AN *Md*-CLASS OF SETS INDEXED BY A REGRESSIVE FUNCTION

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(Received 12 October 1965)

1. Introduction

This paper deals with the study of a particular *md*-class of sets. The underlying theory was introduced and studied by J. C. E. Dekker in [4]. We shall assume that the reader is familiar with the terminology and main results of this paper; in particular with the concepts of *md*-class of sets, *gc*-class of sets, *gc*-set, *gc*-function and the RET of a *gc*-class of sets. We also use the following notations of [4]:

$$\varepsilon = \text{the set of all non-negative integers } (\textit{numbers}),$$
$$R = Req\ (\varepsilon).$$

$\{\rho_n\}$ will stand for the well-known canonical enumeration of the class of all finite sets and r_x for the recursive function defined by r_x = the cardinality of ρ_x. We write $\subset$ for inclusion and $\subsetneqq$ for proper inclusion. For any set α and any number k, we write

$$C(\alpha, k) = \{n \mid \rho_n \subset \alpha \text{ and } r_n = k\},$$
$$\text{Bin}(\alpha) = \{C(\alpha, k) \mid k \geqq 1\}.$$

$C(\alpha, k)$ will also be denoted by $\binom{\alpha}{k}$. The familiar recursive functions j, k and l, such that j maps ε^2 one-to-one onto ε and $j(k(n), l(n)) = n$, will be used.

We recall that a one-to-one function t_n from ε into ε is *regressive*, if the mapping

$$t_n \to t_{n \dot{-} 1}$$

has a partial recursive extension. While regressive functions with finite domains have been recently introduced [cf. 5], we shall always assume that a regressive function is everywhere defined, i.e., has domain ε.

It is known [4, p. 630] that for every set α, Bin(α) is an *md*-class of

[1] The problem considered in this paper was suggested to the author by Professor J. C. E. Dekker at Rutgers University. Research was supported by the U. S. National Science Foundation.

https://doi.org/10.1017/S1446788700004146 Published online by Cambridge University Press

sets. On the other hand, $\mathrm{Bin}(\alpha)$ is a *gc*-class if and only if α is regressive or contains an infinite r.e. subset. Let α denote any set and t_n any regressive function. We are interested in the class of sets obtained by indexing the members of $\mathrm{Bin}(\alpha)$ with the function t_n,

$$\mathrm{Bin}(t_n, \alpha) = \left\{ j\left(t_k, \binom{\alpha}{k+1}\right) \mid k \geqq 0 \right\}.$$

We note that $\mathrm{Bin}(t_n, \alpha)$ is an *md*-class of sets. In addition, it is easily seen that if $\mathrm{Bin}(\alpha)$ is a *gc*-class then so is $\mathrm{Bin}(t_n, \alpha)$. One consequence of this fact is that if α is finite then $\mathrm{Bin}(t_n, \alpha)$ is a *gc*-class; and in this case its RET is readily shown to equal the cardinality of α. The main object of this paper is the following:

Let t_n be any regressive function. We wish to find a necessary and sufficient condition on a set α in order that $\mathrm{Bin}(t_n, \alpha)$ be a *gc*-class; and in the event that it is a *gc*-class, we want to determine its RET.

2. Preliminaries and terminology

We shall henceforth assume that the regressive function t_n is fixed. Also, in view of an earlier remark, we may restrict our attention to the case that α is an infinite set.

REMARK. It is easy to show that if t_n is a recursive function then $\mathrm{Bin}(t_n, \alpha)$ is a *gc*-class if and only if $\mathrm{Bin}(\alpha)$ is a *gc*-class; moreover, if $\mathrm{Bin}(t_n, \alpha)$ is a *gc*-class, it has the same RET as $\mathrm{Bin}(\alpha)$. We could therefore suppose that the function t_n is regressive, yet not recursive; however, this is not necessary for the discussion which follows.

Throughout this paper we let π denote the range of t_n and let $T = \mathrm{Req}(\pi)$. If t_n is a recursive function then π is a recursive set and $T = R$; otherwise, π is an immune regressive set and T is an infinite regressive isol.

If f is a function, then δf will denote its domain and ρf its range. Let u_n and v_n be two one-to-one functions from ε into ε. Then $u_n \leqq * v_n$, if there is a partial recursive function f such that

$$\rho u \subset \delta f \text{ and } (\forall n)[f(u_n) = v_n]. \tag{1}$$

In addition, u_n and v_n are said to be *recursively equivalent* (denoted $u_n \simeq v_n$), if there is a one-to-one partial recursive function f such that (1) holds. Clearly one has,

$$u_n \simeq v_n \rightarrow \rho u_n \simeq \rho v_n.$$

Also, it can be shown [**3**], that

$$u_n \simeq v_n \Leftrightarrow (u_n \leqq * v_n \text{ and } v_n \leqq * u_n).$$

In [4, § 4] the concept of a gc-function of a gc-class is introduced. It turns out to be useful to introduce the following modification of this notion.

DEFINITION. Let f be a partial recursive function and let

$$\gamma = \{j(t_k, e_{k+1}) \mid k \geqq 0\}$$

be a choice set of $\mathrm{Bin}(t_n, \alpha)$. Then f is a *gc-function* of γ, if for every number k and every $w \in \binom{\alpha}{k+1}$,

$$j(t_k, w) \in \delta f \text{ and } fj(t_k, w) = j(t_k, e_{k+1}).$$

Regarding this definition, it is readily seen that γ is a gc-set of $\mathrm{Bin}(t_n, \alpha)$ if and only if γ has at least one gc-function.

3. Fundamental properties of $\mathrm{Bin}(t_n, \alpha)$

INTRODUCTORY REMARK. Let α be any infinite set and let $\mathrm{Bin}(t_n, \alpha)$ be a gc-class. We wish to observe here that the RET of $\mathrm{Bin}(t_n, \alpha)$ is regressive. Let

$$\gamma = \{j(t_k, e_{k+1}) \mid k \geqq 0\}$$

denote a gc-set of $\mathrm{Bin}(t_n, \alpha)$. Clearly, there exists a recursive function $y(x)$ such that

$$y(x) \in \binom{\alpha}{n}, \quad \text{whenever} \quad x \in \binom{\alpha}{n+1}.$$

Using this fact together with the regressiveness of the function t_n, it can readily be shown that

$$j(t_0, e_1), \quad j(t_1, e_2), \quad j(t_2, e_3), \cdots,$$

represents a regressive enumeration of the set γ. Hence γ is a regressive set and therefore the RET of $\mathrm{Bin}(t_n, \alpha)$ is also regressive.

DEFINITION. For any two sets α and β, $\alpha \leqq * \beta$ if there is a partial recursive function g such that

(2) $\alpha \subset \delta g$, g is one-to-one on α, and $g(\alpha) \subset \beta$.

We shall say "$\alpha \leqq * \beta$ by g" if g is a partial recursive function such that (2) holds.

THEOREM 1. *Let α be any set with $\pi \leqq * \alpha$. Then* $\mathrm{Bin}(t_n, \alpha)$ *is a gc-class and its RET is T.*

PROOF. Assume the hypothesis and let $\pi \leqq * \alpha$ by g. We note that α is infinite since π is infinite. Set $a_n = g(t_n)$, and let the function e_n be defined by

$$e_0 = 0 \text{ and } \rho_{e(n+1)} = (a_0, a_1, \cdots, a_n).$$

https://doi.org/10.1017/S1446788700004146 Published online by Cambridge University Press

Since g is one-to-one on π, it follows that a_n and e_n are also one-to-one functions. In addition, $r_{e(n)} = n$ and therefore for each number n,

$$j(t_n, e_{n+1}) \in j\left(t_n, \binom{\alpha}{n+1}\right).$$

Let

$$\delta = \{j(t_k, e_{k+1}) \mid k \geqq 0\}.$$

Then δ is a choice set of $\mathrm{Bin}(t_n, \alpha)$. To complete the proof it suffices to show that

(a) δ is a *gc*-set of $\mathrm{Bin}(t_n, \alpha)$,
(b) $\mathrm{Req}(\delta) = T$.

According to the definition of a_n, we see that $t_n \leqq * a_n$. Combining this with the fact that t_n is a regressive function, it follows that

$$t_n \leqq * e_{n+1}. \tag{3}$$

From (3) we see that the mapping

$$j(t_n, x) \to j(t_n, e_{n+1}), \qquad \text{for } n, x \in \varepsilon,$$

has a partial recursive extension. Any one of these extensions will be a *gc*-function for δ and hence δ is a *gc*-set. This proves (a).

For part (b), consider the two relations,

$$t_n \leqq * j(t_n, e_{n+1}) \text{ and } j(t_n, e_{n+1}) \leqq * t_n.$$

The first follows from (3) and the second is clear. Together they imply that $t_n \simeq j(t_n, e_{n+1})$, which gives $\pi \simeq \delta$ and therefore $\delta \in T$. This proves (b) and completes the proof of Theorem 1.

REMARK. We wish to observe here that there are sets α for which $\mathrm{Bin}(t_n, \alpha)$ is a *gc*-class while $\mathrm{Bin}(\alpha)$ is not. It is well-known that there exist immune sets which are not regressive, yet contain infinite regressive subsets. Let us suppose that α is such a set and that the regressive function t_n ranges over a subset of α. Then clearly $\pi \leqq * \alpha$ and therefore $\mathrm{Bin}(t_n, \alpha)$ is a *gc*-class. On the other hand, $\mathrm{Bin}(\alpha)$ will not be a *gc*-class since α is neither regressive nor contains an infinite r.e. subset.

PROPOSITION 1. *Let α be an infinite set. If* $\mathrm{Bin}(t_n, \alpha)$ *is a gc-class, then it has a gc-set*

$$\delta = \{j(t_k, e_{k+1}) \mid k \geqq 0\}$$

with the property that

$$\rho_{e_1} \subsetneqq \rho_{e_2} \subsetneqq \rho_{e_3} \subsetneqq \cdots. \tag{4}$$

PROOF. Left to the reader.

PROPOSITION 2. *Let* $\mathrm{Bin}(t_n, \alpha)$ *be a gc-class having a gc-set*

$$\delta = \{j(t_k, e_{k+1}) \mid k \geqq 0\}$$

such that (4) *holds and*

$$\sum_1^{\infty} \rho_{e(k)} \subsetneqq \alpha.$$

Then,

(a) $\pi \leqq * \alpha$,

(b) $RET\ (\mathrm{Bin}(t_n, \alpha)) = T$.

PROOF. By Theorem 1, (a) implies (b) and therefore we may restrict our attention to proving (a). Let f denote a *gc*-function of δ and

$$u \in \alpha - \sum_1^{\infty} \rho_{e(k)}.$$

In addition, set

$$(a_0) = \rho_{e(1)},$$

$$(a_0, a_1, \cdots, a_n) = \rho_{e(n+1)}.$$

Note that $u \neq a_n$ for every n. To prove that $\pi \leqq * \alpha$, it is sufficient to show that $t_n \leqq * a_n$, i.e., that the mapping

$$(*) \qquad t_n \rightarrow a_n,$$

has a partial recursive extension. This will be our approach here.

Assume that the value of t_n is given. Using the regressiveness of the function t_x we can find the $n+2$-numbers n and $t_0, t_1, \cdots, t_n$. We can now determine the number

$$w_0 \in \binom{\alpha}{1} \text{ such that } \rho_{w_0} = (u),$$

and then compute $j(t_0, e_1) = fj(t_0, w_0)$. From this value the number e_1 can be found and hence so can the value a_0. We now consider the three numbers t_1, u and a_0 and proceed to determine a_1. First we find the number

$$w_1 \in \binom{\alpha}{2} \text{ such that } \rho_{w_1} = (u, a_0),$$

and then we compute $j(t_1, e_2) = fj(t_1, w_1)$. The number e_2 can now be found and hence also the number a_1, since $(a_1) = \rho_{e(2)} - \rho_{e(1)}$. It is readily seen that by continuing in this fashion we shall be able to find the number a_n. Since the procedure is effective, we conclude that the mapping indicated by $(*)$ has a partial recursive extension. This gives $\pi \leqq * \alpha$ and completes the proof.

4. Two theorems

INTRODUCTORY REMARK. Let a_n be an everywhere defined one-to-one function. Consider the mapping f defined by

$$j(s_m, a_n) \xrightarrow{f} j(s_h, a_h), \text{ where } h = \text{minimum } (m, n).$$

Under the assumption that s_n is a recursive function, it is readily verified that

$$a_n \text{ is a regressive function} \leftrightarrow f \text{ has a partial recursive extension.}$$

We now relativize the notion of a regressive function.

DEFINITION. Let s_n be a regressive function and a_n any one-to-one function from ε into ε. Then a_n is *regressive in* s_n, if the mapping

$$j(s_m, a_n) \rightarrow j(s_h, a_h), \text{ where } h = \text{minimum } (m, n),$$

has a partial recursive extension.

DEFINITION. Let s_n be a regressive function and α any infinite set. Then α is *regressive in* s_n, if there is an everywhere defined one-to-one function a_n ranging over the set α with a_n regressive in s_n.

REMARK. Regarding the above definitions we note that

$$a_n \text{ regressive in } s_n \rightarrow j(s_n, a_n) \text{ is a regressive function.}$$

In addition, if a_n and s_n are both regressive functions then a_n is regressive in s_n. It follows from this fact that every infinite regressive set is regressive in every regressive function.

Finally we wish to note that for each infinite set α there are c regressive functions s_n with α regressive in s_n. To see this, note that for any everywhere defined one-to-one function a_n, the function

$$s_n = 2^{a_0} \cdot 3^{a_1} \cdots p_n^{a_n},$$

where p_n denotes the $n+1^{\text{st}}$ prime, is a regressive function. Moreover, a_n is regressive in s_n. Since there are c choices for a one-to-one function a_n ranging over α, there will be c regressive functions of the type s_n. Clearly, α is regressive in each of these.

THEOREM 2. *Let α be any infinite set which is regressive in t_n. Then*

(*a*) $\text{Bin}(t_n, \alpha)$ *is a gc-class,*

(*b*) $\rho j(t_n, a_n) \in RET(\text{Bin}(t_n, \alpha))$,

where a_n is any everywhere defined one-to-one function ranging over α which is regressive in t_n.

PROOF. Let a_n denote any everywhere defined one-to-one function ranging over α which is regressive in t_n. Let the function e_n be defined by

$$e_0 = 0 \text{ and } \rho_{e(n+1)} = (a_0, a_1, \cdots, a_n).$$

Then

$$\delta = \{j(t_k, e_{k+1}) \mid k \geqq 0\} \text{ is a choice set of } \mathrm{Bin}(t_n, \alpha).$$

We now show that δ is a gc-set of $\mathrm{Bin}(t_n, \alpha)$. For this purpose, let f denote any partial recursive extension of the mapping

$$j(t_m, a_n) \to j(t_h, a_h), \text{ where } h = \text{minimum } (m, n).$$

Set

$$\sigma = \sum_0^\infty j\left(t_k, \binom{\alpha}{k+1}\right),$$

and let $w \in \sigma$ with $w = j(t_n, u)$. We have to show that there exists a partial recursive function at least defined on σ and mapping

$$w = j(t_n, u) \to j(t_n, e_{n+1}).$$

Both numbers t_n and u can be determined from w, and hence also the numbers $t_0, t_1, \cdots, t_n$ together with their respective indices. In addition, the $n+1$-numbers

$$a_{u(0)} < a_{u(1)} < \cdots < a_{u(n)} \text{ such that } \rho_u = (a_{u(0)}, a_{u(1)}, \cdots, a_{u(n)})$$

can be found; however not necessarily their respective indices. We wish to show that we can also determine the number $j(t_n, e_{n+1})$. It is readily seen that this amounts to finding the $n+1$-numbers $a_0, a_1, \cdots, a_n$. We first observe that

$$a_0 = lfj(t_0, a_{u(0)}), \text{ since } 0 = \text{minimum } (0, u(0)),$$

and hence a_0 can be found. To determine a_1, compute the numbers

$$j(t_x, a_x) = fj(t_1, a_{u(0)}),$$
$$j(t_y, a_y) = fj(t_1, a_{u(1)}).$$

Since t_n is a regressive function each of the numbers x and y can be found; moreover $x = 1$ or $y = 1$, since maximum $(u_0, u_1) \geqq 1$. If $x = 1$ then $a_1 = a_x$, and if $y = 1$ then $a_1 = a_y$; in any event the number a_1 can be obtained. By continuing in this fashion it is readily seen that we can determine all of the $n+1$-numbers $a_0, a_1, \cdots, a_n$ and hence also the number $j(t_n, e_{n+1})$. We can conclude therefore that δ is a gc-set of $\mathrm{Bin}(t_n, \alpha)$.

To complete the proof, it remains to prove (b). Since δ is a gc-set of $\mathrm{Bin}(t_n, \alpha)$, this is equivalent to showing that

$$\rho j(t_n, a_n) \simeq \delta.$$

https://doi.org/10.1017/S1446788700004146 Published online by Cambridge University Press

We shall establish this recursive equivalence by proving that

$$j(t_n, a_n) \simeq j(t_n, e_{n+1}).$$

For this purpose, let us first suppose that the number $j(t_n, a_n)$ is given. We can determine the numbers t_n and a_n and hence also the numbers $t_0, t_1, \cdots, t_n$. Moreover, according to the definition of f, we have that

$$a_i = lfj(t_i, a_n), \text{ for } i = 0, 1, \cdots, n \dot{-} 1.$$

Therefore the $n+1$-numbers $a_0, a_1, \cdots, a_n$ can be found; hence also the number e_{n+1}. This means that we can find the number $j(t_n, e_{n+1})$. It follows from these remarks that

$$(5) \qquad j(t_n, a_n) \leqq * j(t_n, e_{n+1}).$$

Now assume that the number $j(t_n, e_{n+1})$ is given. Then the numbers t_n and e_{n+1} can be found as well as the members of the (finite) set $\rho_{e(n+1)}$. We wish to determine which member of $\rho_{e(n+1)}$ is a_n. This can be done by computing the $n+1$-ordered pairs,

$$(kfj(t_n, a), lfj(t_n, a)), \text{ for } a \in \rho_{e(n+1)}.$$

Taking into account the definition of the function f, it follows that exactly one of these pairs will have as its first member the number t_n; the second member of this particular pair will be a_n. Since these pairs can be effectively obtained, we can find a_n and hence also the number $j(t_n, a_n)$. We can conclude from these remarks that

$$(6) \qquad j(t_n, e_{n+1}) \leqq * j(t_n, a_n).$$

Combining (5) and (6) we obtain

$$j(t_n, a_n) \simeq j(t_n, e_{n+1}),$$

as was to be shown. This completes the proof of Theorem 2.

REMARK. Let Λ_R denote the collection of all regressive isols. In [3] Dekker introduced and studied an extension of the function $\min(x, y) : \varepsilon^2 \to \varepsilon$ to a function $\min(X, Y) : \Lambda_R^2 \to \Lambda_R$. In terms of this extension we can give the following corollary to Theorem 2.

COROLLARY. *Let* α *be a regressive immune set, and let* $A = \text{Req}(\alpha)$. *Let* $T = \text{Req}(\rho t_n)$ *be a regressive isol. Then* $Bin(t_n, \alpha)$ *is a gc-class and its RET is* $\min(T, A)$.

PROOF. Note that $A, T \in \Lambda_R$. Let a_n be any regressive function ranging over α. Then a_n is regressive in t_n and hence α is regressive in t_n. Therefore $\text{Bin}(t_n, \alpha)$ is a *gc*-class; let its RET be V. By Theorem 2,

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$$\rho j(t_n, a_n) \in V.$$

In addition, according to the definition of the min(X, Y) function [3, p. 361], the function $j(t_n, a_n)$ ranges over a set in min(T, A). This gives the desired result, $V = \min(T, A)$.

REMARK. The next theorem tells us that the disjunction of the two properties mentioned in Theorems 1 and 2 characterizes the infinite sets α for which Bin(t_n, α) is a gc-class.

THEOREM 3. *Let α be an infinite set. Then Bin(t_n, α) is a gc-class if and only if either $\pi \leqq * \alpha$ or α is regressive in t_n.*

PROOF. In view of Theorems 1 and 2 we only need to show that the condition is necessary. Assume that Bin(t_n, α) is a gc-class and let

$$\delta = \{j(t_k, e_{k+1}) \mid k \geqq 0\}$$

be one of its gc-sets. By Proposition 1, we may assume that δ has property (4). Set

$$\gamma = \sum_{1}^{\infty} \rho_{e(k)}.$$

Clearly, $\gamma \subset \alpha$; also according to Proposition 2, $\gamma \subsetneqq \alpha$ implies $\pi \leqq * \alpha$. To complete the proof it therefore suffices to show that,

$$\text{if } \gamma = \alpha \text{ then } \alpha \text{ is regressive in } t_n.$$

This will be our approach here.

Let the function a_n be defined by

$$(a_0, a_1, \cdots, a_{n-1}) = \rho_{e(n)}, \qquad \text{for } n \geqq 1. \tag{7}$$

Since $\gamma = \alpha$, a_n is an everywhere defined (one-to-one) function which ranges over α. We proceed to show that a_n is regressive in t_n. Assume that the number $j(t_m, a_n)$ is given and let $h =$ minimum (m, n). We wish to show that we can effectively find the number $j(t_h, a_h)$. First of all, we can determine t_m and hence also the numbers $t_0, t_1, \cdots, t_m$ together with their respective indices. We can also find the number a_n, though not immediately its index n. Let w_0 be defined by

$$\rho_{w_0} = (a_n).$$

Then $w_0 \in \binom{\alpha}{1}$ and by computing

$$fj(t_0, w_0) = j(t_0, e_0),$$

we can find e_0. In view of (7), we can also find a_0. Now compare a_n with a_0, and at the same time consider t_m (recall that the value of m can be found).

If $a_n = a_0$ or $t_m = t_0$, then it follows that $h = 0$ and hence

$$j(t_h, a_h) = j(t_0, a_0),$$

and we are done. Otherwise $a_n \neq a_0$ and $h \geqq 1$. Set

$$\rho_{w_1} = (a_0, a_n).$$

By computing

$$fj(t_1, w_1) = j(t_1, e_1),$$

we can effectively find e_1 and hence also a_1. Now compare a_n with a_1 and at the same time consider t_m.

If $a_n = a_1$ or $t_m = t_1$, then it follows that $h = 1$ and hence

$$j(t_h, a_h) = j(t_1, a_1),$$

and we are done. Otherwise $a_n \neq a_1$ and $h \geqq 2$. We would now proceed to determine a_2, etc. By continuing in this way, exactly one of the following two events will occur:

(I) We reach a point where a_k is obtained with $k < m$, $a_n = a_k$ and $t_m \neq t_k$.

(II) We reach a point where a_k is obtained with $k = m$.

Whether (I) or (II) occurs, we see that $h = k$ and therefore

$$j(t_h, a_h) = j(t_k, a_k).$$

In any event $j(t_h, a_h)$ can be found. In view of the effectiveness of this procedure, it follows that a_n is regressive in t_n, as was to be shown. This completes the proof of Theorem 3.

References

[1] J. C. E. Dekker, "Infinite Series of Isols", *Amer. Math. Soc. Proc. of Symposia in Pure Mathematics*, 5 (1962), 77—96.

[2] J. C. E. Dekker, "An Infinite Product of Isols", *Illinois J. Math.*, 7 (1963) 668—680.

[3] J. C. E. Dekker, "The Minimum of Two Regressive Isols", *Math. Zeitschr.*, 83 (1964), 345—366.

[4] J. C. E. Dekker, "The Recursive Equivalence Type of a Class of Sets", *Bull. Amer. Math. Soc.*, 70 (1964), 628—632.

[5] J. C. E. Dekker, "Closure Properties of Regressive Functions", *Proc. London Math. Soc.*, 3, 15 (1965), 226—238.

[6] J. C. E. Dekker and J. Myhill, "Recursive Equivalence Types", *Univ. California Publ. Math.* (*N.S.*), 3 (1960), 67—214.

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