operatorname{Ker} i_{*(C(L)\to C(M))} \cong \mathbb{Z}/2 \times H^0(\operatorname{Gal}(E|F), U_E)/\tau$$

PROOF. From the exactness of the exact S-hexagon associated to $M|E$ one obtains:

$$H^1(\operatorname{Gal}(M|E), C(M)) \cong H^1(\operatorname{Gal}(M|E), U_M) \times \operatorname{Coker} i_0(M|E) \cong \operatorname{Coker} i_0(M|E),$$

where $H^1(\operatorname{Gal}(M|E), U_M)$ is trivial by the bound in (2.1). On the other hand, since $h(F\sqrt{-1})$ is odd, $H^1(\operatorname{Gal}(M|E), C(M)) \cong H^0(\operatorname{Gal}(M|L), C(M))$, which is isomorpic to $H^1(\operatorname{Gal}(M|L), C(M))$. Other groups in the exact S-hexagon associated to $M|L$ are: $H^1(\operatorname{Gal}(M|L), U_M) \cong \operatorname{Ker} i_*$ and $R^0(M|L) \cong 1$ and $R^1(M|L) \cong \mathbb{Z}/2$. From the hexagon, one obtains an exact sequence:

$$1 \longrightarrow \operatorname{Coker} i_0(M|E) \longrightarrow \operatorname{Ker} i_* \xrightarrow{i_1} \mathbb{Z}/2.$$

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Hence, Ker i_* $\cong$ $\mathbb{Z}/2$ $\times$ Coker $i_0(M|E)$ if i_1 is surjective, and Ker i_* $\cong$ Coker $i_0(M|E)$ if i_1 is trivial. The cokernel of $i_0(M|E)$ is determined as in [1] (4.3): If there exists a totally positive S-unit in E that is not a norm from M then Coker $i_0(M|E) \cong H^0/\tau$. If all totally positive S-units of E are norms from M then Coker $i_0(M|E) \cong \mathbb{Z}/2 \times H^0/\tau$. The conclusion follows from the next theorem. ∎

(2.5) THEOREM. *Let F be a number field with property (∗) and $\sigma \in F^*/(F^*)^2$ a non-trivial totally positive square class such that σ is a local square at D_F, let $E = F(\sqrt{\sigma})$ and $L = F(\sqrt{-\sigma})$ and $M = E(\sqrt{-1})$. There exists a totally positive S-unit in E that is not a norm from M if and only if i_1 : Ker $i_* \longrightarrow \mathbb{Z}/2$ is non-trivial.*

REMARK. The map i_1 from the exact S-hexagon associated to the extension $M|L$ takes $H^1(\mathrm{Gal}(M|L), U_M) \cong$ Ker i_* to $R^1(M|L) \cong \mathbb{Z}/2$. If σ is a local square at D_F, i_1 can also be interpreted as the Artin reciprocity law map ω, on the S-ideal class group of L, restricted to Ker i_*; see [2] (7.2).

PROOF. $R^1(M|L) \cong \mathbb{Z}/2$ is generated by the class of any non-dyadic prime $P_0 \subset O_L$ that is inert in $M|L$. To check if any such generator comes from Ker i_*, we use the exactness of the S-hexagon and check where it maps to in $H^0(\mathrm{Gal}(M|L), C(M))$.

Let R_L denote the ring of S-integers of L. Let $P_0 \subset R_L$ be a prime ideal which is inert in $M|L$, and for which $p_0 = P_0 \cap R_F$ is not ramified in L. If p_0 were inert in $L|F$ then $cl(P_0^h) = 1 \in C(L)$, where h is the odd $h(F)$. This is not possible since the Artin reciprocity law map takes $cl(P_0^h) \in C(L)$ to a generator of $\mathrm{Gal}(M|L)$. It follows that p_0 splits in $L|F$, hence $-\sigma$ is a local square at p_0. But -1 is not a local square, hence σ is not a local square at p_0. Therefore, p_0 is inert in $E|F$, and $p_0 \cdot R_E$ splits in $M|E$.

Since F contains S-units with independent signs, there exists a totally positive $x \in F^*$ such that $x \cdot R_F = p_0^h \cdot R_F$. Let $\mathcal{P}_0 = P_0 \cdot R_M$ and $\mathrm{Gal}(M|E) = \langle T_1 \rangle$. In R_M the ideal generated by x is: $(\mathcal{P}_0 \cdot T_1\mathcal{P}_0)^h \cdot R_M$. Therefore $\langle x^{-1}, \mathcal{P}_0^h \cdot R_M \rangle$ is an element of $R^0(M|E)$ and under $j_1(M|E)$ it maps to $cl(\mathcal{P}_0^h)$ in $H^1(\mathrm{Gal}(M|E), C(M))$.

Now, $i_1(M|L)$ is surjective iff $cl(\mathcal{P}_0) = 1 \in H^0(\mathrm{Gal}(M|L), C(M))$ for any, and hence all, $\mathcal{P}_0 = P_0 \cdot R_M$, where P_0 is inert in $M|L$. Since $F(\sqrt{-1})$ has odd S-class number: $H^0(\mathrm{Gal}(M|L), C(M)) \cong H^1(\mathrm{Gal}(M|E), C(M))$ and therefore: $i_1(M|L)$ is surjective iff $cl(\mathcal{P}_0) = 1 \in H^1(\mathrm{Gal}(M|E), C(M))$. The same holds for $cl(\mathcal{P}_0^h)$.

From the exactness of the S-hexagon associated to $M|E$ this is equivalent to: an inverse image under $j_1(M|E)$ of $cl(\mathcal{P}_0^h)$ lies in $Im\, i_0(M|E) \subset R^0(M|E)$ for any $\mathcal{P}_0$ as above. That is, $\langle x^{-1}, \mathcal{P}_0^h \cdot R_M \rangle$ lies in the image of $i_0(M|E)$ for all totally positive x as above. From the isomorphism of $R^0(M|E)$ with $(\mathbb{Z}/2)^{r_1(E)+1}$, it follows that this is equivalent to: there exists a totally positive S-unit in E that is not a norm from M. ∎

The following example shows that both cases from (2.5) can occur. Using the notation from [2]: a prime $l \in \mathbb{Z}$, $l \equiv 1 \bmod 8$, is said to be in $A(2)^-$ if the class number of $\mathbb{Q}(\sqrt{2}, \sqrt{l})$ is odd (example: $l = 17$); and $l \in A(2)^+$ otherwise (example: $l = 41$).

(2.6) EXAMPLE. Let $E = \mathbb{Q}(\sqrt{l})$ and $M = \mathbb{Q}(\sqrt{l}, \sqrt{-1})$ for a prime $l \equiv 1 \bmod 8$. If $l \in A(2)^-$ there exists a totally positive S-unit in E that is not a norm from M.

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If $l \in A(2)^+$ all totally positive S-units of E are norms from M.

PROOF. Consider $L = \mathbb{Q}(\sqrt{-l})$. The 2-primary subgroup of its ideal class group is cyclic, of order at least 4. By [2] (24.1), the exact 2 power dividing the ideal class group is 4 iff $l \in A(2)^-$. Since L has exactly one dyadic prime and it is not principal, it follows that for the S-ideal class group we have: if $l \in A(2)^-$ then $2 \| h(L)$ but if $l \in A(2)^+$ then $4 | h(L)$. The kernel of i_* is a non-trivial elementary abelian subgroup of the cyclic 2-primary subgroup of $C(L)$, hence Ker $i_* \cong \mathbb{Z}/2$. Now consider the surjective Artin reciprocity law map $\omega: C(L) \longrightarrow \mathrm{Gal}(M|L)$. If $2\|h(L)$ then $C(L) = \mathbb{Z}/2 = \mathrm{Ker}\, i_*$, hence the restriction of ω to Ker i_* is surjective. If $4|h(L)$ then Ker $i_* \subseteq (C(L))^2$, hence the restriction of ω to Ker i_* is trivial. ∎

3. **The 4-rank of $K_2(O)$.** Let N be a number field and let $K_2(O_N)$ denote the Milnor K-group of its ring of integers. We are interested in the structure of the 2-primary subgroup of this finite abelian group. It follows from [7] that its rank can be determined as follows:

(3. 1) Tate's 2-rank formula: $$2\,\mathrm{rk}\, K_2(O_N) = r_1(N) + g_2(N) - 1 + 2\,\mathrm{rk}\, C(N)$$

The same source yields p^nrank formulas for $K_2(O_N)$, but only if the p^n-th roots of unity are in N. In particular: If $\sqrt{-1} \in N$ then $4\,\mathrm{rk}\, K_2(O_N) = g_2(N) - 1 + 4\,\mathrm{rk}\, C(N)$.

Kolster [6] (3.1), Conner and Hurrelbrink [3] (1.5) give a 4-rank formula in the complementary case: If $\sqrt{-1} \notin N$, let $M = N(\sqrt{-1})$ then

(3. 2) $$4\,\mathrm{rk}\, K_2(O_N) = g_2(M) - g_2(N) + 2\,\mathrm{rk}\left(\mathrm{Ker}\, N_{M|N} / {}_2Im(C(N) \to C(M))\right).$$

where ${}_2Im(.)$ indicates an elementary abelian 2-group of the same rank as $Im(.)$.

For an imaginary quadratic extension L of a number field with $(*)$ this 4-rank formula can be used to show how the 4-rank of $K_2(O_L)$ is bounded by the 4-rank of $C(E)$, where E is the corresponding real field, see (1.2).

(3.3) THEOREM. *Let F be a number field with property $(*)$ and $\sigma \in F^*/(F^*)^2$ a non-trivial totally positive square class. Let $E = F(\sqrt{\sigma})$, $L = F(\sqrt{-\sigma})$, $M = L(\sqrt{-1})$ and $\alpha = 2\,\mathrm{rk}(H^0/\tau) - 2\,\mathrm{rk}(H^0/\langle t, \mathrm{Ker}\, i_0\rangle)$, where $\langle t, \mathrm{Ker}\, i_0\rangle$ is the subgroup of $H^0(\mathrm{Gal}(E|F), U_E)$ generated by τ and* Ker $i_0(E|F)$.

a) If $-\sigma$ is not a local square at D_F then:

$$\begin{aligned} 4\,\mathrm{rk}\, C(E) &\le 4\,\mathrm{rk}\, K_2(O_L) + (-1 \ {}^{\text{if } \sigma \text{ is a loc.}}_{\text{square at D}}\ ) \\ &\le \min\{4\,\mathrm{rk}\, C(E) + \alpha\,,\ 2\,\mathrm{rk}\, C(E)\} \end{aligned}$$

b) If $-\sigma$ is a local square at D_F then:

$$\begin{aligned} 4\,\mathrm{rk}\, C(E) - 2\,\mathrm{rk}\, \mathrm{Ker}\, I_* &\le 4\,\mathrm{rk}\, K_2(O_L) \\ &\le \min\{4\,\mathrm{rk}\, C(E) + 2\,\mathrm{rk}\, \mathrm{Ker}\, I_* + \alpha\,,\ 2\,\mathrm{rk}\, C(E)\} \end{aligned}$$

PROOF. By [1] (2.3) the 2-primary subgroup of $\operatorname{Ker} N_{M|L}$ is isomorphic to the 2-primary subgroup of $Im I_* \cong C(E)/\operatorname{Ker} I_*$, substituting this into (3.2) yields:

$$4\,\mathrm{rk}\,K_2(O_L) - (g_2(M) - g_2(L)) = 2\,\mathrm{rk}\left(C(E)/\operatorname{Ker} I_* / {}_2 Im\, i_*\right)$$

It follows:

$$4\,\mathrm{rk}\,C(E) - 2\,\mathrm{rk}\,\operatorname{Ker} I_* \leq 4\,\mathrm{rk}\,K_2(O_L) - (g_2(M) - g_2(L)) \leq 2\,\mathrm{rk}\,C(E)$$

with $2\,\mathrm{rk}\,\operatorname{Ker} I_* = 0$ if $-\sigma$ is not a local square at D_F and 0 or 1 otherwise. Furthermore, $g_2(M) - g_2(L) = 1$ if σ is a local square at D_F and 0 otherwise. The upper bound involving $4\,\mathrm{rk}\,C(E)$ is obtained by using the bound for $2\,\mathrm{rk}\,C(E)$ in (3.4) when examining $2\,\mathrm{rk}\left(C(E)/\operatorname{Ker} I_* / {}_2 Im\, i_*\right)$. ∎

(3.4) PROPOSITION. $2\,\mathrm{rk}\,C(E) \leq 2\,\mathrm{rk}\,Im\, i_* + 2\,\mathrm{rk}\,\operatorname{Ker} I_* + \alpha$

PROOF. By [1] (3.1 and 3.2) $2\,\mathrm{rk}\,C(L)$ can be expressed in terms of $2\,\mathrm{rk}\,C(E)$:

$$2\,\mathrm{rk}\,C(L) = 2\,\mathrm{rk}\,C(E) + 2\,\mathrm{rk}(H^0/\langle t, \operatorname{Ker} i_0\rangle) + \begin{cases} +1 & \text{if } \sigma \text{ is a local square at } D_F \\ -1 & \text{if } -\sigma \text{ is a local square at } D_F \end{cases}$$

By (2.2), (2.4): $2\,\mathrm{rk}\,\operatorname{Ker}\, i_* = 2\,\mathrm{rk}\,\operatorname{Ker} I_* + 2\,\mathrm{rk}(H^0/\tau) + \begin{cases} +1 & \text{if } \sigma \text{ is a loc. sq. at } D_F \\ -1 & \text{if } -\sigma \text{ is a loc. sq at } D_F \end{cases}$

Substituting this for $2\,\mathrm{rk}\,C(L)$ and $2\,\mathrm{rk}\,\operatorname{Ker}\, i_*$ into $2\,\mathrm{rk}\,C(L) \leq 2\,\mathrm{rk}\,Im\, i_* + 2\,\mathrm{rk}\,\operatorname{Ker}\, i_*$ yields the result. ∎

(3.5) COROLLARY. *If* $\operatorname{Ker} i_0(E|F) \subseteq \{1, \tau\}$ *and if neither of* $\pm\sigma$ *is a local square at* D_F *then* $4\,\mathrm{rk}\,K_2(O_L) = 4\,\mathrm{rk}\,C(E)$. *This occurs, for example, if E contains S-units with independent signs.* ∎

(3.6) COROLLARY. *If* $C(E)$ *is elementary abelian and if* $\operatorname{Ker} i_0(E|F) \subseteq \{1, \tau\}$ *then*

$$4\,\mathrm{rk}\,K_2(O_L) = \begin{cases} 1 & \text{if } \sigma \text{ is a local square at } D_F \\ 0 & \text{if neither of } \pm\sigma \text{ is a local square at } D_F. \\ 0 \text{ or } 1 & \text{if } -\sigma \text{ is a local square at } D_F \end{cases}$$ ∎

By Dirichlet's S-unit theorem: $\# U_F/U_F^2 = 2^{r_1(F)+1}$, hence $\alpha \leq 2\,\mathrm{rk}(H^0/\tau) \leq r_1(F)$. It follows that the upper bound in (3.3) is quite good if $r_1(F)$ is small.

(3.7) COROLLARY. *Let* σ *be a squarefree positive integer,* $E = \mathbb{Q}(\sqrt{\sigma})$ *and* $L = \mathbb{Q}(\sqrt{-\sigma})$.

a) If $\sigma \not\equiv 1, 7 \bmod 8$ *then* $4\,\mathrm{rk}\,C(E) \leq 4\,\mathrm{rk}\,K_2(O_L) \leq 4\,\mathrm{rk}\,C(E) + 1$.

b) If $\sigma \equiv 1 \bmod 8$ *then* $4\,\mathrm{rk}\,C(E) + 1 \leq 4\,\mathrm{rk}\,K_2(O_L) \leq 4\,\mathrm{rk}\,C(E) + 2$.

c) If $\sigma \equiv 7 \bmod 8$ *then* $4\,\mathrm{rk}\,C(E) - 1 \leq 4\,\mathrm{rk}\,K_2(O_L) \leq 4\,\mathrm{rk}\,C(E) + 2$. ∎

Unfortunately, trying to bound the 4-rank of $K_2(O_E)$ by the 4-rank of $C(L)$ does not seem to work quite as well. The crucial point here is to get an equivalent of [1] (2.3).

(3.8) THEOREM. *Let F be a number field with (∗) and let σ be a non-trivial totally positive square class that is not a local square at D_F. Let $E = F(\sqrt{\sigma})$ and $L = F(\sqrt{-\sigma})$. Then* $2\,\mathrm{prim}(\mathrm{Ker}\, N_{M|E})/2\,\mathrm{prim}(Im\, i_*) \cong \mathrm{Coker}\ i_0(M|E)$

PROOF. By [2] (5.7) there exists a natural homomorphism from $\mathrm{Ker}\, N_{M|E}$ to $H^1(\mathrm{Gal}(M|E), C(M))$ whose image agrees with the image of j_1. From the exactness of the S-hexagon: $Im\, j_1(M|E) \cong \mathrm{Coker}\ i_0(M|E)$, hence

$$(\mathrm{Ker}\, N_{M|E})/\{B \mid j_1(B)=1 \in H^1(\mathrm{Gal}(M|E),C(M))\} \cong \mathrm{Coker}\ i_0(M|E).$$

Let T_1 and T_2 denote the generators of the Galois groups of $M|E$ and $M|L$, respectively. For $B \in C(M) : B \in H^1(\mathrm{Gal}(M|E), C(M))$ iff there exists $C \in C(M)$ with $B = C \cdot T_1(C)^{-1}$. Recall that $h(F((\sqrt{-1}))$ is odd, hence if B is in the 2-primary subgroup of $C(M)$, the above is equivalent to: there exists $C \in C(M)$ with $B = C \cdot T_2(C)$. Since $R^1(M|L) = 1$, an argument like in [1] (2.2.1) shows that there exists a C as above iff $B \in Im\, i_*$. ∎

In particular, when is 2-prim $\mathrm{Ker}\, N_{M|E}$ isomorphic to 2-prim $Im\, i_*$? If neither of $\pm\sigma$ is a local square at D_F then $\mathrm{Coker}\ i_0(M|E) = 1$ iff E contains S-units with independent signs. If $-\sigma$ is a local square at D_F then $\mathrm{Coker}\ i_0(M|E) = 1$ iff E contains S-units with almost independent signs; see [1] (4.3).

(3.9) PROPOSITION. *Let F be a number field with property (∗) and σ a non-trivial totally positive square class that is not a local square at D_F.*

a) If $-\sigma$ is not a local square at D_F either, then

$$4\,\mathrm{rk}\, C(L) - 2\,\mathrm{rk}\,\mathrm{Coker}\ i_0(M|E) \le 4\,\mathrm{rk}\, K_2(O_E) \le 2\,\mathrm{rk}\, C(L) + 2\,\mathrm{rk}\,\mathrm{Coker}\ i_0(M|E).$$

b) If $-\sigma$ is a local square at D_F then,

$$4\,\mathrm{rk}\, C(L) - 2\,\mathrm{rk}\,\mathrm{Coker}\ i_0(M|E) - 2\,\mathrm{rk}\,\mathrm{Ker}\, I_* \le 4\,\mathrm{rk}\, K_2(O_E) - 1$$
$$\le 2\,\mathrm{rk}\, C(L) + 2\,\mathrm{rk}\,\mathrm{Coker}\ i_0(M|E).$$

PROOF. By (3.2), consider $2\,\mathrm{rk}(\mathrm{Ker}\, N_{M|E}/{}_2 Im I_*)$. This is bounded from above by $2\,\mathrm{rk}\,\mathrm{Ker}\, N_{M|E}$ and from below by $4\,\mathrm{rk}\,\mathrm{Ker}\, N_{M|E}$. By (3.8): $2\,\mathrm{rk}\,\mathrm{Ker}\, N_{M|E} \le 2\,\mathrm{rk}\, Im\, i_* + 2\,\mathrm{rk}\,\mathrm{Coker}\ i_0(M|E)$, and $4\,\mathrm{rk}\,\mathrm{Ker}\, N_{M|E} \ge 4\,\mathrm{rk}\, Im\, i_* \ge 4\,\mathrm{rk}\, C(L) - 2\,\mathrm{rk}\,\mathrm{Ker}\ i_*$. By (2.2) $\mathrm{Ker}\ i_* \cong \mathrm{Ker}\, I_* \times \mathrm{Coker}\ i_0(M|E)$. ∎

(3.10) PROPOSITION. *If neither of $\pm\sigma$ is a local square at D_F and if* $\mathrm{Coker}\ i_0(M|E) = 1$ *then* $4\,\mathrm{rk}\, K_2(O_E) = 4\,\mathrm{rk}\, C(L)$.

PROOF. In this case $2\,\mathrm{prim}\,\mathrm{Ker}\, N_{M|E} \cong Im\, i_*$ and $\mathrm{Ker}\, I_* = 1 = \mathrm{Ker}\ i_*$.
By the proof of (3.4): $2\,\mathrm{rk}\, C(E) = 2\,\mathrm{rk}\, C(L)$, hence

$$4\,\mathrm{rk}\, K_2(O_E) = 2\,\mathrm{rk}(\mathrm{Ker}\, N_{M|E}/{}_2 Im I_*) = 2\,\mathrm{rk}(Im C(L)/{}_2 C(L)) = 4\,\mathrm{rk}\, C(L). \quad ∎$$

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