COUNTABLY COMPACT EXTENSIONS AND CARDINAL CHARACTERISTICS OF THE CONTINUUM

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Abstract. In this paper, we show that the existence of certain first-countable compact-like extensions is equivalent to the equality between corresponding cardinal characteristics of the continuum. For instance, $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ if and only if every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a regular first-countable countably compact space.

§1. Introduction. The main motivation for investigation of regular first-countable countably compact spaces is the following problem of Nyikos which is listed among 20 central problems in Set-theoretic Topology by Hrušák and Moore [16].

PROBLEM 1.1 (Nyikos). *Does ZFC imply the existence of a regular separable first-countable countably compact non-compact space*?

Following [3], a regular separable first-countable countably compact space is called a *Nyikos* space. Consistent examples of noncompact Nyikos spaces were constructed by Franklin and Rajagopalan [14] (under $\omega_1 = \mathfrak{t}$) and by Ostaszewski [25] (under $\mathfrak{b} = \mathfrak{c}$). Weiss [35] showed that under MA every perfectly normal Nyikos space is compact. Proper Forcing Axiom or, simply, PFA is a stronger version of Martin's Axiom. Basic information about PFA can be found in [20]. For fruitful applications of PFA in Topology see [10–12, 30, 34]. The following result was proved by Nyikos and Zdomskyy [24].

THEOREM 1.2 (PFA). *Every normal Nyikos space is compact.*

A space X is called $\mathbb{R}$-*rigid* if each continuous real-valued function on X is constant. For more about rigid spaces see [2, 17, 31–33]. Another problem related to Nyikos spaces appeared in [29, Problem C65].

PROBLEM 1.3 (Tzannes). *Does there exist a regular (separable, first countable) countably compact $\mathbb{R}$-rigid space*?

Taking into account the properties in brackets, Tzannes problem can be considered as an ultimate version of Nyikos problem, as $\mathbb{R}$-rigid spaces, being not Tychonoff, are not locally compact.

Bardyla and Osipov [2] constructed a ZFC example of a regular countably compact $\mathbb{R}$-rigid space. Bardyla and Zdomskyy [3] obtained the following answer to Problem 1.3 under the consistent assumption ($\heartsuit\!\!\!\!/$): "$\omega_1 = \mathfrak{t} < \mathfrak{b} = \mathfrak{c}$ and there exists a $P_{\mathfrak{c}}$-point in $\beta(\omega)$".

THEOREM 1.4 (Bardyla, Zdomskyy). ($\heartsuit\!\!\!\!/$) *There exists an $\mathbb{R}$-rigid Nyikos space.*

The proof of Theorem 1.4 had two steps. During the first one we constructed a regular separable first-countable $\mathbb{R}$-rigid space X of cardinality $< \mathfrak{c}$. The second step was to embed densely the space X into a first-countable countably compact space Y. Note that Y will be automatically an $\mathbb{R}$-rigid Nyikos space. This was done using the following result proved in [3] under the assumption ($\heartsuit$): "$\mathfrak{b} = \mathfrak{c}$ and there exists a $P_{\mathfrak{c}}$-point in $\beta(\omega)$".

THEOREM 1.5 (Bardyla, Zdomskyy). ($\heartsuit$) *Each regular first-countable space of cardinality $< \mathfrak{c}$ embeds densely into a regular first-countable countably compact space.*

Similar problems concerning embedding of topological spaces into first-countable compact-like spaces are known in General Topology. For instance, the most relevant to this paper is the following one posed in [27].

PROBLEM 1.6 (Stephenson). *Does every locally feebly compact first-countable regular space embed densely into a feebly compact first-countable regular space?*

Problem 1.6 was solved affirmatively by Simon and Tironi [26]. In case of Tychonoff spaces the following result was obtained in [28].

THEOREM 1.7 (Terada, Terasawa). *Each Tychonoff first-countable locally pseudo-compact space embeds densely into a Tychonoff first-countable pseudocompact space.*

The mentioned results motivate the following general question which we address in this paper.

QUESTION 1.8. *When does a space X embed into a regular first-countable compact-like space?*

§2. Preliminaries. By $\mathbb{N}$ we denote the set of positive integers, i.e., $\mathbb{N} = \omega \setminus \{0\}$. The cardinality of the continuum is denoted by $\mathfrak{c}$. The set of all infinite subsets of a countable set A is denoted by $[A]^\omega$. The set of all finite subsets of a set A is denoted by $[A]^{<\omega}$. A family $\mathcal{A} \subset [\omega]^\omega$ is called *almost disjoint* if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$. A family $\mathcal{S} \subseteq [\omega]^\omega$ is called *splitting* if for any $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ which splits A, i.e., $|A \cap S| = |A \cap (\omega \setminus S)| = \omega$. For two subsets A and B of ω we write $A \subseteq^* B$ or $B \supseteq^* A$ if $|A \setminus B| < \omega$. A family $\mathcal{T} \subset [\omega]^\omega$ is called a *maximal tower* if it is well-ordered by the relation $\supseteq^*$ and there is no set $A \in [\omega]^\omega$ such that $A \subseteq^* T$ for all $T \in \mathcal{T}$. For any functions $f, g \in \omega^\omega$ we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A family $\Phi \subseteq \omega^\omega$ is called *unbounded* if there exists no $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in \Phi$. We write $f \equiv c$ if f is a constant function with value c.

We shall use the following cardinal characteristics of the continuum:
- $\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subset [\omega]^\omega \text{ is a splitting family}\}$;
- $\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \subset [\omega]^\omega \text{ is a maximal tower}\}$;
- $\mathfrak{b} = \min\{|\Phi| : \Phi \subset \omega^\omega \text{ is an unbounded family}\}$.

It is well-known that $\omega_1 \leq \mathfrak{t} \leq \min\{\mathfrak{s}, \mathfrak{b}\}$.

An ultrafilter $\mathcal{F}$ on ω is called a $P_\mathfrak{c}$ point if $\mathcal{F}$ has a base which forms a maximal tower of cardinality $\mathfrak{c}$. The existence of such an ultrafilter is consistent with ZFC and, in particular, follows from $\mathfrak{t} = \mathfrak{c}$.

Let X, Y be topological spaces. A map $f : X \to Y$ is called an *embedding of X into Y* if the map $f' : X \to f(X)$, obtained by restricting the range of f, is a homeomorphism. The *weight* of a topological space X is the smallest cardinal $w(X)$ such that X has a base of cardinality $w(X)$.

A space X is called

- *countably compact* if each closed discrete subset of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function is bounded;
- *feebly compact* if every locally finite family of open subsets of X is finite.

It is known that countable compactness implies feeble compactness. In the case of Tychonoff spaces pseudocompactness is equivalent to feeble compactness.

The *Pixley–Roy hyperspace* $\mathrm{PR}(X)$ of a space X is the set $[X]^{<\omega}$ of all finite subsets of X endowed with the topology generated by the base consisting of the sets

$$[F, U] = \{A \in [X]^{<\omega} : F \subseteq A \subseteq U\},$$

where $F \in [X]^{<\omega}$ and U is open in X. It can be easily checked that $\mathrm{PR}(X)$ is Hausdorff and zero-dimensional, if X is Hausdorff. Moreover, if X is first-countable, then so is $\mathrm{PR}(X)$.

The notions used but not defined in this paper are standard and can be found in [7, 8, 13, 19].

§3. Main results. The majority of the results in this section are equivalences, each being based on an "embedding theorem" combined with a "non-embedding theorem", which are proved in Sections 4 and 5, respectively.

THEOREM 3.1. *The following assertions are equivalent*:

(1) $\omega_1 = \mathfrak{c}$.
(2) *Every first-countable Tychonoff space of weight $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.*
(3) *Each separable first-countable locally compact normal space of cardinality $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.*

THEOREM 3.2. *The following assertions are equivalent.*

(1) $\mathfrak{b} = \mathfrak{c}$.
(2) *Every Hausdorff locally compact first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a Hausdorff first-countable locally compact countably compact space.*
(3) *Every Hausdorff locally compact first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a Hausdorff first-countable countably compact space.*

Observe that $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ follows from $(\heartsuit)$. So, the following result generalizes Theorem 1.5.

THEOREM 3.3. *The following assertions are equivalent.*

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) *Every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.*
(3) *Every regular first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.*

Theorem 3.3 has the following zero-dimensional counterpart.

THEOREM 3.4. *The following assertions are equivalent.*

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) *Every Hausdorff zero-dimensional first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a Hausdorff zero-dimensional first-countable countably compact space.*
(3) *Every Hausdorff zero-dimensional first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a Hausdorff zero-dimensional first-countable countably compact space.*

Combining Theorem 3.3 with the results from [3] we get the following.

THEOREM 3.5. *The following assertions are equivalent*:

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) *Every regular separable first-countable non-normal space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$-rigid Nyikos space.*
(3) *Every regular separable first-countable non-normal space of cardinality $< \mathfrak{c}$ embeds into an $\mathbb{R}$-rigid Nyikos space.*

Theorem 3.5 allows us to prove the following analog of Theorem 1.4 using a milder assumption.

THEOREM 3.6 ($\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c}$). *There exists an $\mathbb{R}$-rigid Nyikos space.*

Theorems 1.2 and 3.5 imply the following corollary which shows a profound contrast in the behavior of normal and non-normal Nyikos spaces under PFA.

COROLLARY 3.7 (PFA). *The following assertions hold*:

(1) *Each normal Nyikos space is compact.*
(2) *Each regular separable first-countable space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$-rigid Nyikos space.*

The following consistency result complements Theorem 1.5.

THEOREM 3.8 ($\heartsuit$). *Every Tychonoff first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a Tychonoff first-countable countably compact space.*

However, the following question remains open:

QUESTION 3.9. *Can the assumption ($\heartsuit$) be weakened to $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ in Theorem 3.8?*

Turning to embeddings into pseudocompact spaces we obtain the following characterization.

THEOREM 3.10. *The following assertions are equivalent*:

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) *Every first-countable zero-dimensional Hausdorff space of weight $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.*
(3) *Every first-countable zero-dimensional Hausdorff space of cardinality $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.*

Recall that a subspace A of the Cantor space is called a *λ-set* if each countable subset of A is G_δ. As a by product we obtain the following characterization of λ-subsets of the Cantor space.

THEOREM 3.11. *A subspace X of the Cantor space is a λ-set if and only if the Pixley–Roy hyperspace $\mathrm{PR}(X)$ embeds densely into a first-countable pseudocompact space.*

§4. Embedding theorems. We start with embeddings into countably compact spaces.

As we mentioned in the previous section, the assumption $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$ is formally weaker than ($\heartsuit$). Indeed, fix a $P_{\mathfrak{c}}$-point p and a splitting family $\mathcal{C}$. To derive a contradiction, assume that $|\mathcal{C}| < \mathfrak{c}$. Consider the family $\mathcal{D} = \mathcal{C} \cap p$ and find an arbitrary pseudointersection $E \in p$ of the family $\mathcal{D} \cup \{\omega \setminus C : C \in \mathcal{C} \setminus p\}$. It is easy to check that there exists no $C \in \mathcal{C}$ which splits E. This contradiction implies that $\mathfrak{s} = \mathfrak{c}$. On the other hand there exists a model of ZFC which satisfies $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$, but contains no $P_{\mathfrak{c}}$-points, see [6, Theorem 8]. Hence the next theorem generalizes Theorem 1.5.

PROPOSITION 4.1 ($\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$). *Let X be a regular first-countable space of weight $\kappa < \mathfrak{c}$. Then X can be densely embedded into a regular first-countable countably compact space.*

PROOF. Without loss of generality we can assume that the underlying set of X is disjoint with $\mathfrak{c}$. The first countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathcal{D} \cup [\mathfrak{c}]^\omega \to \mathfrak{c}$ such that $h(a) \geq \sup(a)$ for any $a \in [\mathfrak{c}]^\omega$. It is easy to see that such a bijection exists. Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on $X_\alpha \subseteq X \cup \alpha$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α. At the end, we will show that the space $Y_{\mathfrak{c}}$ is regular first-countable countably compact and contains X as a dense subspace.

Let $X_0 = X$. Since X is first-countable and regular there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} B_0^x$ of the topology on X, where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is an open neighborhood base at x. With no loss of generality we can assume that $|\mathcal{B}_0| < \mathfrak{c}$; $U_{0,0}^x = X$ for each $x \in X$; and $\overline{U_{n+1,0}^x} \subset U_{n,0}^x$ for every $n \in \omega$ and $x \in X$.

Assume that for each $\alpha < \xi$ regular first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the the topology τ_α, where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is an open neighborhood base at

x. Additionally assume that $X_\alpha \subseteq X_\beta$ for any $\alpha \in \beta$ and the family $\mathcal{B}_\alpha$ satisfies the following conditions:

(1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
(2) $U_{0,\alpha}^x = X_\alpha$ for each $x \in X_\alpha$;
(3) $\mathrm{cl}_{Y_\alpha}(U_{n+1,\alpha}^x) \subset U_{n,\alpha}^x$ for every $n \in \omega$ and $x \in X_\alpha$;
(4) for every $\alpha < \beta < \xi$, $n \in \omega$ and $x \in X_\alpha$ we have that $U_{n,\alpha}^x = U_{n,\beta}^x \cap X_\alpha$.

There are three cases to consider:

1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ;
2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ;
3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$ and $\mathcal{B}_\xi = \mathcal{B}_\gamma$.
2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_n\}_{n \in \omega}$. For each $U_{n,\gamma}^x \in \mathcal{B}_\gamma$ consider the set

$$A_{U_{n,\gamma}^x} = \{k \in \omega : z_k \in U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x\}.$$

Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{s}$ we get that the family $\{A_{U_{n,\gamma}^x} : U_{n,\gamma}^x \in \mathcal{B}_\gamma\}$ is not splitting, i.e., there exists a subset $A \subset \omega$ such that for each $U_{n,\gamma}^x \in \mathcal{B}_\gamma$ the set $d_\gamma = \{z_n : n \in A\}$ is either almost contained in $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$ or almost disjoint with $U_{n,\gamma}^x \setminus U_{n+1,\gamma}^x$. Since the set $\{z_n : n \in \omega\}$ is closed in Y_γ for each $x \in X_\gamma$ there exists $n \in \omega$ such that $|U_{n,\gamma}^x \cap \{z_k : k \in \omega\}| \leq 1$. Since $U_{0,\gamma}^x = X_\gamma$ for all $x \in X_\gamma$, we obtain that for each $x \in X_\gamma$ there exists a unique $m(x) \in \omega$ such that $d_\gamma \subset^* U_{m(x),\gamma}^x \setminus U_{m(x)+1,\gamma}^x$.

For each $x \in X_\gamma$ let $\mathcal{E}(x)$ be the pair $(U_{m(x),\gamma}^x, U_{m(x)+2,\gamma}^x) \in \mathcal{B}_\gamma \times \mathcal{B}_\gamma$. Define a function $f_{\mathcal{E}(x)} \in \omega^A$ as follows: if $z_n \in U_{m(x),\gamma}^x \setminus \mathrm{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x)$, then

$$f_{\mathcal{E}(x)}(n) = \min\{k : U_{k,\gamma}^{z_n} \subset U_{m(x),\gamma}^x \setminus \mathrm{cl}_{Y_\gamma}(U_{m(x)+2,\gamma}^x)\},$$

and $f_{\mathcal{E}(x)}(n) = 0$, otherwise. Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{b}$ there exists a function $f \in \omega^A$ such that $f \geq^* f_{\mathcal{E}(x)}$ for every $U_{m(x),\gamma}^x \in \mathcal{B}_\gamma$. With no loss of generality we can additionally assume that $U_{f(n),\gamma}^{z_n} \cap U_{f(m),\gamma}^{z_m} = \emptyset$ for each distinct $n, m \in A$. It is easy to see that the sequence $\{U_{f(n),\gamma}^{z_n} : n \in A\}$ is locally finite in Y_γ.

Next we define an open neighborhood base at the point $\gamma \in X_\xi$: Put

$$U_{0,\xi}^\gamma = X_\xi \quad \text{and} \quad U_{k,\xi}^\gamma = \bigcup_{n \in A \setminus k} U_{f(n)+k,\gamma}^{z_n} \cup \{\gamma\}$$

for all $k \in \mathbb{N}$.

Now we are going to define an open neighborhood base $\mathcal{B}_\xi^x$ at each point $x \in X_\gamma$. Let $U_{0,\xi}^x = X_\xi$. For each $n \in \mathbb{N}$ let $U_{n,\xi}^x = U_{n,\gamma}^x$ if $n \geq m(x) + 1$ and $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$ if $n \leq m(x)$. It is easy to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ, and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ.

At this point it is a tedious routine to check that the family $\mathcal{B}_\xi$ satisfies conditions (1)–(4).

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$, and for every $n \in \omega$ let us put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x.$$

It is quite straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ, and the family $\mathcal{B}_\xi$ satisfies conditions (1), (2), and (4). In order to check the validity of condition (3) we need the following auxiliary claim.

CLAIM 4.2. *For any $\gamma \in \xi$, $n, m \in \omega$ and distinct points $y_0, y_1 \in Y_\gamma$, if $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$, then $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} = \emptyset$.*

PROOF. Seeking a contradiction, assume that $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} \neq \emptyset$, but $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$ for some $\gamma \in \xi$. It is easy to see that $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} \subset \xi \setminus \gamma$. Let $\delta = \min U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1}$. It follows that $U_{n,\delta}^{y_0} \cap U_{m,\delta}^{y_1} = \emptyset$ and $U_{n,\delta+1}^{y_0} \cap U_{m,\delta+1}^{y_1} = \{\delta\}$. Then the set d_δ (see case 2 above) is closed and discrete in Y_δ. Since $\delta \in U_{n,\delta+1}^{y_0} \cap U_{m,\delta+1}^{y_1}$, the definition of $U_{n,\delta+1}^{y_0}$ and $U_{m,\delta+1}^{y_1}$ implies that $d_\delta \subset^* U_{n,\delta}^{y_0} \cap U_{m,\delta}^{y_1} = \emptyset$, which contradicts our assumption. $\dashv$

Fix any $x \in X_\xi$, $n \in \omega$ and $z \in \overline{U_{n+1,\xi}^x}$. There exists an ordinal $\gamma \in \xi$ such that $x, z \in X_\gamma$. We claim that $z \in \mathrm{cl}_{Y_\gamma}(U_{n+1,\gamma}^x)$. Indeed, otherwise, there exists $m \in \omega$ such that $U_{m,\gamma}^z \cap U_{n+1,\gamma}^x = \emptyset$. The above claim implies that $U_{m,\xi}^z \cap U_{n+1,\xi}^x = \emptyset$, which contradicts the choice of z. Thus $z \in \mathrm{cl}_{Y_\gamma}(U_{n+1,\gamma}^x) \subset U_{n,\gamma}^x \subset U_{n,\xi}^x$, which establishes condition (3).

By the construction, the space $Y_{\mathfrak{c}}$ is regular, first-countable and contains X as a dense subspace. Let A be any countable subset of $Y_{\mathfrak{c}}$. If the set $B = A \cap \mathfrak{c}$ is infinite, then consider $h(B) \in \mathfrak{c}$. By the construction, either B has an accumulation point in $Y_{h(B)}$ or $h(B)$ is an accumulation point of B in $Y_{h(B)+1}$. In both cases B has an accumulation point in $Y_{\mathfrak{c}}$. If $A \subset^* X$, then either it has an accumulation point in X, or A is closed and discrete in X. In the latter case either A has an accumulation point in $Y_{h(A)}$ or $h(A)$ is an accumulation point of A in $Y_{h(A)+1}$. Thus $Y_{\mathfrak{c}}$ is countably compact. $\dashv$

PROPOSITION 4.3 ($\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$). *Let X be a Hausdorff zero-dimensional first-countable space of weight $\kappa < \mathfrak{c}$. Then X embeds densely into a Hausdorff zero-dimensional first-countable countably compact space.*

PROOF. The proof of this theorem is very similar to one of Proposition 4.1, so we give only a sketch of it. Let $\mathcal{D}$ and h be such as in the proof of Proposition 4.1.

Let $X_0 = X$. Since X is first-countable and zero-dimensional there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} \mathcal{B}_0^x$ of the topology on X consisting of clopen sets, where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is a nested open neighborhood base at x. With no loss of generality we can assume that $|\mathcal{B}_0| < \mathfrak{c}$ and $U_{0,0}^x = X$ for each $x \in X$.

Assume that for each $\alpha < \xi$ Hausdorff zero-dimensional first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the topology τ_α on a set $X_\alpha \subseteq X \cup \alpha$, where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is a

nested open neighborhood base at x consisting of clopen sets. Additionally assume that $X_\alpha \subseteq X_\beta$ for any $\alpha \in \beta$ and the family $\mathcal{B}_\alpha$ satisfies the following conditions:

(1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
(2) $U^x_{0,\alpha} = X_\alpha$ for each $x \in X_\alpha$;
(3) for every $\alpha < \beta < \xi$, $n \in \omega$ and $x \in X_\alpha$ we have that $U^x_{n,\alpha} = U^x_{n,\beta} \cap X_\alpha$.

There are three cases to consider:

1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ;

2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ;

3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$ and $\mathcal{B}_\xi = \mathcal{B}_\gamma$.

2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_n\}_{n \in \omega}$. Similarly as in the proof of Theorem 4.1, one can find a subset $A \subset \omega$ such that for each $U^x_{n,\gamma} \in \mathcal{B}_\gamma$ the set $d_\gamma = \{z_n : n \in A\}$ is either almost contained in $U^x_{n,\gamma} \setminus U^x_{n+1,\gamma}$ or almost disjoint with $U^x_{n,\gamma} \setminus U^x_{n+1,\gamma}$. Thus, for each $x \in X_\gamma$ there exists a unique $m(x) \in \omega$ such that $d_\gamma \subset^* U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma}$.

For each $x \in X_\gamma$ let us denote by $\mathcal{E}(x)$ the pair $(U^x_{m(x),\gamma}, U^x_{m(x)+1,\gamma}) \in \mathcal{B}_\gamma \times \mathcal{B}_\gamma$. Define a function $f_{\mathcal{E}(x)} \in \omega^A$ as follows: if $z_n \in U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma}$, then

$$f_{\mathcal{E}(x)}(n) = \min\{k : U^{z_n}_{k,\gamma} \subset U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma}\},$$

and $f_{\mathcal{E}(x)}(n) = 0$, otherwise. Note that $f_{\mathcal{E}(x)}$ is well-defined, as the set $U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma}$ is open for every $x \in X_\gamma$. Since $|\mathcal{B}_\gamma| < \mathfrak{c} = \mathfrak{b}$ we get that there exists a function $f \in \omega^A$ such that $f \geq^* f_{\mathcal{E}(x)}$ for every $\mathcal{E}(x) \in \mathcal{B}_\gamma \times \mathcal{B}_\gamma$. We can additionally assume that $U^{z_n}_{f(n),\gamma} \cap U^{z_m}_{f(m),\gamma} = \emptyset$ for each distinct $n, m \in A$. It is easy to see that the family $\{U^{z_n}_{f(n),\gamma} : n \in A\}$ is locally finite in Y_γ.

Next we define an open neighborhood base at the point $\gamma \in X_\xi$: Put

$$U^\gamma_{0,\xi} = X_\xi \quad \text{and} \quad U^\gamma_{k,\xi} = \bigcup_{n \in A \setminus k} U^{z_n}_{f(n)+k,\gamma} \cup \{\gamma\}$$

for all $k \in \mathbb{N}$. Since the family $\{U^{z_n}_{f(n),\gamma} : n \in A\}$ is locally finite, the set $U^\gamma_{k,\xi}$ is clopen for each $k \in \omega$.

Now we are going to define an open neighborhood base $\mathcal{B}^x_\xi$ at each point $x \in X_\gamma$. Let $U^x_{0,\xi} = X_\xi$ for every $x \in X_\gamma$. For each $x \in X_\gamma$ and $n \in \mathbb{N}$ let $U^x_{n,\xi} = U^x_{n,\gamma}$ if $n \geq m(x) + 1$ and $U^x_{n,\xi} = U^x_{n,\gamma} \cup \{\gamma\}$ if $n \leq m(x)$. It is easy to check that the family $\mathcal{B}_\xi = \{U^x_{n,\xi} : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ, and for each $x \in X_\xi$ the family $\mathcal{B}^x_\xi = \{U^x_{n,\xi} : n \in \omega\}$ forms a nested open neighborhood base at x in Y_ξ. Moreover, the space Y_ξ is Hausdorff and zero-dimensional.

At this point it is a tedious routine to check that the family $\mathcal{B}_\xi$ satisfies conditions (1)–(3).

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$. For each $x \in X_\xi$ and $n \in \omega$ put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x.$$

It is straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ, satisfies conditions (1)–(3) and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ. Hausdorffness of the space Y_ξ follows from the next claim which can be proved similarly as Claim 4.2.

CLAIM 4.4. *For any $\gamma \in \xi$, $n, m \in \omega$ and distinct points $y_0, y_1 \in Y_\gamma$, if $U_{n,\gamma}^{y_0} \cap U_{m,\gamma}^{y_1} = \emptyset$, then $U_{n,\xi}^{y_0} \cap U_{m,\xi}^{y_1} = \emptyset$.*

It remains to check that the space Y_ξ is zero-dimensional which is done in the following claim.

CLAIM 4.5. *For any $x \in X_\xi$ and $n \in \omega$ the set $U_{n,\xi}^x$ is closed.*

PROOF. Fix any $y \in X_\xi \setminus U_{n,\xi}^x$ and find $\gamma < \xi$ such that $x, y \in X_\gamma$. It follows that $y \notin U_{n,\gamma}^x$. By the inductive assumption the set $U_{n,\gamma}^x$ is closed, implying the existence of $m \in \omega$ such that $U_{m,\gamma}^y \cap U_{n,\gamma}^x = \emptyset$. Then the previous claim implies that $U_{m,\xi}^y \cap U_{n,\xi}^x = \emptyset$, witnessing that the set $U_{n,\xi}^x$ is closed. $\dashv$

Similarly as in the proof of Theorem 4.1 it can be checked that the space $Y_{\mathfrak{c}}$ is Hausdorff, zero-dimensional, first-countable, countably compact and contains X as a dense subspace. $\dashv$

Let D be a subset of a set S. An *expansion* of D is a family $\{A_d : d \in D\}$ of subsets of S such that $d \in A_d$ and $A_d \cap D = \{d\}$ for all $d \in D$. A family U of subsets of a space X is called *discrete* if every point of X has an open neighborhood intersecting at most one element of U. A topological space X satisfies (or has) *Property D* if every countable closed discrete subspace $D \subset X$ has an expansion to a discrete family of open sets. A space X is called *pseudonormal* if, whenever F_0 and F_1 are disjoint closed sets, one of which is countable, there are disjoint open sets U_0 and U_1 containing F_0 and F_1 respectively. We will also need the following facts proved in [21, Proposition 3.6] and [21, Theorem 3.7], respectively.

PROPOSITION 4.6 (Nyikos). *Every pseudonormal space X has property D.*

PROPOSITION 4.7 (Nyikos). *Every first-countable regular space of Lindelöf number $< \mathfrak{b}$ is pseudonormal.*

Propositions 4.6 and 4.7 imply the following useful corollary.

COROLLARY 4.8. *Every first-countable regular space of weight $< \mathfrak{b}$ has property D.*

The proof of the following result uses techniques developed in [1, 21, 22, 25].

PROPOSITION 4.9 ($\mathfrak{b} = \mathfrak{c}$). *Each first-countable Hausdorff locally compact space of weight $< \mathfrak{c}$ embeds densely into a first-countable Hausdorff locally compact countably compact space.*

Proof. Fix any Hausdorff locally compact first-countable space X of weight $< \mathfrak{c} = \mathfrak{b}$. Without loss of generality we can assume that the underlying set of X is disjoint with $[0, 1) \times \mathfrak{c}$. First countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D}_0 = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$ such that $h((a, b)) \geq a$.

Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on X_α which will be a union of X and some pairwise disjoint family of copies of the half interval $[0, 1)$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α. At the end, we will show that the space $Y_\mathfrak{c}$ has the desired properties.

Let $X_0 = X$ and fix any bijection $g_0 : \mathcal{D}_0 \to \{0\} \times \mathfrak{c}$. Note that by the definition of h, $h^{-1}(0) \in \{0\} \times \mathfrak{c}$, so $g_0^{-1}(h^{-1}(0))$ is well-defined (and belongs to $\in \mathcal{D}_0$). By Corollary 4.8, the space X has property D. It follows that the closed discrete set $\{z_n : n \in \omega\} = g_0^{-1}(h^{-1}(0))$ can be expanded to a discrete family $\{U_n : n \in \omega\}$ of open sets such that $z_n \in U_n$. Since X is locally compact, we lose no generality assuming that $\mathrm{cl}_X(U_n)$ is compact for every $n \in \omega$. For each $n \in \omega$ fix a continuous function f_n such that $f(z_n) = 0$ and $f(X \setminus U_n) = 1$. Let $X_1 = X_0 \cup ([0, 1) \times \{1\})$, and τ_1 be a topology on X_1 which satisfies the following conditions:

(1) X_0 is an open subspace of X_1;
(2) the sets

$$W_n(0, 1) = [0, 1/n) \times \{1\} \cup \bigcup_{m \geq n} f_m^{-1}([0, 1/n)),$$

$n \in \mathbb{N}$ form an open neighborhood base at the point $(0, 1) \in X_1$;
(3) For $0 < x < 1$ the sets

$$W_n(x, 1) = (x - 1/n, x + 1/n) \times \{1\} \cup \bigcup_{m \geq n} f_m^{-1}((x - 1/n, x + 1/n)),$$

where $n \in \mathbb{N}$ satisfies $(x - 1/n, x + 1/n) \subseteq (0, 1)$, form an open neighborhood base at the point $(x, 1) \in X_1$.

It is easy to check that Y_1 is a first-countable Hausdorff space of weight $< \mathfrak{b}$. Observe that $(0, 1)$ is an accumulation point of the set $\{z_n : n \in \omega\} = g_0^{-1}(h^{-1}(0))$ in Y_1. Clearly, Y_1 is locally compact at each point $x \in X_0$. Let us show that Y_1 is locally compact at $(0, 1)$. We claim that the set $\mathrm{cl}_{Y_1}(W_2(0, 1))$ is compact. Since the family $\{U_n : n \in \omega\}$ is discrete and the sets $U_n, n \in \omega$ have compact closure in X, it is easy to see that the set

$$\mathrm{cl}_{Y_1}(W_2(0, 1)) \subseteq [0, 1/2] \times \{1\} \cup \bigcup_{n \in \omega \setminus \{0, 1\}} \mathrm{cl}_X(U_n)$$

is σ-compact. At this point it is enough to show that $\mathrm{cl}_{Y_1}(W_2(0, 1))$ is countably compact. Fix any infinite subset $A = \{a_n : n \in \omega\} \subset \mathrm{cl}_{Y_1}(W_2(0, 1))$. If for some $n \in \omega$ the set $A \cap \mathrm{cl}_X(U_n)$ is infinite or the set $A \cap ([0, 1/2] \times \{1\})$ is infinite, then A has an accumulation point in Y_1. So, let us assume that all the aforementioned intersections are finite. Shrinking the set A if necessary, we can assume that $A \subset X$ and $|A \cap U_n| \leq 1$ for all $n \in \omega$. For each $n \in \omega$ let $m(n)$ be such that $a_n \in U_{m(n)}$. Put

$y_n = f_{m(n)}(a_n)$. Since $\{y_n : n \in \omega\} \subset [0, 1/2)$ and the interval $[0, 1/2]$ is compact there exists $z \in [0, 1/2]$ and a subsequence $\{y_{n_k} : k \in \omega\}$ which converges to z. It is easy to see that $(z, 1)$ is an accumulation point of the set A in Y_1. Hence the σ-compact set $\mathrm{cl}_{Y_1}(W_2(0, 1))$ is countably compact and thus compact. Similarly one can check that Y_1 is locally compact at $(x, 1)$ for each $0 < x < 1$. Let

$$\mathcal{D}_1 = \{A \in [X_1]^\omega : A \text{ is closed and discrete in } Y_1\}.$$

Fix any bijection $g_1 : \mathcal{D}_1 \to \{1\} \times \mathfrak{c}$.

Assume that locally compact Hausdorff spaces $Y_\alpha = (X_\alpha, \tau_\alpha)$ with weight $< \mathfrak{b}$ are constructed for all $\alpha \in \xi$ such that Y_α is an open dense subspace of Y_η for all $\alpha \in \eta \in \xi$. Additionally assume that if $X_{\alpha+1} \setminus X_\alpha \neq \emptyset$, then $X_{\alpha+1} \setminus X_\alpha = [0, 1) \times \{\alpha + 1\}$ and for each $\alpha \in \xi$ it is defined a bijection $g_\alpha : \mathcal{D}_\alpha \to \{\alpha\} \times \mathfrak{c}$, where

$$\mathcal{D}_\alpha = \{A \in [X_\alpha]^\omega : A \text{ is closed and discrete in } Y_\alpha\}.$$

If ξ is a limit ordinal, then put $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$ and $\tau_\xi = \bigcup_{\alpha \in \xi} \tau_\alpha$. Clearly, Y_ξ is first-countable, Hausdorff, locally compact with weight less than $\mathfrak{b}$. Fix any bijection $g_\xi : \mathcal{D}_\xi \to \{\xi\} \times \mathfrak{c}$.

Assume that $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$. Consider the pair $h^{-1}(\gamma) = (a, b) \in \mathfrak{c} \times \mathfrak{c}$. If the set $g_a^{-1}(h^{-1}(\gamma)) \in \mathcal{D}_a$ already has an accumulation point in Y_γ, then put $Y_\xi = Y_\gamma$. If the set $g_a^{-1}(h^{-1}(\gamma))$ is closed and discrete in Y_γ, then we construct a Hausdorff locally compact first-countable space Y_ξ of weight $< \mathfrak{b}$ similarly as we constructed Y_1. In particular, $X_\xi = X_\gamma \cup [0, 1) \times \{\xi\}$, and $(0, \xi)$ is an accumulation point of the set $g_a^{-1}(h^{-1}(\gamma))$.

Obviously, $Y_\mathfrak{c}$ is a Hausdorff locally compact first-countable space. To derive a contradiction, assume that $Y_\mathfrak{c}$ is not countably compact. Then $Y_\mathfrak{c}$ contains an infinite countable discrete closed subset A. Since the cardinal $\mathfrak{b} = \mathfrak{c}$ is regular, there exists an ordinal $\xi \in \mathfrak{c}$ such that $A \subset X_\xi$. Obviously, A is an infinite closed discrete subset of Y_β for each $\beta \geq \xi$. Then $g_\xi(A) = (\xi, \delta)$ for some $\delta \in \mathfrak{c}$. Let $\mu \in \mathfrak{c}$ be such that $\xi \leq \mu$ and $h((\xi, \delta)) = \mu$. By the construction, the point $(0, \mu + 1) \in X_{\mu+1}$ is an accumulation point of A in $Y_\mathfrak{c}$, which implies the desired contradiction. Hence the space $Y_\mathfrak{c}$ is countably compact. $\dashv$

Next we turn to embeddings into pseudocompact spaces. We adopt the notations from [5].

Let $S = \{S_n : n \in \omega\}$ and $T = \{T_n : n \in \omega\}$ be sequences of subsets of a set X. We say that S *refines* T if there exists an injection $f \in \omega^\omega$ such that $S_n \subseteq T_{f(n)}$ for all $n \in \omega$. We write that the sequence S is *eventually contained* in a subset $A \subset X$ if there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \subset A$. The sequences S and T are called *eventually disjoint* if there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \cap \bigcup_{n \geq k} T_n = \emptyset$. A family $\mathcal{R}$ of sequences of subsets of X is called *eventually disjoint* if any distinct elements $S, T \in \mathcal{R}$ are eventually disjoint.

Let X be a zero-dimensional space and $\mathcal{B}$ be a base of X consisting of clopen sets. By $\mathbf{D}(\mathcal{B})$ we denote the set of all locally finite sequences of elements of $\mathcal{B}$. A sequence $S = \{S_n : n \in \omega\} \in \mathbf{D}(\mathcal{B})$ is called $\mathcal{B}$-*exact* if for each $B \in \mathcal{B}$ which intersects infinitely many S_n there exists $k \in \omega$ such that $\bigcup_{n \geq k} S_n \subset B$. Let $\mathbf{E}(\mathcal{B}) = \{S \in \mathbf{D}(\mathcal{B}) : S \text{ is } \mathcal{B}\text{-exact}\}$.

For any eventually disjoint family $\mathcal{R} \subset \mathbf{E}(\mathcal{B})$ we define a topology τ on the set $X \cup \mathcal{R}$ as follows: for each $B \in \mathcal{B}$ let $B^* = B \cup \{R \in \mathcal{R} : R \text{ is eventually contained in } B\}$. For every $R = \{R_n : n \in \omega\} \in \mathcal{R}$ and $n \in \omega$ let $U_n(R) = \{R\} \cup \bigcup_{m \geq n} R_m^*$. The topology τ is generated by the base $\mathcal{B}^* = \{B^* : B \in \mathcal{B}\} \cup \{U_n(R) : R \in \mathcal{R}, n \in \omega\}$. By $X(\mathcal{R})$ we denote the space $(X \cup \mathcal{R}, \tau)$. It is easy to check that the space $X(\mathcal{R})$ is Hausdorff and zero-dimensional.

The following result is proved by Bell in [5, Claim 3.2].

PROPOSITION 4.10. *Let X be a first-countable zero-dimensional space and $\mathcal{B}$ be a base of X consisting of clopen sets. Suppose that for every $T \in \mathbf{D}(\mathcal{B})$ there exists $S \in \mathbf{E}(\mathcal{B})$ such that S refines T. Then for every maximal eventually disjoint subfamily $\mathcal{R}$ of $\mathbf{E}(\mathcal{B})$ the space $X(\mathcal{R})$ is zero-dimensional, first countable, and pseudocompact.*

PROPOSITION 4.11. *Each first-countable zero-dimensional Hausdorff space X of weight $< \min\{\mathfrak{s}, \mathfrak{b}\}$ embeds densely into a first-countable zero-dimensional pseudo-compact space.*

PROOF. Fix a base $\mathcal{B}$ of X of size $< \min\{\mathfrak{s}, \mathfrak{b}\}$ consisting of clopen sets. We lose no generality assuming that $\mathcal{B} = \{U_n^x : x \in X, n \in \omega\}$, where for every $x \in X$ the family $\{U_n^x : n \in \omega\}$ forms a nested open neighborhood base at x. By Proposition 4.10 it is enough to check that for every $T \in \mathbf{D}(\mathcal{B})$ there exists $S \in \mathbf{E}(\mathcal{B})$ such that S refines T. Consider an arbitrary sequence $T = \{T_n : n \in \omega\} \in \mathbf{D}(\mathcal{B})$ and for each $n \in \omega$ fix a point $x_n \in T_n$. To each $B \in \mathcal{B}$ we assign a function $f_B \in \omega^\omega$ defined as follows: if $x_n \in B$, then put $f_B(n) = \min\{k \in \omega : U_k^{x_n} \subset B \cap T_n\}$; otherwise put $f_B(n) = \min\{k \in \omega : U_k^{x_n} \subset T_n \text{ and } U_k^{x_n} \cap B = \emptyset\}$. Since $|\mathcal{B}| < \mathfrak{b}$, there exists a function $f \in \omega^\omega$ such that $f \geq^* f_B$ for every $B \in \mathcal{B}$. Consider the refinement $\{U_{f(n)}^{x_n} : n \in \omega\}$ of T. Clearly, for each $B \in \mathcal{B}$ there exists $k \in \omega$ such that for every $n \geq k$ either $U_{f(n)}^{x_n} \cap B = \emptyset$ or $U_{f(n)}^{x_n} \subset B$. So, to each $B \in \mathcal{B}$ we can assign a set $C_B = \{n \in \omega : U_{f(n)}^{x_n} \subset B\} \subset \omega$. Since $|\mathcal{B}| < \mathfrak{s}$ we get that the family $\{C_B : B \in \mathcal{B}\}$ is not splitting. Hence there exists an infinite subset A of ω possessing the following property: for every $B \in \mathcal{B}$ there exists $k \in \omega$ such that either $U_{f(n)}^{x_n} \subset B$ for each $n \in A \setminus k$ or $U_{f(n)}^{x_n} \cap B = \emptyset$ for each $n \in A \setminus k$. Hence the sequence $S = \{U_{f(n)}^{x_n} : n \in A\}$ is a $\mathcal{B}$-exact refinement of T. ⊣

§5. Non-embedding theorems.

To each almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ corresponds a *Mrowka space* $\psi(\mathcal{A}) = (\mathcal{A} \cup \omega, \tau)$, where the topology τ is defined as follows: points of ω are isolated and for each $A \in \mathcal{A}$ the family $\{\{A\} \cup (A \setminus n) : n \in \omega\}$ is an open neighborhood base at A in τ. We refer the reader to [15] for basic properties of Mrowka spaces.

PROPOSITION 5.1. *There exists an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{b}$ such that the corresponding Mrowka space $\psi(\mathcal{A})$ doesn't embed into first-countable Hausdorff countably compact spaces.*

PROOF. Fix an unbounded subset $\{b_\alpha : \alpha \in \mathfrak{b}\} \subset \omega^\omega$ which satisfies the following conditions:

(1) the function b_α is increasing for every $\alpha \in \mathfrak{b}$;
(2) $b_\alpha <^* b_\beta$ whenever $\alpha \in \beta$.

For $i \in \omega$ set $v_i = \{(i, n) : n \in \omega\}$. Let $\mathcal{A} \overset{\text{def}}{=} \{b_\alpha : \alpha \in \mathfrak{b}\} \cup \{v_i : i \in \omega\}$. Condition (2) implies that $\mathcal{A}$ is an almost disjoint family of subsets of $\omega \times \omega$. To derive a contradiction, assume that there exists a first-countable Hausdorff countably compact space X which contains the Mrowka space $\psi(\mathcal{A})$ as a subspace. Find an infinite subset C of ω such that the sequence $\{v_n : n \in C\}$ converges to a point $x \in X$. Fix an open neighborhood base $\{U_n : n \in \omega\}$ at x. Without loss of generality we can assume that $\{v_k : k \in C \setminus n\} \subset U_n$ for all $n \in \omega$. Observe that for each $n \in \omega$ there exists a function $f_n \in \omega^C$ such that $\{(k, m) : k \in C \setminus n, m \geq f_n(k)\} \subset U_n$. Since the functions $b_\alpha, \alpha \in \mathfrak{b}$ are increasing, the family $\{b_\alpha \upharpoonright_C : \alpha \in \mathfrak{b}\}$ is unbounded. Then for each $n \in \omega$ there exists $\alpha_n \in \mathfrak{b}$ such that $f_n \not\geq^* b_{\alpha_n} \upharpoonright_C$. It follows that for each $n \in \omega$ and $\xi \geq \alpha_n$,

$$|\{(k, m) : k \in C \setminus n, m \geq f_n(k)\} \cap b_\xi| = \omega.$$

Thus for $\mu = \sup\{\alpha_n : n \in \omega\}$ the set $b_\mu \cap \{(k, m) : k \in C \setminus n, m \geq f_n(k)\}$ is infinite for all $n \in \omega$. Hence $b_\mu \in \overline{U_n}$ for all $n \in \omega$, which contradicts the Hausdorffness of X.   ⊣

The following result is essentially based on the work of van Douwen and Przymusiński [9]. Throughout its proof $\mathbb{N} = \omega \setminus \{0\}$.

PROPOSITION 5.2. *There exists a separable first-countable normal Lindelöf space of cardinality $\mathfrak{s}$ which cannot be densely embedded into a first-countable regular feebly compact space.*

PROOF. By $\mathbb{Q}$ we denote the set of rational numbers from the interval $[0, 1]$. Enumerate the set of all subsets of ω as $\{A_x : x \in [0, 1]\}$, and for each $x \in [0, 1]$ and $m \in \mathbb{N}$ choose $q_x(m) \in \mathbb{Q}$ such that $0 < |x - q_x(m)| < 1/m$ and put $Q_x = \{q_x(m) : m \in \mathbb{N}\}$. Let X be the set $[0, 1] \times \{0\} \cup \mathbb{Q} \times \mathbb{N}$ endowed with a topology τ satisfying the following conditions:

- $\mathbb{Q} \times \mathbb{N}$ is the set of isolated points;
- if $x \in \mathbb{Q}$, then basic neighborhoods of a point $(x, 0) \in [0, 1] \times \{0\}$ have the form

$$B_m(x) = \{(a, b) \in X : |x - a| < 1/m\} \setminus (Q_x \times A_x \cup \{x\} \times \mathbb{N}) \qquad \text{for} \quad m \in \mathbb{N};$$

- if $x \in [0, 1] \setminus \mathbb{Q}$, then basic neighborhoods of a point $(x, 0) \in [0, 1] \times \{0\}$ have the form

$$B_m(x) = \{(a, b) \in X : |x - a| < 1/m\} \setminus (Q_x \times A_x) \qquad \text{for} \quad m \in \mathbb{N}.$$

Clearly, the space[1] X is regular, separable and first-countable.

Fix an arbitrary splitting family $\mathcal{S} \subset [\omega]^\omega$ of cardinality $\mathfrak{s}$ and consider the subspace

$$Y = \{x \in [0, 1] : A_x \in \mathcal{S}\} \times \{0\} \cup \mathbb{Q} \times \mathbb{N}$$

of X. It is clear that Y is regular and first-countable. Taking into account that the subspace $([0, 1] \times \{0\}) \cap Y$ of Y is second-countable, we deduce that Y is separable, Lindelöf and subsequently normal.

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[1]This space was originally constructed by van Douwen and Przymusiński [9] in order to investigate remainders of Čech–Stone compactifications.

Let us show that Y doesn't embed densely into a first-countable regular feebly compact space. Assuming the contrary, fix any first-countable regular feebly compact space Z which contains Y as a dense subspace. The following claim is crucial.

CLAIM 5.3. *For each $B \in [\mathbb{N}]^\omega$ there exists $q \in \mathbb{Q}$ such that $\{q\} \times B$ is not a convergent sequence in Z.*

PROOF. Fix an arbitrary infinite subset $B \subseteq \mathbb{N}$ and find $S \in \mathcal{S}$ that splits B. Let $x \in [0,1]$ be such that $A_x = S$. Fix an arbitrary open neighborhood W of $(x,0)$ in Z such that $W \cap Y \subseteq B_2(x)$. By the regularity of Z there exists an open neighborhood V of $(x,0)$ in Z such that $\mathrm{cl}_Z(V) \subseteq W$. Find $m \in \mathbb{N}$ such that $B_m(x) \subset V$. Assume that the set $\{q_x(m)\} \times B$ converges to some $b \in Z$. Then $b \in \mathrm{cl}_Z(V) \subseteq W$. Recall that $W \cap Y \subseteq B_2(x)$ and $B_2(x) \cap (\{q_x(m)\} \times \mathbb{N}) = \{q_x(m)\} \times (\mathbb{N} \setminus S)$. Since the set $B \cap S$ is infinite, the open set W contains no open neighborhood of b which contradicts the choice of W.   ⊣

Fix any enumeration $\mathbb{Q} = \{q_n : n \in \omega\}$. Since the set $Q_0 \stackrel{\mathrm{def}}{=} \{q_0\} \times \mathbb{N}$ is open and discrete in a feebly compact space Z, there exists an accumulation point z_0 of Q_0. Since Z is first-countable, there exists $A_0 \subseteq \mathbb{N}$ such that the sequence $\{(q_0, n) : n \in A_0\}$ converges to z_0. Assume that for each $i \le n$ we constructed a subset A_i of $\mathbb{N}$ satisfying:

(1) $A_i \subset A_j$ for every $i \le j$;
(2) the sequence $\{(q_i, n) : n \in A_i\}$ converges to some $z_i \in Z$.

Consider the discrete open set $Q_{n+1} \stackrel{\mathrm{def}}{=} \{q_{n+1}\} \times A_n$. Since Z is first-countable and feebly compact, there exists an infinite subset $A_{n+1} \subset A_n$ such that the sequence $\{(q_{n+1}, n) : n \in A_{n+1}\}$ converges to some point $z_{n+1} \in Z$.

After completing the induction we obtain a decreasing chain $\{A_i : i \in \omega\}$ of infinite subsets of $\mathbb{N}$ and a sequence of points $\{z_i : i \in \omega\}$ such that z_i is the limit of the sequence $\{(q_i, n) : n \in A_i\}$. Let A be any pseudointersection of the family $\{A_i : i \in \omega\}$, i.e., $A \subseteq^* A_i$ for all $i \in \omega$. It is easy to see, that the sequence $\{(q_i, n) : n \in A\}$ converges to z_i for each $i \in \omega$. But this contradicts Claim 5.3.   ⊣

The following trivial fact will be useful in the next proposition.

LEMMA 5.4. *Suppose that $A \cup B$ is a G_δ subset of a Hausdorff space X. If B is countable, then A is G_δ.*

PROOF. Let $B \setminus A = \{b_n : n \in \omega\}$ and $A \cup B = \bigcap_{n \in \omega} U_n$, where U_n is open for each $n \in \omega$. Then the set $V_n \stackrel{\mathrm{def}}{=} U_n \setminus \{b_n\}$ is open for each $n \in \omega$, and $\bigcap_{n \in \omega} V_n = A$, i.e., A is G_δ.   ⊣

For each $i \in \omega$ let $\omega^{\le i} = \bigcup_{n \le i} \omega^n$ and $\omega^{<i} = \bigcup_{n < i} \omega^n$. Clearly, the sets $\omega^{\le i}$ and $\omega^{<i}$ ordered by the inclusion are subtrees of the complete ω-ary tree. Recall that a subspace A of the Cantor space is called a *λ-set* if each countable subset of A is G_δ. For the definition of a Pixley–Roy hyperspace see Section 2. The proof of the next result uses the ideas of Bell [5].

PROPOSITION 5.5. *Let X be a subspace of the Cantor space which is not a λ-set. Then the Pixley–Roy hyperspace $\mathrm{PR}(X)$ is a zero-dimensional Tychonoff first-countable space that doesn't embed densely into first-countable regular feebly-compact spaces.*

PROOF. Let A be a countable infinite subset of X which is not G_δ. Seeking a contradiction, assume that $\mathrm{PR}(X)$ embeds densely into a regular first-countable feebly compact space Z. Fix any countable dense subset A' of X. Lemma 5.4 implies that $Q = A \cup A'$ is not G_δ. Fix an increasing sequence $\{F_k^\emptyset : k \in \omega\}$ of finite subsets of Q such that $\bigcup_{k \in \omega} F_k^\emptyset = Q$. Let $B_\emptyset \overset{\text{def}}{=} X$. Consider the family $\{\mathrm{cl}_Z[F_k^\emptyset, X] : k \in \omega\}$. Taking into account that $\mathrm{PR}(X)$ is dense in Z and $[F_k^\emptyset, B_\emptyset]$ is clopen in $\mathrm{PR}(X)$, it is easy to check that $[F_k^\emptyset, B_\emptyset] \subset \mathrm{int}_Z(\mathrm{cl}_Z[F_k^\emptyset, B_\emptyset])$ for all $k \in \omega$. So, the family $\{\mathrm{int}_Z(\mathrm{cl}_Z[F_k^\emptyset, B_\emptyset]) : k \in \omega\}$ consists of open subsets of Z and is centered. By feeble compactness of Z, there exists

$$z_\emptyset \in \bigcap_{k \in \omega} \mathrm{cl}_Z(\mathrm{int}_Z(\mathrm{cl}_Z[F_k^\emptyset, B_\emptyset])) = \bigcap_{k \in \omega} \mathrm{cl}_Z[F_k^\emptyset, B_\emptyset].$$

Fix a countable open neighborhood base $\{O_k^\emptyset : k \in \omega\}$ at $z_\emptyset$. Since $\mathrm{PR}(X)$ is dense in Z, for each $k \in \omega$ there exists a basic clopen subset $[G_k^\emptyset, W_k^\emptyset]$ of $\mathrm{PR}(X)$ such that $[G_k^\emptyset, W_k^\emptyset] \subseteq O_k^\emptyset \cap [F_k^\emptyset, B_\emptyset]$. Clearly, $F_k^\emptyset \subseteq G_k^\emptyset$ and the sequence $\{G_k^\emptyset : k \in \omega\} \subset \mathrm{PR}(X)$ converges to $z_\emptyset$.

Suppose that for some $i \geq 0$ we accomplished the following things:

(1) constructed a subtree $\mathbb{T}_i$ of $\omega^{\leq i}$;
(2) constructed a family $\{B_s : s \in \mathbb{T}_i\}$ of clopen subsets of X satisfying the following conditions:
    (2.1) for every $s \in \mathbb{T}_i \cap \omega^{<i}$, $Q_s \overset{\text{def}}{=} Q \cap B_s \subseteq \bigcup_{n \in \beta_s} B_{s\hat{\ }n} \subseteq B_s$, where $\beta_s \overset{\text{def}}{=} \{n \in \omega : s\hat{\ }n \in \mathbb{T}_i\} \in \omega + 1$;
    (2.2) $B_{s\hat{\ }n} \cap B_{s\hat{\ }m} = \emptyset$ for every $s \in \mathbb{T}_i \cap \omega^{<i}$ and distinct $n, m \in \beta_s$;
    (2.3) for each $s \in \mathbb{T}_i$ the diameter of B_s doesn't exceed $1/|s|$.
(3) $\forall s \in \mathbb{T}_i$ fixed an increasing sequence $\{F_k^s : k \in \omega\}$ of finite subsets of $Q_s \overset{\text{def}}{=} B_s \cap Q$ such that $\bigcup_{k \in \omega} F_k^s = Q_s$;
(4) $\forall s \in \mathbb{T}_i$ fixed a point $z_s \in \bigcap_{k \in \omega} \mathrm{cl}_Z[F_k^s, B_s]$ and a nested open neighborhood base $\{O_k^s : k \in \omega\}$ at z_s;
(5) $\forall s \in \mathbb{T}_i$, $\forall k \in \omega$ fixed a subset $[G_k^s, W_k^s] \subset O_k^s \cap [F_k^s, B_s]$ such that the following condition holds:
    (5.1) $\forall s \in \omega^{<i} \cap \mathbb{T}_i$, $\forall n \in \beta_s$, $\exists k(s, n) \geq |s| + 1$ such that $B_{s\hat{\ }n} \subseteq \bigcap_{j \leq |s|} \times W_{k(s,n)}^{s\upharpoonright j}$.

Fix any $s \in \mathbb{T}_i \cap \omega^i$. Since B_s is open in X and Q is dense, the set $Q_s \overset{\text{def}}{=} Q \cap B_s$ is not empty. Note that since X could contain isolated points, it is possible that $|Q_s| < \omega$. So, let us fix an enumeration $Q_s = \{q_n^s : n \in \alpha_s\}$, where $\alpha_s \in \omega + 1$. Observe that $q_0^s \in Q \cap B_{s\upharpoonright j}$ for all $j \leq |s|$. Since $\{F_k^{s\upharpoonright j} : k \in \omega\}$ is an increasing sequence of finite subsets of $Q \cap B_{s\upharpoonright j}$, there exists $n_j \in \omega$ such that $q_0^s \in F_n^{s\upharpoonright j}$ for all $n \geq n_j$. Put $k(s, 0) = \max\{n_j : j \leq |s|\} + |s| + 1$ and note that the open set $\bigcap_{j \leq |s|} W_{k(s,0)}^{s\upharpoonright j}$ contains q_0^s. So fix an arbitrary clopen set $B_{s\hat{\ }0} \subseteq \bigcap_{j \leq |s|} W_{k(s,0)}^{s\upharpoonright j}$ which contains q_0^s and has diameter $\leq 1/(i + 1)$. By condition (5), we get that $W_{k(s,0)}^s \subseteq B_s$ which implies that $B_{s\hat{\ }0} \subseteq B_s$. If $Q_s \subset B_{s\hat{\ }0}$, we put $\beta_s = 1$ and move to another $s \in \mathbb{T}_i \cap \omega^i$. If $Q_s \setminus B_{s\hat{\ }0} \neq \emptyset$, find minimal $m \in \omega$ such that $q_m^s \notin B_{s\hat{\ }0}$. Similarly as above there exists $k(s, 1) \geq |s| + 1$ such that the open set $\bigcap_{j \leq |s|} W_{k(s,1)}^{s\upharpoonright j}$ contains q_m^s. Since the

set $B_{s^{\smallfrown}0}$ is clopen and doesn't contain q_m^s, we can find a clopen neighborhood $B_{s^{\smallfrown}1}$ of q_m^s of diameter $\leq 1/(i+1)$ which is contained in $(B_s \setminus B_{s^{\smallfrown}0}) \cap \bigcap_{j \leq |s|} W_{k(s,1)}^{s \restriction j}$. If $Q_s \subset B_{s^{\smallfrown}0} \cup B_{s^{\smallfrown}1}$, we put $\beta_s = 2$ and move to another $s \in \mathbb{T}_i \cap \omega^i$. Otherwise, we can find minimal $m \in \omega$ such that $q_m^s \in B_s \setminus (B_{s^{\smallfrown}0} \cup B_{s^{\smallfrown}1})$ and repeat the previous steps. Proceeding this way we shall construct a family $\{B_{s^{\smallfrown}k} : k \in \beta_s \in \omega + 1\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3), and (5.1). Note that $\beta_s \leq \alpha_s$ can be finite. Let $\mathbb{T}_{i+1}^s = \mathbb{T}_i \cup \{s^{\smallfrown}k : k \in \beta_s\}$.

After making the above construction for all $s \in \mathbb{T}_i \cap \omega^i$, we obtain a tree $\mathbb{T}_{i+1} = \bigcup_{s \in \mathbb{T}_i \cap \omega^i} \mathbb{T}_{i+1}^s$ and a family $\{B_{s^{\smallfrown}k} : k \in \beta_s \text{ and } s \in \mathbb{T}_i \cap \omega^i\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3), and (5.1).

For each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ fix an increasing sequence $\{F_k^s : k \in \omega\}$ of finite subsets of Q_s such that $\bigcup_{k \in \omega} F_k^s = Q_s$. Analogous arguments to those we used to find $z_\emptyset$ imply that for each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ there exists a point $z_s \in \bigcap_{k \in \omega} \mathrm{cl}_Z[F_k^s, B_s]$. By the first-countability of Z, we can fix a nested open neighborhood base $\{O_k^s : k \in \omega\}$ at z_s. For each $s \in \mathbb{T}_{i+1} \cap \omega^{i+1}$ and $k \in \omega$ fix a subset $[G_k^s, W_k^s] \subset O_k^s \cap [F_k^s, B_s]$. This finishes step $i+1$ of the induction.

Upon completing the induction we obtain a tree $\mathbb{T} = \bigcup_{i \in \omega} \mathbb{T}_i$, and a family $\{B_s : s \in \mathbb{T}\}$ of clopen subsets of X which satisfies conditions (2.1), (2.2), (2.3), and (5.1). Also, for each $s \in \mathbb{T}$ we found an element $z_s \in Z$ which satisfies condition (4) and constructed the families $\{W_k^s : k \in \omega, s \in \mathbb{T}\}$ (consisting of open subsets of X) and $\{G_k^s : k \in \omega, s \in \mathbb{T}\}$ (consisting of finite subsets of X), which satisfy conditions (5) and (5.1).

Let

$$P = \bigcap_{i \in \omega} \bigcup_{s \in \omega^i \cap \mathbb{T}} B_s.$$

By the construction, P is a G_δ-set containing Q. It follows that $|P| > \omega$. Indeed, otherwise we would have that $Q = P \cap \bigcap_{x \in P \setminus Q}(X \setminus \{x\})$ is a G_δ subset of X, which is not the case. Thus, we can fix

$$p_\infty \in P \setminus \bigcup_{s \in \mathbb{T}} \bigcup_{k \in \omega} G_k^s. \tag{1}$$

It is easy to see that there exists $t \in \omega^\omega$ such that $p_\infty \in B_{t \restriction i}$ for any $i \in \omega$. Note that since the diameter of $B_{t \restriction i}$ doesn't exceed $1/i$, we get that the family $\{B_{t \restriction i} : i \in \omega\}$ forms an open neighborhood base at p_∞ in X. It follows that the family $\{[\{p_\infty\}, B_{t \restriction i}] : i \in \omega\}$ forms an open neighborhood base of $\{p_\infty\}$ in $\mathrm{PR}(X)$. Put $R = \mathrm{int}_Z(\mathrm{cl}_Z([\{p_\infty\}, X]))$. One can check that $R \cap \mathrm{PR}(X) = [\{p_\infty\}, X]$. By the regularity of Z, there exists an open in Z neighborhood W of $\{p_\infty\}$ such that $\mathrm{cl}_Z(W) \subseteq R$. Thus, there exists $i \in \omega$ such that $\mathrm{cl}_Z([\{p_\infty\}, B_{t \restriction i}]) \subseteq R$.

CLAIM 5.6. $z_{t \restriction i} \in \mathrm{cl}_Z([\{p_\infty\}, B_{t \restriction i}])$.

PROOF. Pick any integer $l > i$. Using condition (5.1) for $s = t \restriction l$ and $n = t(l)$ we can find an integer $k(t \restriction l, t(l)) \geq l + 1$ such that $B_{t \restriction (l+1)} \subseteq W_{k(t \restriction l, t(l))}^{t \restriction i}$. Then $p_\infty \in W_k^{t \restriction i}$ for infinitely many $k \in \omega$. It follows that $[\{p_\infty\}, B_{t \restriction i}] \cap [G_k^{t \restriction i}, W_k^{t \restriction i}] \neq \emptyset$ for infinitely many k. Since for each $k \in \omega$, $[G_k^{t \restriction i}, W_k^{t \restriction i}] \subset O_k^{t \restriction i}$ and the family

$\{O_k^{t \restriction i} : k \in \omega\}$ forms an open neighborhood base at $z_{t \restriction i}$, we deduce that $z_{t \restriction i} \in$ $\mathrm{cl}_Z([\{p_\infty\}, B_{t \restriction i}])$.   ⊣

Since $\mathrm{cl}_Z([\{p_\infty\}, B_{t \restriction i}]) \subseteq R$, Claim 5.6 implies that $z_{t \restriction i} \in R$. Condition (5) implies that for each $s \in \mathbb{T}$ the sequence $\{G_k^s : k \in \omega\} \subseteq \mathrm{PR}(X)$ converges to z_s. Since the set R is an open neighborhood of $z_{t \restriction i}$, there exists $k \in \omega$ such that $G_k^{t \restriction i} \in R \cap \mathrm{PR}(X) = [\{p_\infty\}, X]$, which is impossible, as $p_\infty \notin \bigcup_{s \in \mathbb{T}} \bigcup_{k \in \omega} G_k^s$ (see Equation 1).   ⊣

By [7, Corollary 8.51] there exists a subspace A of the Cantor space which is not a λ-set and has cardinality $\mathfrak{b}$. Note that $|\mathrm{PR}(A)| = |A| = \mathfrak{b}$. Hence Proposition 5.5 implies the following.

COROLLARY 5.7.   *There exists a Tychonoff zero-dimensional first-countable space X of cardinality $\mathfrak{b}$ which doesn't embed densely into regular first-countable feebly compact spaces.*

§6. Proofs of the main results.

Proof of Theorem 3.1. We need to show that the following assertions are equivalent:
(1) $\omega_1 = \mathfrak{c}$.
(2) Every first-countable Tychonoff space of weight $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.
(3) Each separable first-countable locally compact normal space of cardinality $< \mathfrak{c}$ embeds into a Hausdorff first-countable compact space.

PROOF.   Implication $(1) \Rightarrow (2)$ follows from the fact that Each Tychonoff second-countable space is a subspace of the Tychonoff cube $[0, 1]^\omega$. Implication (2) $\Rightarrow (3)$ follows from the fact that the weight of a first-countable space doesn't exceed its cardinality. Reference [4, Proposition 2.11] implies the existence of a separable normal locally compact first-countable space X which contains a homeomorphic copy of the cardinal ω_1 endowed with the order topology. Since the only compactification of ω_1 is $\omega_1 + 1$, which is not first-countable, we deduce that X cannot be embedded into a Hausdorff first-countable compact space.   ⊣

Proof of Theorem 3.2. We need to show that the following statements are equivalent.
(1) $\mathfrak{b} = \mathfrak{c}$.
(2) Every Hausdorff, locally compact, first-countable space of weight $< \mathfrak{c}$ can be (densely) embedded in a Hausdorff first-countable locally compact countably compact space.
(3) Every Hausdorff, locally compact, first-countable space of cardinality $< \mathfrak{c}$ can be (densely) embedded in a Hausdorff first-countable countably compact space.

PROOF.   The implication $(1) \Rightarrow (2)$ follows from Proposition 4.9. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication $(2) \Rightarrow$ (3) follows. Taking into account that Mrowka spaces are first-countable and locally compact, implication $(3) \Rightarrow (1)$ follows from Proposition 5.1.   ⊣

Proof of Theorem 3.3. We need to show that the following assertions are equivalent.

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) Every regular first-countable space of weight $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.
(3) Every regular first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded in a regular first-countable countably compact space.

PROOF. Implication $(1) \Rightarrow (2)$ follows from Proposition 4.1. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication $(2) \Rightarrow (3)$ holds true.

$(3) \Rightarrow (1)$. Proposition 5.1 implies that $\mathfrak{b} = \mathfrak{c}$. Proposition 5.2 implies that $\mathfrak{s} = \mathfrak{c}$.   ⊣

Proof of Theorem 3.4. We need to show that the following assertions are equivalent.

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) Every Hausdorff zero-dimensional first-countable space of weight $< \mathfrak{c}$ can be densely embedded into a Hausdorff zero-dimensional first-countable countably compact space.
(3) Every Hausdorff zero-dimensional first-countable space of cardinality $< \mathfrak{c}$ can be densely embedded into a Hausdorff zero-dimensional first-countable countably compact space.

PROOF. Implication $(1) \Rightarrow (2)$ follows from Proposition 4.3. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication $(2) \Rightarrow (3)$ holds true.

$(3) \Rightarrow (1)$ Proposition 5.1 implies that $\mathfrak{b} = \mathfrak{c}$. Recall that a Tychonoff first-countable space Y constructed in the proof of Proposition 5.2 has cardinality $\mathfrak{s}$. If $\mathfrak{s} < \mathfrak{c}$, then the space Y is zero-dimensional (as every Tychonoff space of cardinality $< \mathfrak{c}$ is zero-dimensional), and Y doesn't embed into regular first-countable countably compact spaces, which contradicts assertion (3). Hence $\mathfrak{s} = \mathfrak{c}$, i.e., assertion (1) holds.   ⊣

In order to prove Theorem 3.5 we need the following auxiliary lemma.

LEMMA 6.1. *There exists a regular zero-dimensional separable first-countable non-normal space of cardinality ω_1.*

PROOF. We shall use ideas of Nyikos and Vaughan [23]. Consider a Hausdorff (ω_1, ω_1) gap, i.e., a family $\{A_\alpha : \alpha \in \omega_1\} \cup \{B_\alpha : \alpha \in \omega_1\} \subset [\omega]^\omega$ which satisfies the following conditions:

- $A_\alpha \subset^* B_\beta$ for any $\alpha, \beta \in \omega_1$;
- $A_\alpha \subset^* A_\beta$ for any $\alpha < \beta < \omega_1$;
- $B_\alpha \subset^* B_\beta$ for any $\beta < \alpha < \omega_1$;
- there exists no $C \in [\omega]^\omega$ such that $A_\alpha \subset^* C \subset^* B_\alpha$ for all $\alpha \in \omega_1$.

For a finite set F and ordinals $\alpha < \beta < \omega_1$ let us introduce two sorts of sets:

$$A(\alpha, \beta, F) = \{A_\xi \mid \alpha < \xi \le \beta\} \cup ((A_\beta \setminus A_\alpha) \setminus F) \quad \text{and}$$

$$B(\alpha, \beta, F) = \{B_\xi \mid \alpha < \xi \leq \beta\} \cup ((B_\alpha \setminus B_\beta) \setminus F).$$

Let N be the set $\{A_\alpha : \alpha \in \omega_1\} \cup \{B_\alpha : \alpha \in \omega_1\} \cup \omega$ endowed with the topology τ defined by the subbase:

$$\mathcal{B} = \{\{n\} : n \in \omega\} \cup \{A_0\} \cup \{B_0\} \cup \{A(\alpha, \beta, F) : \alpha < \beta < \omega_1, F \in [\omega]^{<\omega}\} \cup$$
$$\cup \{B(\alpha, \beta, F) : \alpha < \beta < \omega_1, F \in [\omega]^{<\omega}\}.$$

It is straightforward to check that the space N is regular, separable, first-countable and has cardinality ω_1. Also, the closed sets $\{A_\alpha : \alpha \in \omega_1\}$ and $\{B_\alpha : \alpha \in \omega_1\}$ cannot be separated by disjoint open sets. Hence the space N is not normal. $\dashv$

Proof of Theorem 3.5. We need to show the equivalence of the following assertions:

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) Every regular separable first-countable non-normal space of weight $< \mathfrak{c}$ embeds into a Nyikos $\mathbb{R}$-rigid space.
(3) Every regular separable first-countable non-normal space of cardinality $< \mathfrak{c}$ embeds into a Nyikos $\mathbb{R}$-rigid space.

PROOF. $(1) \Rightarrow (2)$. Let X be a regular separable first countable non-normal space of weight $< \mathfrak{c}$. Applying Jones machine (see [3] or [18] for details) we embed X into a first-countable regular separable space $J(X)$ of weight $< \mathfrak{c}$ containing two points a, b which cannot be separated by any real-valued continuous function. Applying the modified van Douwen's extension $E_{a,b}(J(X))$ of $J(X)$ (see [3, Section 5]) we embed $J(X)$ into a regular separable first-countable $\mathbb{R}$-rigid space of weight $< \mathfrak{c}$. By Theorem 4.1 we get a regular first-countable countably compact space Z which contains the separable $\mathbb{R}$-rigid space $E_{a,b}(J(X))$ as a dense subspace. Hence Z is an $\mathbb{R}$-rigid Nyikos space which contains a homeomorphic copy of X.

The implication $(2) \Rightarrow (3)$ holds true, because the weight of an infinite first-countable space doesn't exceed its cardinality.

$(3) \Rightarrow (1)$. Let N_1 be the topological sum of the space N constructed in Lemma 6.1 and the Mrowka space $\psi(\mathcal{A})$ constructed in Proposition 5.1. It is clear that N_1 is a regular separable first-countable non-normal space of cardinality $\mathfrak{b}$ which doesn't embed into Nyikos spaces. This example implies that $\mathfrak{b} = \mathfrak{c}$.

Let N_2 be the topological sum of the space N constructed in Lemma 6.1 and the space Y constructed in Proposition 5.2. It can be checked that N_2 is a separable regular first-countable non-normal space of cardinality $\mathfrak{s}$ which doesn't embed into Nyikos spaces. Hence $\mathfrak{s} = \mathfrak{c}$. $\dashv$

Proof of Theorem 3.6. Assuming that $\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ we need to construct an $\mathbb{R}$-rigid Nyikos space.

PROOF. Consider the space N constructed in Lemma 6.1 and apply item (3) of Theorem 3.5 to it. $\dashv$

Proof of Corollary 3.7. We need to show that under PFA the following assertions hold:

(1) Every normal Nyikos space is compact.
(2) Every regular separable first-countable space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$-rigid Nyikos space.

PROOF. Assertion (1) follows from Theorem 1.2. In order to prove assertion (2) fix any regular separable first-countable space X of weight $< \mathfrak{c}$. Let N be the space constructed in Lemma 6.1. Recall that PFA implies $\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c}$. Hence the topological sum $X \cup N$ is a regular separable first-countable non-normal space of weight $< \mathfrak{c}$. By Theorem 3.5(2), $X \cup N$ embeds into an $\mathbb{R}$-rigid Nyikos space. $\dashv$

In order to prove Theorem 3.8 we shall need the following auxiliary lemma.

LEMMA 6.2. *Let X be a first-countable Tychonoff space of weight κ. Then for each $x \in X$ there exists an open neighborhood base $\mathcal{B}_x = \{U_n^x : n \in \omega\}$ at x such that $U_{n+1}^x \subseteq U_n^x$ for any $n \in \omega$, and $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a base of X of size κ satisfying the following condition*:

(i) *for each $x \in X$ and $U_n^x \in \mathcal{B}_x$ there exists $f_{U_n^x} \in C(X, [0,1])$ such that $f_{U_n^x} {\restriction}_{X \setminus U_n^x} \equiv 0$ and $f_{U_n^x} {\restriction}_{U_{n+1}^x} \equiv 1$.*

PROOF. Let $\mathcal{D}$ be a base of X of size κ. Put

$$\mathcal{Y} = \{\langle D_0, D_1 \rangle \in \mathcal{D}^2 : \exists f_{\langle D_0, D_1 \rangle} \in C(X, [0,1]) \text{ such that}$$
$$f_{\langle D_0, D_1 \rangle} {\restriction}_{D_1} \equiv 1 \text{ and } f_{\langle D_0, D_1 \rangle} {\restriction}_{X \setminus D_0} \equiv 0\}.$$

CLAIM 6.3. *For every $x \in X$ and open neighborhood U of x there exists $\langle D_0, D_1 \rangle \in \mathcal{Y}$ such that $x \in D_1 \subset D_0 \subset U$.*

PROOF. Fix any $x \in X$ and an open neighborhood U of x. Since $\mathcal{D}$ is a base, there exists an open set $D_0 \in \mathcal{D}$ such that $x \in D_0 \subseteq U$. Then there exists a continuous function $h \in C(X, [0,1])$ such that $h {\restriction}_{X \setminus D_0} \equiv 0$ and $h(x) = 1$. Since the set $h^{-1}([1/2, 1])$ contains x in its interior, there exists $D_1 \in \mathcal{D}$ such that $D_1 \subset h^{-1}([1/2, 1])$. Clearly, $f(z) \stackrel{\text{def}}{=} 2 \min\{h(z), 1/2\} \in C(X, [0,1])$. It is easy to see that $f {\restriction}_{D_1} \equiv 1$ and $f {\restriction}_{X \setminus D_0} \equiv 0$, i.e., $\langle D_0, D_1 \rangle \in \mathcal{Y}$. $\dashv$

Fix any $x \in X$ and an open neighborhood base $\mathcal{V} = \{V_n : n \in \omega\} \subset \mathcal{D}$ of x such that $V_{n+1} \subset V_n$ for all $n \in \omega$. Wlog we can assume that $V_0 = X$. Let $U_0^x = X$ and $f_{U_0^x}$ be the constant map from X to $[0,1]$ with value 1. Claim 6.3 implies that there exist a pair $\langle D_0^1, D_1^1 \rangle \in \mathcal{Y}$ such that $x \in D_0^1 \subset D_1^1 \subseteq V_1$ and a function $f_{\langle D_0^1, D_1^1 \rangle} \in C(X, [0,1])$ such that $f_{\langle D_0^1, D_1^1 \rangle} {\restriction}_{D_1^1} \equiv 1$ and $f_{\langle D_0^1, D_1^1 \rangle} {\restriction}_{X \setminus D_0^1} \equiv 0$. Put $U_1^x = D_0^1$ and $f_{U_1^x} = f_{\langle D_0^1, D_1^1 \rangle}$. Using Claim 6.3 again we can find a pair $\langle D_0^2, D_1^2 \rangle \in \mathcal{Y}$ such that $x \in D_0^2 \subset D_1^2 \subseteq V_2 \cap D_1^1$ and a function $f_{\langle D_0^2, D_1^2 \rangle} \in C(X, [0,1])$ such that $f_{\langle D_0^2, D_1^2 \rangle} {\restriction}_{D_1^2} \equiv 1$ and $f_{\langle D_0^2, D_1^2 \rangle} {\restriction}_{X \setminus D_0^2} \equiv 0$. Put $U_2^x = D_0^2$ and $f_{U_2^x} = f_{\langle D_0^2, D_1^2 \rangle}$. Proceeding this way, for each $x \in X$ we obtain an open neighborhood base $\{U_n^x : n \in \omega\}$ at x and a family $\{f_{U_n^x} : U_n^x \in \mathcal{B}_x\} \subset C(X, [0,1])$ such that $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ satisfies condition (i). Since $\mathcal{B} \subseteq \mathcal{D}$ we get that $|\mathcal{B}| \leq \kappa$. $\dashv$

Recall that by $(\heartsuit)$ we denote the following consistent assumption: "$\mathfrak{b} = \mathfrak{c}$ and there exists a $P_{\mathfrak{c}}$-point in $\beta(\omega)$".

Proof of Theorem 3.8. We need to show that assuming $(\heartsuit)$ every Tychonoff first-countable space X of weight $\kappa < \mathfrak{c}$ embeds densely into a Tychonoff first-countable countably compact space.

PROOF.   Let X be a first-countable Tychonoff space such that $w(X) < \mathfrak{c}$. Without loss of generality we can assume that the underlying set of X is disjoint with $\mathfrak{c}$. The first countability of X implies that $|X| \leq \mathfrak{c}^\omega = \mathfrak{c}$. If X is countably compact, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{A \in [X]^\omega : A \text{ is closed and discrete in } X\}.$$

Fix any bijection $h : \mathcal{D} \cup [\mathfrak{c}]^\omega \to \mathfrak{c}$ such that $h(a) \geq \sup(a)$ for any $a \in [\mathfrak{c}]^\omega$. Next, for every $\alpha \leq \mathfrak{c}$ we shall recursively construct a topology τ_α on $X_\alpha \subseteq X \cup \alpha$. For the sake of brevity we denote the space (X_α, τ_α) by Y_α. At the end, we will show that the space $Y_\mathfrak{c}$ has the desired properties.

Let $X_0 = X$. By Lemma 6.2, there exists a base $\mathcal{B}_0 = \bigcup_{x \in X} \mathcal{B}_0^x$ of size $< \mathfrak{c}$ of the topology on X, where for each $x \in X$, the collection $\mathcal{B}_0^x = \{U_{n,0}^x : n \in \omega\}$ is a nested open neighborhood base at x and satisfies the following: $U_{0,0}^x = X$; for every $n > 0$ and $x \in X_0$ there exists a continuous function $f_{(U_{n,0}^x, U_{n+1,0}^x)} : X \to [0, 1]$ such that $f_{(U_{n,0}^x, U_{n+1,0}^x)} \restriction X \setminus U_{n,0}^x \equiv 0$ and $f_{(U_{n,0}^x, U_{n+1,0}^x)} \restriction U_{n+1,0}^x \equiv 1$. We also set $f_{(U_{0,0}^x, U_{1,0}^x)} \equiv 1$.

Assume that for each $\alpha < \xi$ the Tychonoff first-countable spaces Y_α are already constructed by defining a base $\mathcal{B}_\alpha = \bigcup_{x \in X_\alpha} \mathcal{B}_\alpha^x$ of the the topology τ_α on a set $X_\alpha \subseteq X \cup \alpha$, where for each $x \in X_\alpha$, the collection $\mathcal{B}_\alpha^x = \{U_{n,\alpha}^x : n \in \omega\}$ is a nested open neighborhood base at x. Additionally assume that $X_\alpha \subset X_\beta$ for any $\alpha \in \beta$, and we have in parallel constructed a family

$$\Phi_\alpha = \{f_{(U_{n,\alpha}^x, U_{n+1,\alpha}^x)} : n \in \omega, x \in X_\alpha\},$$

where for each $(U_{n,\alpha}^x, U_{n+1,\alpha}^x)$ the function $f_{(U_{n,\alpha}^x, U_{n+1,\alpha}^x)} : Y_\alpha \to [0, 1]$ is continuous and

$$f_{(U_{n,\alpha}^x, U_{n+1,\alpha}^x)} \restriction X_\alpha \setminus U_{n,\alpha}^x \equiv 0.$$

In what follows, for brevity, we put $E(x, n, \alpha) = (U_{n,\alpha}^x, U_{n+1,\alpha}^x)$ for all $n \in \omega$, $\alpha < \xi$ and $x \in X_0$.

The families $\mathcal{B}_\alpha$ and Φ_α are presumed to satisfy the following conditions for all $\alpha < \xi$:

(1) $|\mathcal{B}_\alpha| < \mathfrak{c}$;
(2) $U_{0,\alpha}^x = X_\alpha$ for each $x \in X_\alpha$;
(3) $f_{E(x,0,\alpha)} \equiv 1$ for any $x \in X_\alpha$;
(4) for every $x \in X_\alpha$ and $n \in \omega$ we have $f_{E(x,n,\alpha)} \restriction U_{n+1,\alpha}^x \equiv 1$ and
$$f_{E(x,n,\alpha)} \restriction X_\alpha \setminus U_{n,\alpha}^x \equiv 0;$$
(5) for every $\beta < \alpha$, $n \in \omega$ and $x \in X_\beta$ we have $U_{n,\beta}^x = U_{n,\alpha}^x \cap X_\beta$ and $\mathrm{cl}_{Y_\beta}(U_{n,\beta}^x) = \mathrm{cl}_{Y_\alpha}(U_{n,\alpha}^x)$;
(6) for every $\beta < \alpha$, $n \in \omega$ and $x \in X_\beta$ we have $f_{E(x,n,\alpha)} \restriction X_\beta = f_{E(x,n,\beta)}$.

There are three cases to consider:

1) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is not an infinite closed discrete subset of Y_γ;

2) $\xi = \gamma + 1$ for some $\gamma \in \mathfrak{c}$ and $h^{-1}(\gamma) \cap X_\gamma$ is an infinite closed discrete subset of Y_γ;

3) ξ is a limit ordinal.

1) Let $X_\xi = X_\gamma$, $\mathcal{B}_\xi = \mathcal{B}_\gamma$ and $\Phi_\xi = \Phi_\gamma$.

2) Put $X_\xi = X_\gamma \cup \{\gamma\}$. Let $h^{-1}(\gamma) = \{z_i\}_{i \in \omega}$. Fix any $P_\mathfrak{c}$-point p which exists by the assumption. Since the set $h^{-1}(\gamma)$ is closed and discrete, for any $x \in X_\gamma$ there exists a positive integer $k(x)$ such that $|U^x_{k(x),\gamma} \cap h^{-1}(\gamma)| \leq 1$. Since $U^x_{0,\gamma} = X_\gamma$ and the ultrafilter p is free, for any $x \in X_\gamma$ there exists $m(x) < k(x)$ such that

$$F_x = \{i \in \omega : z_i \in U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma}\} \in p.$$

Note that the set F_x actually depends on the pair $\left(U^x_{m(x),\gamma}, U^x_{m(x)+1,\gamma}\right) = E(x, m(x), \gamma)$.

Fix an arbitrary $f \in \Phi_\gamma$. By the compactness of $[0, 1]$, the ultrafilter $f(p)$ on $[0, 1]$ generated by the family $\{f(P) : P \in p\}$ converges to a point $b \in [0, 1]$. That is for each $n \in \mathbb{N}$

$$P_{f,n} = \{i \in \omega : f(z_i) \in (b - 1/n, b + 1/n)\} \in p.$$

Since p is a P-point, there exists $G_f \in p$ such that $G_f \subset^* P_{f,1/n}$ for every $n \in \mathbb{N}$. It is easy to see that the sequence $\{f(z_i) : i \in G_f\}$ converges to b.

Since $|\mathcal{B}_\gamma \cup \Phi_\gamma| < \mathfrak{c}$ we obtain that $|\{F_x : x \in X_\gamma\} \cup \{G_f : f \in \Phi_\gamma\}| < \mathfrak{c}$, and hence there exists $F \in p$ such that $F \subset^* F_x \cap G_f$ for any $x \in X_\gamma$ and $f \in \Phi_\gamma$.

For any $n \in \omega$ and $x \in X_\gamma$ set

$$b^x_n = \lim_{i \in F} f_{E(x,n,\gamma)}(z_i)$$

and note that $b^x_n = 0$ for $n > m(x)$.

Let $d_\gamma = \{z_i : i \in F\}$. Condition (4) implies that $\mathrm{cl}_{Y_\gamma}(U^x_{n+1,\gamma}) \subseteq U^x_{n,\gamma}$ for every $x \in X_\gamma$ and $n \in \omega$. Then for any $x \in X_\gamma$ we have

$$d_\gamma \subset^* U^x_{m(x),\gamma} \setminus U^x_{m(x)+1,\gamma} \subset U^x_{m(x),\gamma} \setminus \mathrm{cl}_{Y_\gamma}\left(U^x_{m(x)+2,\gamma}\right).$$

It follows that for every $x \in X_\gamma$ there exists a function $g_x \in \omega^F$ such that for all but finitely many $i \in F$ the following holds:

$$\mathrm{cl}_{Y_\gamma}(U^{z_i}_{g_x(i),\gamma}) \subset U^x_{m(x),\gamma} \setminus \mathrm{cl}_{Y_\gamma}\left(U^x_{m(x)+2,\gamma}\right).$$

By the continuity, for any $f \in \Phi_\gamma$ there is a function $\pi_f \in \omega^F$ such that

$$f(U^{z_i}_{\pi_f(i),\gamma}) \subset \left(f(z_i) - \frac{1}{2^{i+1}}, f(z_i) + \frac{1}{2^{i+1}}\right) \cap [0, 1]. \qquad (\circ_0)$$

Let $h_\gamma \in \omega^F$ be a function such that $h_\gamma \geq^* g_x$ and $h_\gamma \geq^* \pi_f$ for each $x \in X_\gamma$ and $f \in \Phi_\gamma$. Such a function h_γ exists because $|\mathcal{B}_\gamma \cup \Phi_\gamma| < \mathfrak{b} = \mathfrak{c}$. Without loss of generality we can additionally assume that $U^{z_i}_{h_\gamma(i),\gamma} \cap U^{z_j}_{h_\gamma(j),\gamma} = \emptyset$ for each distinct $i, j \in \omega$.

We are in a position now to define the open neighborhood base $\mathcal{B}_\xi^\gamma$ at the point γ in the space Y_ξ: put

$$U_{0,\xi}^\gamma = X_\xi \quad \text{and} \quad U_{n,\xi}^\gamma = \bigcup_{i \in F \setminus n} U_{h_\gamma(i)+n,\gamma}^{z_i} \cup \{\gamma\} \tag{$\circ_1$}$$

for all $n \in \mathbb{N}$.

Next, we define $\mathcal{B}_\xi^x$ for $x \in X_\gamma$. Let $U_{0,\xi}^x = X_\xi$ for every $x \in X_\gamma$. For each $n \in \mathbb{N}$ let $U_{n,\xi}^x = U_{n,\gamma}^x$ if either $n \geq m(x) + 1$ or $n = m(x)$ and $b_n^x = 0$, otherwise, put $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$. It is easy to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : x \in X_\xi, n \in \omega\}$ forms a base of a topology τ_ξ, and for each $x \in X_\xi$ the family $\mathcal{B}_\xi^x = \{U_{n,\xi}^x : n \in \omega\}$ forms an open neighborhood base at x in Y_ξ. Clearly, the first equality in condition (5) is satisfied for every $\beta < \alpha \leq \xi$ and $x \in X_\beta$. Fix an arbitrary $x \in X_\gamma$. The choice of g_x and the inequality $g_x \leq^* h_\gamma$ ensure the existence of large enough $n \in \omega$ such that $U_{m(x)+2,\xi}^x \cap U_{n,\xi}^\gamma = \emptyset$. Thus Y_ξ is Hausdorff. The next claim proves the second equality in condition (5) for $\alpha = \gamma$, and therefore also for any $\alpha < \xi$.

CLAIM 6.4. *In the space Y_ξ, $U_{n,\gamma}^x$ is dense in $U_{n,\xi}^x$ for all $n \in \omega$ and $x \in X_\gamma$.*

PROOF. It suffices to consider the case $U_{n,\xi}^x = U_{n,\gamma}^x \cup \{\gamma\}$. Thus either $n < m(x)$ or $n = m(x)$ and $b_n^x > 0$. In both of these cases $z_i \in U_{n,\gamma}^x$ for almost all $i \in F$, and hence $U_{n,\gamma}^x$ intersects all neighborhoods of γ in Y_ξ. ⊣

Similarly as before, for each $n \in \omega$ and $x \in X_\xi$ put $E(x, n, \xi) = (U_{n,\xi}^x, U_{n+1,\xi}^x)$. Now we shall define a family $\Phi_\xi = \{f_{E(x,n,\xi)} : n \in \omega, x \in X_\xi\}$ such that conditions (3), (4) and (6) are satisfied when α is replaced with ξ. Put $f_{E(x,0,\xi)} \equiv 1$ for all $x \in X_\xi$. For each $x \in X_\gamma$ and $n \in \mathbb{N}$ set

$$f_{E(x,n,\xi)} = f_{E(x,n,\gamma)} \cup \{(\gamma, b_n^x)\}.$$

Conditions $(\circ_0)$, $(\circ_1)$ together with the choice of b_n^x as the limit of the sequence $\{f_{E(x,n,\gamma)}(z_i) : i \in F\}$, ensure the continuity of functions $f_{E(x,n,\xi)}$ for all $n \in \omega$ and $x \in X_\gamma$. Finally, we set

$$f_{E(\gamma,n,\xi)}(y) = \begin{cases} f_{E(z_i, h_\gamma(i)+n,\gamma)}(y) & \text{if } y \in U_{h_\gamma(i)+n,\gamma}^{z_n} \text{ and } i \in F \setminus n; \\ 1 & \text{if } y = \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Since the family $\{U_{h_\gamma(i),\gamma}^{z_i} : i \in \omega\}$ is locally finite in Y_γ, it remains to check that $f_{E(\gamma,n,\xi)}$ is continuous at γ. For this it is enough to prove the first equality of condition (4) for $x = \gamma$ and $\xi = \alpha$. This follows from the following equation:

$$f_{E(\gamma,n,\xi)}(U_{n+1,\xi}^\gamma) = \{1\} \cup \bigcup_{i \in F \setminus (n+1)} f_{E(\gamma,n,\xi)}(U_{h_\gamma(i)+n+1,\gamma}^{z_i}) =$$

$$= \{1\} \cup \bigcup_{i \in F \setminus (n+1)} f_{E(z_i, h_\gamma(i)+n,\gamma)}(U_{h_\gamma(i)+n+1,\gamma}^{z_i}) = \{1\}.$$

The fact that functions from Φ_ξ satisfy the first equation from (4) also for $x \in X_\gamma$ follows from (5) established in Claim 6.4 and from the continuity of functions in Φ_ξ.

It is clear that any function from Φ_ξ satisfies the second equation of condition (4). Finally, it is a tedious routine to check that the families $\mathcal{B}_\xi$ and Φ_ξ satisfy conditions (1), (2), (3), and (6), which concludes case 2.

3) Let $X_\xi = \bigcup_{\alpha \in \xi} X_\alpha$. For each $x \in X_\xi$ let $\theta_x = \min\{\alpha : x \in X_\alpha\}$. For each $x \in X_\xi$ and $n \in \omega$ put

$$U_{n,\xi}^x = \bigcup_{\theta_x \leq \alpha < \xi} U_{n,\alpha}^x \quad \text{and} \quad f_{E(x,n,\xi)} = \bigcup_{\theta_x \leq \alpha < \xi} f_{E(x,n,\alpha)}.$$

It is straightforward to check that the family $\mathcal{B}_\xi = \{U_{n,\xi}^x : n \in \omega, x \in X_\xi\}$ forms a base of a topology τ_ξ on X_ξ, and the families $\Phi_\xi = \{f_{E(x,n,\xi)} : n \in \omega, x \in X_\xi\}$ and $\mathcal{B}_\xi$ satisfy conditions (1)–(6). Thus, it remains to establish the continuity of functions in Φ_ξ, which we do next.

Assume to the contrary that there exists $f_{E(x,n,\xi)} \in \Phi_\xi$ which is not continuous at a point $z \in X_\xi$. Then there exists a sequence $\{a_k\}_{k \in \omega}$ such that $\lim_{k \in \omega} a_k = z$ but

$$\lim_{k \in \omega} f_{E(x,n,\xi)}(a_k) \neq f_{E(x,n,\xi)}(z).$$

For each $k \in \omega$ let $\mu_k \in \xi$ be such that $\{a_k, x, z\} \subset X_{\mu_k}$. Since Y_ξ is first-countable, X_0 is a dense subset of X_ξ and the functions $f_{E(x,n,\mu_k)}$, $k \in \omega$ are continuous on Y_{μ_k}, for each $k \in \omega$ we can pick $b_k \in X_0$ such that

$$\lim_{k \in \omega} b_k = \lim_{k \in \omega} a_k = z \qquad \text{and} \qquad |f_{E(x,n,\mu_k)}(b_k) - f_{E(x,n,\mu_k)}(a_k)| \leq 1/k.$$

For each $k \in \omega$, by condition (6), $f_{E(x,n,\xi)}\!\restriction_{X_{\mu_k}} = f_{E(x,n,\mu_k)}$ implying that

$$|f_{E(x,n,\xi)}(b_k) - f_{E(x,n,\xi)}(a_k)| = |f_{E(x,n,\mu_k)}(b_k) - f_{E(x,n,\mu_k)}(a_k)| \leq 1/k. \quad (\bullet)$$

Since the function $f_{E(x,n,\mu_0)}$ is continuous on Y_{μ_0} condition (6) implies the following:

$$\lim_{k \in \omega} f_{E(x,n,\xi)}(b_k) = \lim_{k \in \omega} f_{E(x,n,\mu_0)}(b_k) = f_{E(x,n,\mu_0)}(\lim_{k \in \omega} b_k)$$
$$= f_{E(x,n,\mu_0)}(z) = f_{E(x,n,\xi)}(z).$$

But then

$$0 \neq f_{E(x,n,\xi)}(z) - \lim_{k \in \omega} f_{E(x,n,\xi)}(a_k) = \lim_{k \in \omega} f_{E(x,n,\xi)}(b_k) - \lim_{k \in \omega} f_{E(x,n,\xi)}(a_k) =$$
$$= \lim_{k \in \omega} \left(f_{E(x,n,\xi)}(b_k) - f_{E(x,n,\xi)}(a_k) \right) = 0,$$

where the latter equality follows from equation ($\bullet$). The obtained contradiction ensures that all functions in Φ_ξ are continuous, and thus concludes case 3.

By the construction, the space $Y_{\mathfrak{c}}$ is Tychonoff, first-countable and contains X as a dense subspace. Let A be any countable subset of $Y_{\mathfrak{c}}$. If the set $B = A \cap \mathfrak{c}$ is infinite, then consider $h(B) \in \mathfrak{c}$. By the construction, either B has an accumulation point in $Y_{h(B)}$ or $h(B)$ is an accumulation point of B in $Y_{h(B)+1}$. In both cases B has an accumulation point in $Y_{\mathfrak{c}}$. If $A \subset^* X$, then either it has an accumulation point in X, or A is closed and discrete in X. In the latter case either A has an accumulation point in $Y_{h(A)}$ or $h(A)$ is an accumulation point of A in $Y_{h(A)+1}$. Thus $Y_{\mathfrak{c}}$ is countably compact, which completes our proof. ⊣

Proof of Theorem 3.10. We need to show that the following assertions are equivalent:

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
(2) Every first-countable zero-dimensional Hausdorff space of weight $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.
(3) Every first-countable zero-dimensional Hausdorff space of cardinality $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.

PROOF.   The implication $(1) \Rightarrow (2)$ follows from Proposition 4.11. Since the weight of an infinite first-countable space doesn't exceed its cardinality, implication $(2) \Rightarrow (3)$ holds.

$(3) \Rightarrow (1)$. Corollary 5.7 implies that $\mathfrak{b} = \mathfrak{c}$. Assuming that $\mathfrak{s} < \mathfrak{c}$, we get that the normal first-countable space Y constructed in Proposition 5.2 has cardinality $\mathfrak{s} < \mathfrak{c}$ and, thus, is zero-dimensional. Then, by the assumption, Y embeds densely into a first-countable zero-dimensional pseudocompact space. But this contradicts Proposition 5.2. Hence $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.   ⊣

Proof of Theorem 3.11. We need to show that a subspace X of the Cantor space is a λ-set if and only if the Pixley-Roy hyperspace $\mathrm{PR}(X)$ embeds densely into a first-countable pseudocompact space.

PROOF.   The implication $(\Rightarrow)$ follows from [5, Theorem 4.2].
The implication $(\Leftarrow)$ follows from Proposition 5.5.   ⊣

REFERENCES

[1] U. ABRAHAM, I. GORELIC, and I. JUHÁSZ, *On Jakovlev spaces*. **Israel Journal of Mathematics**, vol. 152 (2006), pp. 205–219.

[2] S. BARDYLA and A. OSIPOV, *On regular κ-bounded spaces admitting only constant continuous mappings into T_1 spaces of pseudo-character $\leq \kappa$*. **Acta Mathematica Hungarica**, vol. 163 (2021), pp. 323–333.

[3] S. BARDYLA and L. ZDOMSKYY, *On regular separable countably compact $\mathbb{R}$-rigid spaces*. **Israel Journal of Mathematics**, vol. 255 (2023), pp. 783–810.

[4] S. BARDYLA, J. SUPINA, and L. ZDOMSKYY, *Open filters and measurable cardinals*, preprint, arXiv:2301.08704.

[5] M. BELL, *First countable pseudocompactifications*. **Topology and Its Applications**, vol. 21 (1985), pp. 159–166.

[6] J. BRENDLE and S. SHELAH, *Ultrafilters on ω—Their ideals and their cardinal characteristics*. **Transactions of the American Mathematical Society**, vol. 351 (1999), no. 7, pp. 2643–2674.

[7] L. BUKOVSKÝ, **The Structure of the Real Line**, Monografie Matematyczne, 71, Birkhäuser, Basel, 2011.

[8] E. VAN DOUWEN, *The integers and topology*, **Handbook of Set-Theoretic Topology** (K. Kunen, J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 111–167.

[9] E. VAN DOUWEN and T. PRZYMUSINSKI, *First countable and countable spaces all compactifications of which contain $\beta(\mathbb{N})$*. **Fundamenta Mathematicae**, vol. 102 (1979), pp. 229–234.

[10] A. DOW, *Countable π-character, countable compactness and PFA*. **Topology and Its Applications**, vol. 239 (2018), pp. 25–34.

[11] A. DOW and F. D. TALL, *PFA(S)[S] and countably compact spaces*. **Topology and Its Applications**, vol. 230 (2017), pp. 393–416.

[12] ———, *Hereditarily normal manifolds of dimension greater than one may all be metrizable*. **Transactions of the American Mathematical Society**, vol. 372 (2019), pp. 6805–6851.

[13] R. ENGELKING, **General Topology**, Heldermann Verlag, Berlin, 1989.

[14] S. FRANKLIN and M. RAJAGOPALAN, *Some examples in topology*. **Transactions of the American Mathematical Society**, vol. 155 (1971), no. 2, pp. 305–314.

[15] M. HRUŠÁK, *Almost disjoint families in topology*, **Recent Progress in General Topology III** (K. P. Hart, J. van Mill, and P. Simon, editors), Atlantis Press, Paris, 2014, pp. 601–638.

[16] M. HRUŠÁK and J. MOORE, *Twenty problems in set-theoretic topology*, **Open Problems in Topology II** (E. Pearl, editor), Elsevier, Amsterdam, 2007, pp. 111–113.

[17] S. ILIADIS and V. TZANNES, *Spaces on which every continuous map into a given space is constant*. **Canadian Journal of Mathematics**, vol. 38 (1986), no. 6, pp. 1281–1298.

[18] F. B. JONES, *Hereditary separable, non-completely regular spaces*, **Topology Conference: Virginia Polytechnic Institute and State University** (Raymond F. Dickman, Jr. and Peter Fletcher, editors), Lecture Notes in Mathematics, 375, Springer-Verlag, Berlin, 1974, pp. 149–152.

[19] K. KUNEN, **Set Theory, Studies in Logic: Mathematical Logic and Foundations**, *vol. 34*, College Publications, London, 2011, viii + 401 pp.

[20] J. T. MOORE, *The proper forcing axiom*, **Proceedings of the International Congress of Mathematicians**, Hyderabad, 2010.

[21] P. NYIKOS, *On first-countable, countably compact spaces III. The problem of obtaining separable noncompact examples*, **Open Problems in Topology** (J. van Mill and G. M. Reed, editors), North-Holland, Amsterdam, 1990, pp. 127–161.

[22] ———, *First-countable, countably compact, noncompact spaces*, **Open Problems in Topology II** (E. Pearl, editor), Elsevier, Amsterdam, 2007, pp. 217–224.

[23] P. NYIKOS and J. VAUGHAN, *On first-countable, countably compact spaces I: $\left(\omega_1, \omega_1^*\right)$-gaps*. **Transactions of the American Mathematical Society**, vol. 279 (1983), no. 2, pp. 463–469.

[24] P. NYIKOS and L. ZDOMSKYY, *Locally compact, ω_1-compact spaces*. **Annals of Pure and Applied Logic**, vol. 175 (2024), no. 1, p. 03324.

[25] A. OSTASZEWSKI, *On countably compact, perfectly normal spaces*. **Journal of the London Mathematical Society**, vols. 2–14 (1976), no. 3, pp. 505–516.

[26] P. SIMON and G. TIRONI, *First countable extensions of regular spaces*. **Proceedings of the American Mathematical Society**, vol. 132 (2004), no. 9, pp. 2783–2792.

[27] R. M. STEPHENSON, *Moore-closed and first countable feebly compact extension spaces*. **Topology and Its Applications**, vol. 27 (1987), pp. 11–28.

[28] T. TERADA and J. TERASAWA, *Maximal extensions of first-countable spaces*. **Proceedings of the American Mathematical Society**, vol. 85 (1982), no. 1, pp. 95–99.

[29] E. PEARL, **Problems from Topology Proceedings**, Topology Atlas, Toronto, ON, 2003.

[30] F. D. TALL, *PFA(S)[S]: More mutually consistent topological consequences of PFA and $V = L$*. **Canadian Journal of Mathematics**, vol. 64 (2012), no. 5, pp. 1182–1200.

[31] V. TZANNES, *A Moore strongly rigid space*. **Canadian Mathematical Bulletin**, vol. 34 (1991), no. 4, pp. 547–552.

[32] ———, *Two Moore spaces on which every continuous real-valued function is constant*. **Tsukuba Journal of Mathematics**, vol. 16 (1992), no. 1, pp. 203–210.

[33] ———, *A Hausdorff countably compact space on which every continuous real-valued function is constant*. **Topology Proceedings**, vol. 21 (1996), pp. 239–244.

[34] M. VIALE and C. WEISS, *On the consistency strength of the proper forcing axiom*. **Advances in Mathematics**, vol. 228 (2011), no. 5, pp. 2672–2687.

[35] W. WEISS, *Countably compact spaces and Martin's axiom*. **Canadian Journal of Mathematics**, vol. 30 (1978), no. 2, pp. 243–249.

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